

# Analysis of a model of elastic plastic mixtures (“Prandtl-Reuss-mixtures”)

Project of Josef Málek and Jens Frehse

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# Outline

Introduction

KMRS model

Prandtl-Reuss

Prandtl-Reuss mixtures

We refer to a paper

J. Kratochvíl, J. Málek, K.R. Rajagopal and A.R. Srinivasa:  
*Modelling of the response of elastic plastic materials treated as  
mixture of hard and soft regions*, ZAMP 55 (2004), 500-518

and to

Phd-Thesis of Luba Khasina

advised by Kratochvíl-Málek-Frehse, where first steps are done to  
formulate the model of KMRS in the framework of Sobolev spaces  
nad variational inequalities.

KMRS consider a body  $\Omega$  consisting of soft and hard material and loading.

During the loading process, soft material is assumed to be perfectly elastic plastic, the hard material is assumed to satisfy a certain hardening rule in the non elastic region.

The beauty of the theory consists in the fact, that no artificial internal variables enter, the history of the plastic deformation of the soft material replaces the internal variable.

## Purpose of the present talk

Presenting a mathematical formulation in the framework of quasi-variational inequalities and Sobolev spaces.

The theory turns out to be very similar to the Prandtl-Reuss-law for single materials; all known regularity results hold also for the mixture model.

## KMRS-model after simplification and choice of example for illustration

$\Omega$	basic domain $\subset \mathbb{R}^3$
$[0, T]$	loading interval
$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s(x, t), x \in \Omega, t \in [0, T]$	stress of the soft material
$\boldsymbol{\sigma}_h$	stress of the hard material
$\mathbf{e} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$	strain
$\mathbf{e}_{ps}$	plastic strain of the soft material
$\mathbf{e}_{hs}$	plastic strain of the hard material
$\alpha = \alpha(x, t)$	volume fraction of the soft material
$1 - \alpha$	volume fraction of the hard material

$$0 \leq \alpha \leq 1, \text{ (later } 0 < \varepsilon_0 < \alpha < 1 - \varepsilon_0)$$

$$\alpha(x, 0) = 0, \text{ (resp. } \varepsilon_0)$$

$$\alpha(x, t) = \alpha(x, t; \mathbf{e}_{ps}), \alpha \text{ monotone increasing in } t$$

## Governing equations

### Balance of forces

$$-\operatorname{div}(\alpha \boldsymbol{\sigma}_s + (1 - \alpha) \boldsymbol{\sigma}_h) = \mathbf{f} \text{ in } \Omega$$

### Weak formulation with boundary values

$$(\alpha \boldsymbol{\sigma}_s + (1 - \alpha) \boldsymbol{\sigma}_h, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \quad \forall \varphi \in H_{\Gamma}^{1,2}(\Omega; \mathbb{R}^3)$$

### Symmetry

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

### Hooke's law in the elastic region

$$\boldsymbol{\sigma}_s = \mathbf{C}(\mathbf{e} - \mathbf{e}_{ps}), \quad \boldsymbol{\sigma}_h = \mathbf{C}(\mathbf{e} - \mathbf{e}_{ph}),$$

$\mathbf{C}$  elasticity tensor, of course different  $\mathbf{C}_s$  and  $\mathbf{C}_h$  can be used.

### Plastic incompressibility

$$\operatorname{trace} \mathbf{e}_{ps} = \operatorname{trace} \mathbf{e}_{ph} = 0.$$

## Yield function and yield condition

Let

$$\mathbf{B}_D = \mathbf{B} - \frac{1}{3}(\text{trace } \mathbf{B})\mathbf{I} \quad \text{“Deviator”}$$

$$F_s(\boldsymbol{\sigma}_s) = |\boldsymbol{\sigma}_s| - \kappa_s$$

$$F_h(\boldsymbol{\sigma}_h) = |\boldsymbol{\sigma}_h| - \kappa_h(t), \quad \kappa_h(t) = \kappa_h(t, \mathbf{e}_{ps})$$

These are v Mises type yield functions.

