Limity funkcí I

1. Dokažte z definice, že
   a) $\lim\limits_{x\to 1}\left(\frac{x}{2}\right)^3=\frac{1}{8}$ b) $\lim\limits_{x\to 1^+}[x]=1$ c) $\lim\limits_{x\to 1^-}[x]=0$

   Spočtěte

2. (a) $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}$ (b) $\lim\limits_{x\to 1}\frac{x^2-1}{2x^2-x-1}$

3. $\lim\limits_{x\to 2}\left(\frac{1}{x^2-2x}-\frac{x}{x^2-4}\right)$

4. $\lim\limits_{x\to 0}\frac{(1+x)(1+2x)\dots(1+nx)-1}{x}$, $n\in\mathbb{N}$

5. $\lim\limits_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1}$

6. $\lim\limits_{x\to 0}\frac{(1+mx)^n-(1+nx)^m}{x^2}$, $m,n\in\mathbb{N}$

7. $\lim\limits_{x\to 1}\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$, $n\in\mathbb{N}$

8. $\lim\limits_{x\to 1}\frac{x+x^2+\dots x^n-n}{x-1}$, $n\in\mathbb{N}$

9. $\lim\limits_{x\to 1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)$, $m,n\in\mathbb{N}$

10. $\lim\limits_{x\to 0}\frac{\frac{2}{x^2}+1}{\sqrt{\frac{3}{x^4}-\frac{6}{x^2}+5}}$

11. $\lim\limits_{x\to 0+}\frac{(\sqrt{\frac{1}{x^2}+1}-\sqrt{\frac{1}{x^2}-1})}{x}$

12. $\lim\limits_{x\to 0^+}\left(\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}-\sqrt{\frac{1}{x}-\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}}\right)$

13. (a) $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4}$ (b) $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$

14. $\lim\limits_{x\to 0} \dfrac{\sqrt{1-2x-x^2}-(1-x)}{x}$

15. $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}}$

16. $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}$, $m, n \in \mathbb{N}$

17. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt[3]{1-x}}{\sqrt[3]{1+x}-\sqrt{1-x}}$

18. $\lim\limits_{x\to a^+} \dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$, $a \in \mathbb{R}_0^+$

19. $\lim\limits_{x\to 0} \dfrac{\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1}{x}$, $m, n \in \mathbb{N}$, $a, b \in \mathbb{R}$

Limity funkcí II

Základní limity

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{\mathrm{e}^x-1}{x}=1 \qquad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

Pro výpočet limit typu "1^∞":

$$\lim_{x\to a}(f(x))^{g(x)}=\mathrm{e}^{\lim_{x\to a} g(x)\ln(f(x))}\,.$$

Příklady

1. $\displaystyle\lim_{x\to a}\frac{\operatorname{tg} x-\operatorname{tg} a}{x-a}$, $a\in\mathbb{R}$
2. $\displaystyle\lim_{x\to 0}\frac{\sqrt{1-\cos x^2}}{1-\cos x}$
3. $\displaystyle\lim_{x\to 0}\frac{\operatorname{tg} x-\sin x}{x^3}$
4. $\displaystyle\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x}$
5. $\displaystyle\lim_{x\to \pi}\frac{\sin nx}{\sin mx}$, $n,\ m\in\mathbb{N}$
6. $\displaystyle\lim_{x\to 1}\frac{\sin \pi x}{1-x}$
7. $\displaystyle\lim_{x\to \frac{\pi}{4}}\operatorname{tg}(2x)\operatorname{tg}\left(\frac{\pi}{4}-x\right)$
8. $\displaystyle\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$, $a\in\mathbb{R}$
9. $\displaystyle\lim_{x\to 0}\frac{\operatorname{cotg}(a+2x)-2\operatorname{cotg}(a+x)+\operatorname{cotg} a}{x^2}$, $\sin a\neq 0$
10. $\displaystyle\lim_{x\to 0^+}\frac{\arccos(1-x)}{\sqrt{x}}$
11. $\displaystyle\lim_{x\to 0^+}\frac{\left(\frac{\pi}{2}-\arcsin\frac{1}{\sqrt{x^2+1}}\right)}{x}$

