Partial differential equations 2

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1. Vector-valued functions

Notation. We consider $u(t) : I \to X$, where $I = [0, T]$ is time interval, X is Banach space with norm $\|u\|_X$, X^* is dual of X, $\langle x^*, x\rangle_{X^*,X}$ is the duality between $x^* \in X^*$ and $x \in X$. We usually omit the subscripts.

1.1. Vector-valued integrable functions – Bochner integral

Definition. Function $u(t) : I \to X$ is called

1. *simple*, if $u(t) = \sum_{j=1}^{N} \chi_{A_j}(t) x_j$, where $A_j \subset I$ are (Lebesgue) measurable, and $x_j \in X$
2. *measurable* (strongly measurable), if there are $u_n(t)$ simple such that $u_n(t) \to u(t)$ (strongly in X) for a.e. $t \in I$
3. *weakly measurable*, if the (scalar) function $t \mapsto \langle x^*, u(t)\rangle$ is (Lebesgue) measurable for any $x^* \in X^*$ fixed

Remarks.

- (strongly) measurable $\implies$ weakly measurable
- $u(t)$ is simple $\iff$ $u(t)$ is measurable and $u(I) \subset X$ finite

Theorem 1.1.* [1] [Pettis] Function $u(t) : I \to X$ is measurable iff $u(t)$ is weakly measurable and there is $N \subset I$ of measure zero such that $u(I \setminus N) \subset X$ is separable ("essentially separably-valued").

Corollaries. ① For X separable weak measurability implies measurability.
② $u_n(t)$ measurable, $u_n(t) \to u(t)$ a.e. $\implies$ $u(t)$ measurable
③ Continuity implies measurability.

Definition. Function $u(t) : I \to X$ is called (Bochner) integrable, provided there exist $u_n(t)$ simple such that $\int_I \|u(t) - u_n(t)\|_X \, dt \to 0$ for $n \to \infty$. The (Bochner) integral of $u(t) : I \to X$ is defined as follows:

1. $\int_I u(t)\, dt = \sum_{j=1}^{N} x_j \lambda(A_j)$, if $u(t)$ is simple
2. $\int_I u(t)\, dt = \lim_{n\to\infty} \int_I u_n(t)\, dt$, if $u(t)$ is (Bochner) integrable

Remark. One has to check these definitions is correct (i.e. independent of x_j, A_j in the first part, and of $u_n(t)$ in the second part).
One also proves that $\|\int_I u(t)\, dt\|_X \le \int_I \|u(t)\|_X \, dt$ for any $u(t)$ integrable.

Theorem 1.2.* [Bochner] Function $u(t) : I \to X$ is Bochner integrable iff $u(t)$ is measurable and $\int_I \|u(t)\|_X \, dt < \infty$.

[1] Theorems marked with $\star$ were not proven in this class.

Theorem 1.3.* [Lebesgue] Let $u_n(t) : I \to X$ be measurable, $u_n(t) \to u(t)$ for a.e. $t \in I$, and let there exist $g(t) : I \to \mathbb{R}$ integrable such that $\|u_n(t)\| \le g(t)$ for a.e. t and all n. Then $u(t)$ is Bochner integrable and $\int_I u_n(t)\,dt \to \int_I u(t)\,dt$; in fact one even has $\int_I \|u_n(t) - u(t)\|\,dt \to 0$, $n \to \infty$.

Recall. For a scalar $x(t) : I \to \mathbb{R}$ we say that t is a Lebesgue point, if $\lim_{h\to 0} \frac{1}{2h}\int_{-h}^{h} |x(t+s)-x(t)|\,ds = 0$. Lebesgue's theorem: if $x(t) : I \to \mathbb{R}$ is (locally) integrable, then a.e. $t \in I$ is a Lebesgue point.

Definition. We say that $t \in I$ is a Lebesgue point of a function $u(t) : I \to X$, provided that $\lim_{h\to 0} \frac{1}{2h}\int_{-h}^{h} \|u(t+s) - u(t)\|_X\,ds = 0$.

Theorem 1.4. [Lebesgue] If $u(t) : I \to X$ is Bochner integrable, then a.e. $t_0 \in I$ is a Lebesgue point.