Yield conditions

$$F_s(\boldsymbol{\sigma}_s) \leq 0, \quad F_h(\boldsymbol{\sigma}_h) \leq 0$$

Kuhn-Tucker-conditions

$$\dot{\mathbf{e}}_s = \lambda_s \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}, \quad \dot{\mathbf{e}}_h = \lambda_h \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|},$$

$$\lambda_s, \lambda_h \geq 0, \quad \lambda_s F_s(\boldsymbol{\sigma}_s) = 0, \quad \lambda_h F_h(\boldsymbol{\sigma}_h) = 0.$$



## Model for the volume fraction $\alpha$

$$\ell = \int_0^t |\dot{\mathbf{e}}_{ps}|$$
$$\alpha(\cdot, t) = \alpha_0 + (1 - \alpha_0)e^{-c_0 \ell}$$

Model for the hardening rule and yield parameters

$$\kappa_s = \text{const} > 0, \quad \kappa_h = \kappa_s + r_0 \ell.$$

Later we will work with Lipschitz  $\alpha$  and  $\kappa_h$ .

## Prandtl-Reuss law

Formulation as variational inequality

Find  $\sigma \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{3 \times 3}))$  such that  $\dot{\sigma} \in L^\infty(L^2)$  and

$$\mathbb{K} \left\{ \begin{array}{l} \sigma(0) = \sigma_0 \\ \sigma = \sigma^T \\ (\sigma, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \quad \forall \varphi \in H_{\Gamma}^{1,2}(\Omega; \mathbb{R}^3) \\ F(\sigma) = |\sigma_D| - \kappa \\ F(\sigma) \leq 0 \end{array} \right.$$

$$(\mathbf{A}\dot{\sigma}, \sigma - \tilde{\sigma})_{L^2(\Omega)} \leq 0 \quad \forall \tilde{\sigma} \text{ satisfying } \mathbb{K}$$

### Theorem

$\exists$  unique solution

Regularity results:  $\dot{\sigma} \in L^\infty(L^{2+\delta})$ ,  $\sigma \in L^\infty(H_{loc}^{1,2})$  (1992),

$\dot{\sigma} \in H^{1/2,2}(L^2)$  (2011)

Strain and plastic strain can be derived via penalty approximation.

# Prandtl-Reuss law. Penalty approximation.

(Penalty)

$$(\mathbf{A}\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau}) + (\mu^{-1} \max[|\boldsymbol{\sigma}_D| - \kappa]_+ \frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|}, \boldsymbol{\tau}) = 0 \quad \mu \rightarrow 0_+$$

$$\forall \boldsymbol{\tau} \text{ such that } \boldsymbol{\tau} = \boldsymbol{\tau}^T$$

$$(\boldsymbol{\tau}, \nabla \varphi) = 0$$

Initial condition, symmetry, balance of force remain as before, yield condition is replaced by the penalty term.

## How to prove existence of solutions to the Prandtl-Reuss law. Estimates

- Step 0** Solve the “ODE” in an abstract Hilbert space setting or use Rothe approximation  $\Rightarrow$  (Penalty) is solvable
- Step 1** Assume safe load condition, i.e. existence of a “good”  $\hat{\sigma}$  satisfying  $\mathbb{K} \Rightarrow L^\infty(L^2)$  estimate for  $\sigma$  and  $L^1(L^1)$  estimate for the penalty term.
- Step 2** Test by  $\dot{\sigma} - \dot{\hat{\sigma}} \Rightarrow L^2(L^2)$  estimate for  $\dot{\sigma}$
- Step 3** Test by  $\ddot{\sigma} \Rightarrow L^\infty(L^2)$  estimate for  $\dot{\sigma}$
- Step 4** Go to Step 1 once more  $\Rightarrow L^\infty(L^1)$  estimate for penalty term (all estimates uniformly for  $\mu \rightarrow 0$ )

