Regularity for systems of PDEs arising in continuum thermodynamics

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Challenges in analysis and modeling - K. R. Rajagopal

March 31, 2012

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Regularity for systems of PDEs

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to have fun

- to learn something from physics
- what kind of regularity? weak & strong & classical solution
- to justify the model (in case regularity holds)
- to justify the numerical scheme and the error estimate (in case regularity holds)
- to show that the model is wrong

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Outline

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We demonstrate all results and open problems on the prototype:

$$\begin{split} & \operatorname{div} \boldsymbol{\mathsf{v}} = \boldsymbol{\mathsf{0}} \\ \boldsymbol{\mathsf{v}}_t + \operatorname{div}(\boldsymbol{\mathsf{v}} \otimes \boldsymbol{\mathsf{v}}) - \operatorname{div} \boldsymbol{\mathsf{T}} = \boldsymbol{\mathsf{f}}, \\ & \cdots = \cdots . \end{split}$$

Navier-Stokes equations (neglect coupling)

- full problem unsteady vs. steady
- neglect inertia \implies full regularity
- power-law like models & more general situation (neglect coupling)
 - neglect inertia vs. full system
 - regularity of stress vs. velocity (displacement gradient)
- coupled problems (only with the equation for temperature/internal energy)
 - Newtonian fluid (with and without inertia) -
 - non-Newtonian models -nonlinearity may help

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• $\mathbf{T} := -p\mathbf{I} + 2\nu_0 \mathbf{D}(\mathbf{v})$, where $\mathbf{D}(\mathbf{v})$ is the symmetric part of $\nabla \mathbf{v}$

Navier-Stokes equations

$$\mathbf{v}_t - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu_0 \triangle \mathbf{v} = -\nabla p + \mathbf{f}$$

$$\operatorname{div} \mathbf{v} = \mathbf{0}.$$
 (N-S)

d = 2 - regularity, $d = 3, \ldots$ - regularity partial & conditional & special geometries & small data & short time

Stokes equations

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maximal regularity in any d

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d = 2, 3, 4 - regularity,

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Challenge

If data are smooth, is there a smooth solution to (N-S)?

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• $\mathbf{T} := -p\mathbf{I} + \mathbf{S}$, where

$$\begin{split} \hline \mathbf{G}(\mathbf{S},\mathbf{D}(\mathbf{v})) &= \mathbf{0} & \text{prototype: } \mathbf{S} \sim \nu_0 (1 + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v}) \\ \hline \mathbf{v}_t - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \nu_0 \operatorname{div} \left((1 + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v}) \right) &= -\nabla p + \mathbf{f} \\ & \operatorname{div} \mathbf{v} = \mathbf{0}. \end{split}$$
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 $r \ge rac{3d+2}{d+2}$ - strong solution & uniqueness (for smooth data) d=2 - full regularity

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$$\mathbf{v}_t - \nu_0 \operatorname{div} \left(\left(1 + \left| \mathbf{D}(\mathbf{v}) \right|^2 \right)^{\frac{t-2}{2}} \mathbf{D}(\mathbf{v}) \right) = -\nabla \rho + \mathbf{f}$$

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for all $r \in (1,\infty)$ the same as for (N_r) , i.e., NO higher regularity for d = 3, ...

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Power-law like implicit

•
$$T := -\rho I + S$$
, where
• prototype I
• prototype II
 $D = \frac{S}{|S|} + \tilde{\nu}(|S|)S$

• prototype I: higher regularity of $\nabla \mathbf{v}$

• prototype II: higher regularity of S

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$$\fbox{G(S,D(v))=0}$$

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$$-\operatorname{div}(\nu(|\varepsilon|)\varepsilon) = \mathbf{f}$$

- $|\nabla \mathbf{u}| \leq \varepsilon_0$, one can justify such a "model" due to the work of prof. **Rajagopal**
- no regularity for $d \ge 3$
- $|\nabla \mathbf{u}| \leq \varepsilon_1 \implies$ regularity
- $\varepsilon_0 \leq \varepsilon_1 \implies$ perfect model:)
- $\varepsilon_1 \ll \varepsilon_0$ singularity maybe be there, the model is incorrect

• Consider "nonlinear linearized elasticity", very simplified i.e.

$$\boxed{\mathbf{T} = \nu(|\boldsymbol{\varepsilon}|)\boldsymbol{\varepsilon}} \qquad \boldsymbol{\varepsilon} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^{T})$$

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Challenge

Are the solutions to (S_r) smooth (or at least $C^{1,\alpha}$)?