Remarks. Let $u(t) : I \to X$ be Bochner integrable.

- Fix $t_0 \in I$ and set $U(t) = \int_{t_0}^{t} u(s)\,ds$. Then $U'(t) = u(t)$ in each Lebesgue point of $u(t)$, in particular a.e. in I.
- Let $\psi_0(t) : \mathbb{R} \to \mathbb{R}$ be convolution kernel, i.e. a bounded measurable function supported in $[-1, 1]$ such that $\int_{-1}^{1} \psi(s)\,ds = 1$ (and possibly with additional regularity and symmetries).

  Define $\psi_n(t) = n\psi_0(nt)$, and $u * \psi_n(t) = \int_{\mathbb{R}} u(t-s)\psi_n(s)\,ds$. (Note that this only makes sense for $t \in [1/n, T - 1/n]$, or one has to define $u(t)$ outside of I e.g. by zero).

  Then $u * \psi_n(t) \to u(t)$ at each Lebesgue point of $u(t)$, in particular a.e. in I.

Definition. For $p \in [1, \infty)$ we set

$$L^p(I; X) = \left\{u(t) : I \to X;\ u(t) \text{ is measurable and } \int_I \|u(t)\|_X^p\,dt < \infty\ \right\}$$

For $p = \infty$ we set

$$L^\infty(I, X) = \left\{u(t) : I \to X;\ u(t) \text{ is measurable and } t \mapsto \|u(t)\|_X \text{ is essentially bounded}\right\}$$

Essential boundedness means: there is $c > 0$ such that $\|u(t)\|_X \le c$ pro a.e. $t \in I$.

Remarks. These are Banach spaces – with the usual norm, and the convention that $u(t)$, $v(t)$ are considered identical whenever $u(t) = v(t)$ a.e. This is proved just as in the scalar case.
If X is a Hilbert space with scalar product $(\cdot,\cdot)_X$, then $L^2(I; X)$ is Hilbert space with scalar product $\int_I (u(t), v(t))_X\,dt$.
Note that $L^1(I; X)$ is just the space of integrable functions, and $L^p(I; X) \subset L^q(I; X)$ if $p \ge q$ thanks to the boundedness of I.

Lemma 1.1. [Approximation and density.] Let $p \in [1, \infty)$. Then:

1. Simple functions are dense in $L^p(I; X)$.
2. Functions of the from $u(t) = \sum_{j=1}^{N} \varphi_j(t)x_j$, where $\varphi_j(t) \in C_c^\infty(I; \mathbb{R})$, are dense in $L^p(I; X)$.
3. If the space Y is dense in X, then the space $C_c^\infty(I; Y)$ is dense in $L^p(I; X)$.
4. Let $\psi_n(t)$ be a sequence of regularizing kernels, and let $u(t)$ be extended by 0 outside of I. Then $u * \psi_n(t) \to u(t)$ in $L^p(I; X)$, for $n \to \infty$.

Remarks. It follows that if X is separable, then also $L^p(I; X)$ is separable for $p < \infty$. But none of these holds for $p = \infty$. We will talk on these spaces (duality, geometry) a little bit later.

Recall. Function $x(t) : I \to \mathbb{R}$ is called absolutely continuous, provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any disjoint, finite collection of intervals $(\alpha_j, \beta_j) \subset I$, if $\sum_j (\beta_j - \alpha_j) < \delta$ then $\sum_j |x(\beta_j) - x(\alpha_j)| < \varepsilon$.
Proposition 1: If $x(t) \in AC(I;\mathbb{R})$, then $x'(t)$ exists finite a.e. in I, $x'(t) \in L^1(I;\mathbb{R})$ and $x(t_2) - x(t_1) = \int_{t_1}^{t_2} x'(s)\,ds$ for any t_1, $t_2 \in I$.
Proposition 2: Let $g(t) \in L^1(I;\mathbb{R})$. Fix $t_0 \in I$ and set $x(t) = \int_{t_0}^{t} g(s)\,ds$ for $t \in I$. Then $x(t)$ is AC and $x'(t) = g(t)$ a.e. in I.
We want to generalize this to vector-valued case. In fact Proposition 2 follows quite easily from the above (cf. Remark after Theorem 1.4). Proposition 1 is generalized in Theorem 1.5 below.