## Converge and existence for Prandtl-Reuss

**Step 5** Use symmetric Helmholtz decomposition  $\Rightarrow \exists \mathbf{v} \in H_1^{1,2}(\Omega; \mathbb{R}^3)$   
 such that  $\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) = \mathbf{A}\dot{\boldsymbol{\sigma}} + \text{Penalty term}$

with  $\|\nabla \mathbf{v} + (\nabla \mathbf{v})^T\|_{L^\infty(L^1)} \leq K$  uniformly as  $\mu \rightarrow 0$

$\Rightarrow$  The strain velocities are only measures for  $\mu = 0$

$\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \in C^*$ , similar, for the penalty term

$$\mu^{-1} \max[|\boldsymbol{\sigma}_D| - \kappa]_+ \frac{\boldsymbol{\sigma}_D}{|\boldsymbol{\sigma}_D|} \rightharpoonup \dot{\mathbf{A}} \in C^*(0, T; \mathbb{R}^{3 \times 3})$$

$$\frac{\mu^{-1} \max[|\boldsymbol{\sigma}_D| - \kappa]_+}{|\boldsymbol{\sigma}_D|} \rightharpoonup \lambda$$

$\dot{\mathbf{A}}$  is the plastic strain velocity

If one had  $\boldsymbol{\sigma}_\mu \rightarrow \boldsymbol{\sigma}$  uniformly  $\Rightarrow$  representation as multiplier

$$\dot{\mathbf{A}} = \lambda \boldsymbol{\sigma}, \quad \lambda \geq 0, \quad \lambda F(\boldsymbol{\sigma}) = 0.$$

## Theorem

$\sigma_\mu \rightarrow \sigma$ ,  $\sigma$  solution of the Prandtl-Reuss variational inequality and

$$\frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) = \mathbf{A}\dot{\sigma} + \dot{\boldsymbol{\lambda}}.$$

A similar procedure can be done for the mixture MDE model of KMRS.

## Constitutive law with elastic interaction

Let

$$\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_s \\ \boldsymbol{\tau}_h \end{pmatrix}, \quad \hat{\boldsymbol{\tau}} = \begin{pmatrix} \hat{\boldsymbol{\tau}}_s \\ \hat{\boldsymbol{\tau}}_h \end{pmatrix}$$

and

$$\mathbf{Q}(\hat{\boldsymbol{\tau}}, \boldsymbol{\tau}) = \int_{\Omega} [\mathbf{A}_{ss} \hat{\boldsymbol{\tau}}_s : \boldsymbol{\tau}_s + \mathbf{A}_{sh} \hat{\boldsymbol{\tau}}_h : \boldsymbol{\tau}_s + \mathbf{A}_{sh} \hat{\boldsymbol{\tau}}_s : \boldsymbol{\tau}_h + \mathbf{A}_{hh} \hat{\boldsymbol{\tau}}_h : \boldsymbol{\tau}_h] dx$$

Here  $\mathbf{A}_{ss}$ ,  $\mathbf{A}_{hh}$ ,  $\mathbf{A}_{sh} = \mathbf{A}_{hs}$  are inverse elasticity tensors, say Lamé-Navier structure. They model the elastic interaction between the soft and hard material.

## Penalty approximation for Prandtl-Reuss mixtures

Find  $\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_h \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^{3 \times 3}))$  such that  $\dot{\boldsymbol{\sigma}}_s, \dot{\boldsymbol{\sigma}}_h \in L^\infty(L^2)$   
and

$$\boldsymbol{\sigma}_s(0) = \boldsymbol{\sigma}_{s0}, \boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{h0}$$

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}_s^T, \boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^T$$

$$(\alpha \boldsymbol{\sigma}_s + (1 - \alpha) \boldsymbol{\sigma}_h, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \quad \forall \varphi \in H_\Gamma^{1,2}(\Omega; \mathbb{R}^3)$$

(Penmix)

$$\mathbf{Q} \left( \begin{array}{c} \alpha \dot{\boldsymbol{\sigma}}_s, \alpha \boldsymbol{\tau}_s \\ (1 - \alpha) \dot{\boldsymbol{\sigma}}_h, (1 - \alpha) \boldsymbol{\tau}_h \end{array} \right) +$$

$$\left( \alpha \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}, \boldsymbol{\tau}_s \right) +$$

$$\left( (1 - \alpha) \mu^{-1} [|\boldsymbol{\sigma}_{hD}| - \kappa_h]_+ \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|}, \boldsymbol{\tau}_h \right) = 0$$

for all  $\boldsymbol{\tau}_s = \boldsymbol{\tau}_s^T, \boldsymbol{\tau}_h = \boldsymbol{\tau}_h^T$  such that  $\operatorname{div}(\alpha \boldsymbol{\tau}_s + (1 - \alpha) \boldsymbol{\tau}_h) = 0$ .