12. $\lim\limits_{x\to 0} \dfrac{\ln\cos ax}{\ln\cos bx}$, $a,\ b\in\mathbb{R},\ b\neq 0$

13. $\lim\limits_{x\to 0} \dfrac{\ln(a+x)+\ln(a-x)-2\ln a}{x^2}$, $a>0$

14. $\lim\limits_{x\to 0} \dfrac{\ln(\operatorname{tg}(\frac{\pi}{4}+ax))}{\sin bx}$, $a,\ b\in\mathbb{R},\ b\neq 0$

15. $\lim\limits_{x\to 0^+} \ln(x\ln a)\ln\left(\dfrac{\ln ax}{\ln\frac{x}{a}}\right)$, $a>0$

16. $\lim\limits_{x\to 0} \dfrac{\ln(1+x\mathrm{e}^x)}{\ln(x+\sqrt{1+x^2})}$

17. $\lim\limits_{x\to 1}(1-x)\log_x 2$

18. $\lim\limits_{x\to 0^+}(1+x)^{\frac{1}{x}}$

19. $\lim\limits_{x\to \frac{\pi}{2}}(\sin x)^{\operatorname{tg} x}$

20. $\lim\limits_{x\to 0}\left(\dfrac{1+\operatorname{tg} x}{1+\sin x}\right)^{\frac{1}{\sin^3 x}}$

21. $\lim\limits_{x\to 1}(1+\sin\pi x)^{\operatorname{cotg}\pi x}$

22. $\lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}$

23. $\lim\limits_{x\to 0}(1+x^2)^{\operatorname{cotg}\pi x}$

24. $\lim\limits_{x\to \frac{\pi}{4}}(\operatorname{tg} x)^{\operatorname{tg} 2x}$

25. $\lim\limits_{x\to 1} \dfrac{\sin\pi x^{\alpha}}{\sin\pi x^{\beta}}$, $\alpha,\ \beta\in\mathbb{R},\ \beta\neq 0$

26. $\lim\limits_{x\to 0} \dfrac{\mathrm{e}^{\alpha x}-\mathrm{e}^{\beta x}}{\sin\alpha x-\sin\beta x}$, $\alpha,\ \beta\in\mathbb{R},\ \alpha\neq\beta$

27. $\lim\limits_{x\to a} \dfrac{a^x-x^a}{x-a}$, $a\in\mathbb{R}^+$

28. $\lim\limits_{x\to 0}\left(\dfrac{1+x2^x}{1+x3^x}\right)^{\frac{1}{x^2}}$

29. $\lim\limits_{x\to 0}\left(\dfrac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{\frac{1}{x}}$, $a,\ b\in\mathbb{R}^+$

(4) $(1+x)(1+2x)\dots(1+nx) = 1 + x(1+2+\dots+n) + O(x^2)$

$\longrightarrow \dfrac{n(n+1)}{2}$.

(5) $x^{100} - 2x + 1 = x^{100} - x - (x-1) = x(x^{99}-1) - (x-1)$;

$x^{99} - 1 = (x-1)(x^{98} + x^{97} + \dots + x + 1)$;

čitatel: $(x-1)\left(x(x^{98} + \dots + 1) - 1\right)$

jmenovatel: $(x-1)\left(x(x^{48} + \dots + 1) - 1\right)$; $\longrightarrow \dfrac{98}{48} = \dfrac{49}{24}$

(6) $(1+mx)^n = 1 + n\cdot mx + \binom{n}{2} m^2x^2 + O(x^2)$

$(1+nx)^m = 1 + m\cdot nx + \binom{m}{2} n^2x^2 + O(x^2)$

$\longrightarrow \dfrac{n(n-1)}{2} m^2 - \dfrac{m(m-1)}{2} n^2 = \dfrac{mn}{2}(n-m)$.