Definition. Function $u(t) : I \to X$, is called absolutely continuous, writing $u(t) \in AC(I;X)$, provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any disjoint, finite collection of intervals $(\alpha_j, \beta_j) \subset I$, if $\sum_j (\beta_j - \alpha_j) < \delta$ then $\sum_j \|u(\beta_j) - x(\alpha_j)\|_X < \varepsilon$.

Theorem 1.5. Let $u(t) : I \to X$ be absolutely continuous, let X be reflexive and separable space. Then $u'(t)$ (strong derivative) exists for a.e. $t \in I$. Moreover, $u'(t)$ is (Bochner) integrable and $u(t_2) - u(t_1) = \int_{t_1}^{t_2} u'(s)\,ds$ for any t_1, $t_2 \in I$.

Notation. Let $\mathcal{D}(I) = C_c^\infty(I;\mathbb{R})$ be the space of test functions, i.e. infinitely smooth functions with support strictly inside of I.

Lemma 1.2. Let $u(t) \in L^1(I;X)$.

1. If $\int_I u(t)\varphi(t)\,dt = 0$ for all $\varphi(t) \in \mathcal{D}(I)$, then $u(t) = 0$ a.e. in I.
2. If $\int_I u(t)\varphi'(t)\,dt = 0$ for all $\varphi(t) \in \mathcal{D}(I)$, then there is $x_0 \in X$ such that $u(t) = x_0$ a.e. in I.

Lemma 1.3. Let $u(t)$, $g(t) \in L^1(I;X)$. Then the following are equivalent:

1. There exist $x_0 \in X$ such that $u(t) = x_0 + \int_0^t g(s)\,ds$ for a.e. $t \in I$.
2. $\int_I u(t)\varphi'(t)\,dt = -\int_I g(t)\varphi(t)\,dt$ for all $\varphi(t) \in \mathcal{D}(I)$.
3. $\frac{d}{dt}\langle x^*, u(t)\rangle = \langle x^*, g(t)\rangle$ in the sense of distributions on $(0,T)$, for every $x^* \in X^*$ fixed.

Definition. Let $u(t)$, $g(t) \in L^1(I;X)$. We say that $g(t)$ is a weak derivative of $u(t)$, if one (hence all) of the assertions of Lemma 1.3 hold. We write $\frac{d}{dt}u(t) = g(t)$. We further define the space

$$W^{1,p}(I;X) = \left\{u(t) \in L^p(I;X);\ \frac{d}{dt}u(t) \in L^p(I;X)\right\}$$

Remarks. By Lemma 1.3 part 1, weakly differentiable functions are just AC functions; equivalently, primitive functions to integrable functions (up to a modification a.e.).
In applications we often have $u(t) \in L^p(I;Y)$ with $\frac{d}{dt}u(t) \in L^q(I;Z)$ with some spaces Y, Z. This requires there is some space X such that $Y \subset X$, $Z \subset X$ (often simply $Y \subset Z = X$), so that $u(t)$ is weakly differentiable as a function $I \to X$, and moreover $u(t)$, $\frac{d}{dt}u(t)$ have the above-mentioned higher integrability.

Recall. X is called *reflexive*, if the canonical embedding $J : X \to X^{**}$ is isometrically *onto*. The sequence u_n converges weakly to u in X, if $\langle x^*, u_n\rangle \to \langle x^*, u\rangle$ for any $x^* \in X^*$ fixed. We denote weak convergence by $u_n \rightharpoonup u$.
The key application (as far as the PDE theory goes) of these concepts is the Eberlein-Šmulian theorem: if X is reflexive, and $u_n \subset X$ is a bounded sequence, then there is a subsequence $\tilde{u}_n$ and $u \in X$ such that $\tilde{u}_n \rightharpoonup u$.

Definition. Space X is called *strictly convex*, if $\|x\|$, $\|y\| \le 1$ and $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$.
It is called *uniformly convex*, if for any $\varepsilon > 0$ there is $\delta > 0$ such that $\|x\|$, $\|y\| \le 1$ and $\|x - y\| \ge \varepsilon$ implies $\|\frac{x+y}{2}\| \le 1 - \delta$.