## Consequence of (Penmix)

Choose  $\boldsymbol{\tau}_h = \mathbf{0}$ ,  $\operatorname{div}(\alpha\boldsymbol{\tau}_s) = 0$ ,  $\alpha\boldsymbol{\tau}_s = \boldsymbol{\tau}_0 \Rightarrow$

$$\int_{\Omega} \left( \alpha \mathbf{A}_{ss} \dot{\boldsymbol{\sigma}}_s + (1 - \alpha) \mathbf{A}_{sh} \dot{\boldsymbol{\sigma}}_h + \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|} \right) : \boldsymbol{\tau}_0 dx = 0$$

By symmetric Helmholtz decomposition

$$\frac{1}{2} ((\nabla \dot{\mathbf{u}}_s) + (\nabla \dot{\mathbf{u}}_s)^T) = \underbrace{\alpha \mathbf{A}_{ss} \dot{\boldsymbol{\sigma}}_s + (1 - \alpha) \mathbf{A}_{sh} \dot{\boldsymbol{\sigma}}_h}_{\text{rate of stress acting on soft material}} + \underbrace{\mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|}}_{\text{approximate plastic strain velocity of the soft material}}$$

Similarly

$$\frac{1}{2} ((\nabla \dot{\mathbf{u}}_h) + (\nabla \dot{\mathbf{u}}_h)^T) = \alpha \mathbf{A}_{sh} \dot{\boldsymbol{\sigma}}_s + (1 - \alpha) \mathbf{A}_{hh} \dot{\boldsymbol{\sigma}}_h + \mu^{-1} [|\boldsymbol{\sigma}_{hD}| - \kappa_h]_+ \frac{\boldsymbol{\sigma}_{hD}}{|\boldsymbol{\sigma}_{hD}|}$$

## Theorem

Assume  $\alpha, \kappa_h$  Lipschitz,  $\mathbf{Q}$  positively definite and smooth data. Assume a safe load condition with smooth stresses  $\hat{\sigma}_s$  and  $\hat{\sigma}_h$ . Let  $0 \leq \varepsilon_0 \leq \alpha \leq 1 - \varepsilon_0$ . Then

1. The solutions  $\sigma_s^n, \sigma_h^n$  are uniformly bounded in the norms  $L^\infty(L^2), L^\infty(L^{2+\delta}), H^{1,\infty}(L^2)$ . Furthermore  $\dot{\sigma} \in L^2(H_\Gamma^{1/2,2})$ .
2. The partial strain velocities and the partial plastic strain velocities are bounded in  $L^\infty(L^1)$ .
3. The approximate stresses converge weakly  $\sigma_s^n \rightharpoonup \sigma_s, \sigma_h^n \rightharpoonup \sigma_h$  ( $\mu \rightarrow 0_+$ ) in  $H^{1,2}(L^2) \cap L^2(H_{loc}^{1,2})$ .

## Theorem (continuation)

The limit satisfies the variational inequality

$$\mathbf{Q} \left( \left( \begin{array}{c} \alpha \dot{\boldsymbol{\sigma}}_s \\ (1 - \alpha) \dot{\boldsymbol{\sigma}}_h \end{array} \right), \left( \begin{array}{c} \alpha (\boldsymbol{\sigma}_s - \boldsymbol{\tau}_s) \\ (1 - \alpha) (\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) \end{array} \right) \right) \leq 0 \quad (\text{V})$$

with respect to the yield conditions and the balance of forces defining the convex set.

