# Regularity for systems of PDEs arising in continuum thermodynamics

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Challenges in analysis and modeling - **K. R. Rajagopal**

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# Why to study regularity of PDEs

- to have fun
- to learn something from physics
- what kind of regularity? weak & strong & classical solution
- to justify the model (in case regularity holds)
- to justify the numerical scheme and the error estimate (in case regularity holds)
- to show that the model is wrong

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# Outline

We demonstrate all results and open problems on the prototype:

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0 \\ \mathbf{v}_t + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbf{T} &= \mathbf{f}, \\ \cdots &= \cdots . \end{aligned}$$

- Navier-Stokes equations (neglect coupling)
  - full problem unsteady vs. steady
  - neglect inertia $\Longrightarrow$ full regularity
- power-law like models & more general situation (neglect coupling)
  - neglect inertia vs. full system
  - regularity of stress vs. velocity (displacement gradient)
- coupled problems (only with the equation for temperature/internal energy)
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$$\boxed{\begin{aligned} \mathbf{v}_t - \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) - \nu_0\triangle\mathbf{v} &= -\nabla p + \mathbf{f} \\ \operatorname{div}\mathbf{v} &= 0. \end{aligned}} \tag{N-S}$$

  $d = 2$ - regularity,
  $d = 3, \dots$ - regularity partial & conditional & special geometries & small data & short time
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  $d = 5$ - the same scaling as for (N-S) in $d = 3$ - maybe a hint to (N-S)

- **J. Frehse** & coauthors - existence of a regular solution for $d = 5, \dots, 10$
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# Challenge

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*If data are smooth, is there a smooth solution to* (N-S)*?*

# Power-law like models

- $\mathbf{T} := -p\mathbf{I} + \mathbf{S}$, where

$$\boxed{\mathbf{G}(\mathbf{S},\mathbf{D}(\mathbf{v})) = \mathbf{0}} \qquad \text{prototype: } \mathbf{S} \sim \nu_0(1+|\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}}\mathbf{D}(\mathbf{v})$$

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$r \geq \frac{3d+2}{d+2}$ - strong solution & uniqueness (for smooth data)
$d = 2$ - full regularity

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for all $r \in (1,\infty)$ the same as for $(\mathrm{N}_r)$, i.e., NO higher regularity for $d = 3, \ldots$

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$$\mathbf{S} = \frac{\mathbf{D}}{|\mathbf{D}|} + \nu(|\mathbf{D}|)\mathbf{D}$$

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$$\mathbf{D} = \frac{\mathbf{S}}{|\mathbf{S}|} + \tilde{\nu}(|\mathbf{S}|)\mathbf{S}$$

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# No pressure - "no" advantage

- Consider "nonlinear linearized elasticity", very simplified i.e.

$$\boxed{\mathbf{T} = \nu(|\varepsilon|)\varepsilon} \qquad \varepsilon := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$$

- the resulting equations are similar

$$\boxed{-\operatorname{div}(\nu(|\varepsilon|)\varepsilon) = \mathbf{f}}$$

- $|\nabla\mathbf{u}| \leq \varepsilon_0$, one can justify such a "model" - due to the work of prof. **Rajagopal**
- no regularity for $d \geq 3$
- $|\nabla\mathbf{u}| \leq \varepsilon_1 \implies$ regularity
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*Are the solutions to* $(\mathsf{S}_r)$ *smooth (or at least* $\mathcal{C}^{1,\alpha}$*)?*

**Nightmare**

*If* $\mathbf{S} = \nu(|\nabla \mathbf{v}|)\nabla \mathbf{v}$ *then the solution is regular.*

- $|\nabla \mathbf{v}|$ is sub-solution to an elliptic problem

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*Is* $|\mathbf{D}(\mathbf{v})|$ *or any other relevant quantity a sub- or super-solution to something? Is there something behind the structure?*

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# Coupled systems - basic framework

- $\mathbf{T} := -p\mathbf{I} + \mathbf{S}$, where

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- standard sets of equations (equation for internal energy $e$)

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*For very special $\nu$, for solution to (S-F) we know that $\nabla^2\mathbf{v} \in L^2$ in any $d$; for $d = 2$ existence of classical solution.*

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no regularity known in general; quadratic term on the right hand side

**Lemma (M.B & Kaplický & Málek)**

*For very special $\nu$, for solution to (S-F) we know that $\nabla^2\mathbf{v} \in L^2$ in any $d$; for $d = 2$ existence of classical solution.*

# Coupled systems - Newtonian models

- $\mathbf{S} = \nu(e)\mathbf{D}(\mathbf{v})$
- Navier-Stokes-Fourier

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# Navier-Stokes-Fourier

**Challenge**

*Are solution to* (S-F) *regular for more general* $\nu$*'s? What with* (N-S-F)*?*

# Eddy viscosity - turbulence model - hope for regularity?

- $\nu(e) = \kappa(e) \sim \nu_0 e^{\alpha}$ with some $\alpha \geq 0$
- fluid dynamics - nonsense; TKE-models (simplified Kolmogorov model) with $\alpha = \frac{1}{2}$
- different scaling than in Navier-Stokes

**Conjecture**

*There exists $\varepsilon_0 >$ such that any solution to (N-S-F) satisfying*

$$\int_0^1 \int_{B_1(0)} \nu(e)|\mathbf{D}(\mathbf{v})|^2 \leq \varepsilon_0$$

*is regular in $(\frac{1}{2}, 1) \times B_{\frac{1}{2}}(0)$.*

**Lemma (M.B., Lewandowski, Málek)**

*Let the conjecture hold. Then for any $\alpha \geq \frac{1}{2}$ the solution to (N-S-F) is regular.*

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# Coupled system - nonlinearity helps

- $\mathbf{S} \sim (\nu(e) + |\mathbf{D}(\mathbf{v})|^2)^{\frac{r-2}{2}} \mathbf{D}(\mathbf{v})$

**Lemma (M.B., Málek, Shilkin)**

*Let* $d = 2$ *then the solution to* (N-S-F) *are regular.*
*Let* $d \geq 3$ *and* $r \geq \frac{3d+2}{d+2}$ *then there exists a strong solution to* (N-S-F).