(7) $x^{n+1} - (n+1)x + n = x(x^n - 1) - n(x-1)$

$= (x-1)\left(x(x^{n-1} + x^{n-2} + \dots + x + 1) - n\right)$

$= (x-1)\left((x^n - 1) + (x^{n-1} - 1) + \dots + (x-1)\right)$

$= (x-1)^2\left((x^{n-1} + \dots + 1) + (x^{n-2} + \dots + 1) + \dots + 1\right)$

— n závorek délky $1 \dots n$.

$\longrightarrow n + (n-1) + \dots + 2 + 1 = \dfrac{n(n+1)}{2}$

(8) $x + \dots + x^n - n = (x-1) + (x^2 - 1) + \dots + (x^n - 1)$

$= (x-1)\left(1 + (x+1) + \dots + (x^{n-1} + \dots + 1)\right)$

$\longrightarrow \dfrac{n(n+1)}{2}$.

(9) 1. krok: $m=1$, $n \in \mathbb{N}$ obecné

$$\frac{1}{1-x}-\frac{n}{1-x^n}=\frac{1+x+\dots+x^{n-1}-n}{1-x^n}=\frac{(x-1)+(x^2-1)+\dots+(x^{n-1}-1)}{1-x^n}$$

$$=\frac{x-1}{1-x}\cdot\left(\frac{1+(x+1)+\dots+(x^{n-2}+\dots+1)}{1+x+\dots+x^{n-1}}\right)\to-\frac{n(n-1)}{2n}=\frac{1-n}{2}$$

2. krok: $$\frac{m}{1-x^m}-\frac{n}{1-x^n}=\frac{1}{1-x}-\frac{n}{1-x^n}-\left(\frac{1}{1-x}-\frac{m}{1-x^m}\right)$$

$$\to\frac{1-n}{2}-\frac{1-m}{2}=\frac{m-n}{2}.$$

(16)

1. krok: $$\frac{1-\sqrt[n]{1+x}}{x}=\frac{x}{x}\cdot\frac{1}{(1+x)^{\frac{n-1}{n}}+\dots+1}\to\frac{1}{n}$$

2. krok: $$\frac{\sqrt[m]{1+x}-\sqrt[n]{1+x}}{x}=\frac{1-\sqrt[n]{1+x}}{x}-\frac{1-\sqrt[m]{1+x}}{x}\to\frac{1}{n}-\frac{1}{m}$$

(18) $$=\frac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x+a}\cdot\sqrt{x-a}}=\frac{\sqrt{x}-\sqrt{a}+1}{\sqrt{x+a}}\to\frac{1}{\sqrt{2a}},\ (a>0)$$

$a=0$: $\to+\infty$

(19) $$\sqrt[m]{1+ax}\,\sqrt[n]{1+bx}-1=\sqrt[m]{1+ax}\left(\sqrt[n]{1+bx}-1\right)+\sqrt[m]{1+ax}-1$$

a podobně jako 16, 1. krok

$$\to\frac{b}{n}+\frac{a}{m}.$$

(1) čit: $\dfrac{\sin x}{\cos x} - \dfrac{\sin a}{\cos a} = \dfrac{\sin x \cos a - \cos x \sin a}{\cos x \cos a} = \dfrac{\sin(x-a)}{\cos x \cdot \cos a}$

$\to + \dfrac{1}{\cos^2 a} = tg'(a)$.

(3) čit: $\dfrac{\sin x}{\cos x} - \sin x = \sin x \cdot (1-\cos x) \cdot \dfrac{1}{\cos x}$

$\to \dfrac{1}{2}$

(4) čit: $(1-\cos x) + \cos x\,(1-\cos 2x) + \cos x \cdot \cos 2x \cdot (1-\cos 3x)$

$\dfrac{1-\cos(ax)}{x^2} \to \dfrac{a^2}{2}$; celkem: $\to 14$

(5) $\sin(y \pm k\pi) = (-1)^k \sin y$

$= \dfrac{(-1)^n \sin n(x-\pi)}{(-1)^m \sin m(x-\pi)} \to (-1)^{n+m} \dfrac{n}{m}$