Since in the application  $\alpha = \alpha(t, x; \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_h)$  and similar  $\kappa_h$ , inequality (V) is a quasi-variational inequality.

## Safe load condition for mixtures

$\exists \hat{\boldsymbol{\sigma}}_s, \hat{\boldsymbol{\sigma}}_h$  such that

$$\hat{\boldsymbol{\sigma}}_s = \hat{\boldsymbol{\sigma}}_s^T, \hat{\boldsymbol{\sigma}}_h = \hat{\boldsymbol{\sigma}}_h^T$$

$$(\alpha \hat{\boldsymbol{\sigma}}_s + (1 - \alpha) \hat{\boldsymbol{\sigma}}_h, \nabla \varphi)_{L^2(\Omega)} = (\mathbf{f}, \varphi)_{L^2(\Omega)} + \int_{\partial\Omega} p_0 \varphi d\mathbf{o}, \quad \forall \varphi \in H_F^{1,2}(\Omega; \mathbb{R}^3)$$

$$F_s(\boldsymbol{\sigma}_s) \leq -\varepsilon_0 < 0, \quad F_h(\boldsymbol{\sigma}_h) \leq -\varepsilon_0 < 0$$

Smoothness:  $\hat{\boldsymbol{\sigma}}, \ddot{\boldsymbol{\sigma}}, \nabla \dot{\boldsymbol{\sigma}} \in L^\infty$  is more than necessary for the regularity theory.

## $L^1(L^1)$ estimate using safe load

Test the penalty equation with

$$\boldsymbol{\tau}_s = \boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s, \quad \boldsymbol{\tau}_h = \boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_s$$

This admissible since

$$\operatorname{div}(\alpha(\boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s) + (1 - \alpha)(\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_s)) = 0$$

Hence

$$\mathbf{Q} \left( \left( \begin{array}{c} \alpha \dot{\boldsymbol{\sigma}}_s \\ (1 - \alpha) \dot{\boldsymbol{\sigma}}_h \end{array} \right), \left( \begin{array}{c} \alpha(\boldsymbol{\sigma}_s - \hat{\boldsymbol{\sigma}}_s) \\ (1 - \alpha)(\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \end{array} \right) \right) + \\ \left( \alpha \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|} \right) + \text{penalty for } \boldsymbol{\sigma}_{hD} = 0$$

$$\frac{1}{2} \frac{d}{dt} \mathbf{Q}(\dots) + \text{pollution} +$$

$$\left( \alpha \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ \left( |\boldsymbol{\sigma}_{sD}| - \underbrace{\frac{\boldsymbol{\sigma}_{sD}}{|\boldsymbol{\sigma}_{sD}|} \hat{\boldsymbol{\sigma}}_{sD}}_{\geq -\kappa_s + \varepsilon_0} \right) \right) + \text{penalty for } \boldsymbol{\sigma}_{hD} = 0$$

Integration with respect to  $t$

$$\frac{1}{2} \mathbf{Q} \left( \begin{array}{cc} \alpha \boldsymbol{\sigma}_s, & \alpha \boldsymbol{\sigma}_s \\ (1 - \alpha) \boldsymbol{\sigma}_h, & (1 - \alpha) \boldsymbol{\sigma}_h \end{array} \right) \Big|_T - \int_0^T K |\boldsymbol{\sigma}|^2 dx dt +$$

$$\varepsilon_0 \int \int \alpha \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ dx dt \leq K(\hat{\boldsymbol{\sigma}})$$

Gronwall  $\Rightarrow L^\infty(L^2)$  estimate for  $\boldsymbol{\sigma}$ ,  $L^1$  estimate for  $\mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+$

Inspection return  $\Rightarrow \mu^{-1} [|\boldsymbol{\sigma}_{sD}| - \kappa_s]_+ |\boldsymbol{\sigma}_{sD}| \in L^1(L^1)$

Similar for  $\boldsymbol{\sigma}_h$ .

आपका ध्यान के लिए धन्यवाद

Děkuji za pozornost.

Thank you for your attention.