Nightmare

If $\mathbf{S} = \nu(|\nabla \mathbf{v}|) \nabla \mathbf{v}$ then the solution is regular.

• $|\nabla \mathbf{v}|$ is sub-solution to an elliptic problem

Challenge

Is $|\mathbf{D}(\mathbf{v})|$ or any other relevant quantity a sub- or super-solution to something? Is there something behind the structure?

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Coupled systems - basic framework

• $\mathbf{T} := -p\mathbf{I} + \mathbf{S}$, where

$$G(e, S, D(v)) = 0$$

• standard sets of equations (equation for internal energy *e*)

$$\begin{aligned} \mathbf{v}_t - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla \rho + \mathbf{f} \\ \operatorname{div} \mathbf{v} &= \mathbf{0} \\ e_t - \operatorname{div}(e\mathbf{v}) - \operatorname{div}(\kappa(e)\nabla e) &= \mathbf{S} \cdot \mathbf{D}(\mathbf{v}) \end{aligned}$$

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• standard sets of equations (equation for internal energy *e*)

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• "better" sets of equations (equation for global energy $E := \frac{1}{2} |\mathbf{v}|^2 + e$)

$$\begin{aligned} \mathbf{v}_t - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{S} &= -\nabla p + \mathbf{f} \\ \operatorname{div} \mathbf{v} &= \mathbf{0} \end{aligned} \tag{N-S-F_E} \\ E_t - \operatorname{div}(\mathbf{v}(E+p)) - \operatorname{div}(\kappa(e)\nabla e + \mathbf{Sv}) &= \mathbf{f} \cdot \mathbf{v} \end{aligned}$$

• $\mathbf{S} = \nu(e)\mathbf{D}(\mathbf{v})$

• Navier-Stokes-Fourier

$$\mathbf{v}_t - \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}(\nu(e)\mathbf{D}(\mathbf{v})) = -\nabla p + \mathbf{f}$$
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no regularity known in any d; convective term, presence of D(v), no-Hölder continuity of e, quadratic term on the right hand side

Stokes-Fourier

$$\mathbf{v}_t - \operatorname{div}(\nu(e)\mathbf{D}(\mathbf{v})) = -\nabla \rho + \mathbf{f}$$

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Lemma (M.B & Kaplický & Málek)

For very special ν , for solution to (S-F) we know that $\nabla^2 \mathbf{v} \in L^2$ in any d; for d = 2 existence of classical solution.

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$$\begin{split} \mathbf{v}_t - \operatorname{div}(\nu(e) \mathbf{D}(\mathbf{v})) &= -\nabla p + \mathbf{f} \\ \operatorname{div} \mathbf{v} &= \mathbf{0} \\ e_t - \operatorname{div}(\kappa(e) \nabla e) &= \nu(e) |\mathbf{D}(\mathbf{v})|^2 \end{split}$$

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Regularity for systems of PDEs

(N-S-F)

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Coupling with internal energy

Navier-Stokes-Fourier

Challenge

Are solution to (S-F) regular for more general ν 's? What with (N-S-F)?

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Regularity for systems of PDEs

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- $u(e) = \kappa(e) \sim \nu_0 e^{\alpha}$ with some $\alpha \ge 0$
- fluid dynamics nonsense; TKE-models (simplified Kolmogorov model) with $\alpha = \frac{1}{2}$
- different scaling than in Navier-Stokes

Conjecture

There exists $\varepsilon_0 >$ such that any solution to (N-S-F) satisfying

$$\int_{0}^{1}\int_{B_{1}(0)}\nu(e)|\mathsf{D}(\mathbf{v})|^{2}\leq\varepsilon_{0}$$

is regular in $(rac{1}{2},1) imes B_{rac{1}{2}}(0).$

Lemma (M.B., Lewandowski, Málek)

Let the conjecture hold. Then for any $\alpha \geq \frac{1}{2}$ the solution to (N-S-F) is regular.

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Coupled system - nonlinearity helps

•
$$\mathbf{S} \sim (\nu(e) + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v})$$

Lemma (M.B., Málek, Shilkin)

Let d = 2 then the solution to (N-S-F) are regular. Let $d \ge 3$ and $r \ge \frac{3d+2}{d+2}$ then there exists a strong solution to (N-S-F).

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