(6) $\sin \pi x = -\sin \pi(x-1)$; $\to +\pi$

(7) $= \dfrac{\sin(\frac{\pi}{4}-x)}{\cos 2x} \cdot \underbrace{\dfrac{\sin 2x}{\cos(\frac{\pi}{4}-x)}}_{\to 1}$; $\cos y = \sin(\frac{\pi}{2}-y)$; $\to \frac{1}{2}$

$\cos 2x = \sin(2(\frac{\pi}{4}-x))$;

(8) čit: $\sin(a+2x) - \sin(a+x) - (\sin(a+x) - \sin a)$

$= 2\cos(a+\frac{3}{2}x)\sin\frac{x}{2} - 2\cos(a+\frac{1}{2}x)\sin\frac{x}{2}$

$= 2\sin\frac{x}{2}\underbrace{\left(\cos(a+\frac{3}{2}x) - \cos(a+\frac{1}{2}x)\right)}_{-2\sin(a+x)\sin\frac{x}{2}} \to -1.$

(14) $\ln\left(\underbrace{\operatorname{tg}\left(\frac{\pi}{4}+ax\right)}_{\to \operatorname{tg}\frac{\pi}{4}=1}\right) = \frac{\ln \operatorname{tg}(\cdots)}{\operatorname{tg}(\cdots)-1}\cdot\left(\operatorname{tg}(\cdots)-1\right)$

kde $\operatorname{tg}\left(\frac{\pi}{4}+ax\right)-1 = \operatorname{tg}\left(\frac{\pi}{4}+ax\right) - \operatorname{tg}\left(\frac{\pi}{4}\right)$ .. dále jako př. (1)

(15) $\ln\left(\frac{\ln ax}{\ln\frac{x}{a}}\right) = \ln\left(\underbrace{\frac{\ln x+\ln a}{\ln x-\ln a}}_{\to 1\,;\ x\to 0+}\right) = \ln\left(1+\underbrace{\frac{2\ln a}{\ln x-\ln a}}_{\to 0}\right)$

odtud $= \underbrace{\frac{\ln(1+\cdots)}{\cdots}}_{\to 1}\cdot\frac{2\ln a}{\underbrace{\ln x-\ln a}_{\downarrow\ -\infty}}\cdot\left(\underbrace{\ln x}_{\downarrow\ -\infty}+\ln\ln a\right) \to 2\ln a.$

(16) čitatel: $\frac{\ln(1+xe^x)}{xe^x}\cdot xe^x$; jm.: $\frac{\ln\left(x+\sqrt{1+x^2}\right)}{x+\sqrt{1+x^2}-1}\cdot(\cdots)$,

kde $(\cdots) = \sqrt{1+x^2}-(1-x) = \frac{1+x^2-(1-x)^2}{\sqrt{1+x^2}+(1-x)^2} = \frac{2x}{\sqrt{\ }+\sqrt{\ }}$ ..

(17) $\log_x 2 = \frac{\ln 2}{\ln x}$

(21) $\begin{matrix}\sin\\ \cos\end{matrix}\ \pi x = -\begin{matrix}\sin\\ \cos\end{matrix}\ \pi(x-1)$

(25) $\sin \pi x^\alpha = -\sin\pi(x^\alpha-1)$; $\frac{x^\alpha-1}{x-1}\to \ln\alpha,\ x\to 1$

(26) čit.: $e^{\alpha x}-e^{\beta x} = e^{\beta x}\left(e^{(\alpha-\beta)x}-1\right)$; BÚNO $\alpha\neq\beta$

jm.: $\sin\alpha x - \sin\beta x = 2\sin\left(\frac{(\alpha-\beta)x}{2}\right)\cos\left(\frac{(\alpha+\beta)x}{2}\right)$

(27) čit.: $a^x - x^a = (a^x-a^a)-(x^a-a^a)$

$= a^a\left(\underbrace{a^{x-a}}_{e^{\cdots}}-1\right) - a^a\left(\underbrace{\left(\frac{x}{a}\right)^a}_{e^{\cdots}}-1\right)$