Theorem 1.6. Let X be uniformly convex, let $x_n \rightharpoonup x$, and let $\|x_n\| \to \|x\|$. Then $x_n \to x$.

Remark. It is easy to verify that Hilbert space is uniformly convex, and it is elementary to prove Theorem 1.6 if X is Hilbert. Uniformly convex spaces have a number of good properties (e.g. they are always reflexive).
As a typical example, spaces $L^p(\Omega)$ are uniformly convex for $p \in (1, \infty)$. An obvious generalization is

Theorem 1.7.* Let X be uniformly convex, let $p \in (1, \infty)$. Then $L^p(I;X)$ is uniformly convex.

Recall. We call p, $p' \in [1, \infty]$ Hölder conjugate, if $\frac{1}{p} + \frac{1}{p'} = 1$. By Hölder's inequality we have

$$\int_\Omega |u(x)v(x)|\, dx \le \left(\int_\Omega |u(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_\Omega |v(x)|^{p'}\, dx\right)^{\frac{1}{p'}}$$

It follows that any $v(x) \in L^{p'}(\Omega)$ fixed defines an element $F \in (L^p(\Omega))^*$ by the formula $F : u(\cdot) \mapsto \int_\Omega u(x)v(x)$. Conversely, for $p \in [1, \infty)$, any element of $(L^p(\Omega))^*$ has such a representation, and in this sense $(L^p(\Omega))^* = L^{p'}(\Omega)$.
Consequently, $L^p(\Omega)$ are reflexive if $p \in (1, \infty)$. But $L^1(\Omega)$, $L^\infty(\Omega)$ are not reflexive, and $(L^\infty(\Omega))^*$ contains elements that cannot be represented by functions from $L^1(\Omega)$.
Once again, we have a vector-valued version of these results.

Theorem 1.8. [Hölder's inequality.] Let $u(t) \in L^p(I;X)$, $v(t) \in L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Then $t \mapsto \langle v(t), u(t)\rangle$ is measurable and

$$\int_I |\langle v(t), u(t)\rangle| \le \left(\int_I \|u(t)\|_X^p dt\right)^{\frac{1}{p}} \left(\int_I \|v(t)\|_{X^*}^{p'} dt\right)^{\frac{1}{p'}}$$

Theorem 1.9.* [Dual space to $L^p(I;X)$.] Let X be reflexive, separable and $p \in [1, \infty)$. Denote $\mathscr{X} = L^p(I;X)$. Then for any $F \in \mathscr{X}^*$ there is $v(t) \in L^{p'}(I, X^*)$ such that

$$\langle F, u(\cdot)\rangle_{\mathscr{X}^*, \mathscr{X}} = \int_I \langle v(t), u(t)\rangle_{X^*, X}\, dt \qquad \forall u(t) \in \mathscr{X}.$$

Moreover, $v(t)$ is uniquely defined, and its norm in $L^{p'}(I;X^*)$ equals to the norm of F in $\mathscr{X}^*$.

Corollaries. If X is reflexive, separable, and $p \in (1, \infty)$, then $L^p(I;X)$ is also reflexive, separable. Any sequence bounded in $L^p(I;X)$ has a weakly convergent subsequence.

Lemma 1.4. Let $u(t) : I \to X$ be weakly differentiable.

1. If $\eta(t) : I \to \mathbb{R}$ is Lipschitz, then $u(t)\eta(t) : I \to X$ is weakly differentiable, and $\frac{d}{dt}\big(u(t)\eta(t)\big) = \frac{d}{dt}u(t)\eta(t) + u(t)\eta'(t)$ a.e. in I.

2. If $\psi(t) \in \mathcal{D}(I)$, then $u * \psi(t)$ is smooth and moreover, $(u*\psi)'(t) = \frac{d}{dt}u * \psi(t)$ whenever $t - \operatorname{supp}\psi \subset (0,T)$.

Theorem 1.10. [Extension operator.] Let $u(t) \in L^p(I;Y)$ with $\frac{d}{dt}u(t) \in L^q(I;Z)$, where $I = [0,T]$. Denote $I_\Delta = [-\Delta, T+\Delta]$ for some $\Delta > 0$.
Then there is a function $Eu(t) \in L^p(I_\Delta;Y)$ with $\frac{d}{dt}Eu(t) \in L^q(I_\Delta;Z)$ such that $Eu(t) = u(t)$ and $\frac{d}{dt}Eu(t) = \frac{d}{dt}u(t)$ a.e. in I.

Remarks. It also follows from the proof that the norms of Eu, $\frac{d}{dt}Eu$ are estimated by the corresponding norms of u, $\frac{d}{dt}u$, the mapping $u \mapsto Eu$ is linear and we can also have $Eu = 0$ outside (say) $I_{\Delta/2}$ by multiplication of some cut-off function.

Theorem 1.11. [Smooth approximation.] Let $u(t) \in L^p(I;Y)$ with $\frac{d}{dt}u(t) \in L^q(I;Z)$. Then there exist functions $u_n(t) \in C^1(I;Y)$ such that $u_n(t) \to u(t)$ in $L^p(I;Y)$ and $u_n'(t) \to \frac{d}{dt}u(t)$ in $L^q(I;Z)$.

Notation. Symbol $X \hookrightarrow Y$ means *embedding*: $X \subset Y$ and there is $c > 0$ such that $\|u\|_Y \le c\|u\|_X$ for all $u \in X$. Symbol $X \hookrightarrow\hookrightarrow Y$ means *compact embedding*: $X \hookrightarrow Y$ and any sequence bounded in X has a subsequence converging strongly in Y.

Definition. Let X be separable, reflexive, densely embedded into a Hilbert space H. By *Gelfand triple* we mean $X \hookrightarrow H \cong H^* \hookrightarrow X^*$.
Note that $X \hookrightarrow H$ implies $H^* \hookrightarrow X^*$. Thanks to identification of H with H^* (via Riesz theorem), we have also "embedding" $\iota : X \to X^*$ is defined by

$$\langle \iota u, v\rangle_{X^*,X} = (u,v)_H \qquad u,\ v \in X$$

where $(\cdot,\cdot)_H$ is the scalar product in H. In this sense, duality $\langle \cdot,\cdot\rangle_{X,X^*}$ can be seen as a generalization of $(\cdot,\cdot)_H$.

Lemma 1.5. $W^{1,p}(I;X) \hookrightarrow C(I;X)$ in the sense of representative: for any $u(t) \in W^{1,p}(I;X)$ there is $\tilde{u}(t) \in C(I;X)$ such that

$$\|\tilde{u}\|_{C(I;X)} \le c\|u\|_{W^{1,p}(I;X)}$$

and $u(t) = \tilde{u}(t)$ a.e. in I.

Remark. It can be shown that even $W^{1,p}(I;X) \hookrightarrow C^{0,\alpha}(I;X)$, the space of α-Hölder functions with $\alpha = 1 - 1/p$. Also $W^{1,\infty}(I;X) = C^{0,1}(I;X)$, the space of Lipschitz functions.

Theorem 1.12. [Continuous representative.] Let $X \hookrightarrow H \cong H^* \hookrightarrow X^*$ be Gelfand triple, let $u(t) \in L^p(I;X)$, $\frac{d}{dt}u(t) \in L^{p'}(I;X^*)$, where p, p' are Hölder conjugate. Then:

1. $u(t) \in C(I;H)$ in the sense of representative; more precisely, there is $\tilde{u}(t)$ such that

$$\|\tilde{u}\|_{C(I;H)} \le C\left(\|u(t)\|_{L^p(I;X)} + \|\frac{d}{dt}u(t)\|_{L^{p'}(I;X^*)}\right)$$

   and $u(t) = \tilde{u}(t)$ a.e. in I.

2. function $t \mapsto \|u(t)\|_H^2$ is weakly differentiable with $\frac{d}{dt}\|u(t)\|_H^2 = 2\left\langle \frac{d}{dt}u(t), u(t)\right\rangle_{X^*,X}$ a.e. In particular

$$\|\tilde{u}(t_2)\|_H^2 = \|\tilde{u}(t_1)\|_H^2 + 2\int_{t_1}^{t_2} \left\langle \frac{d}{dt}u(t), u(t)\right\rangle_{X^*,X} dt$$

for any t_1, $t_2 \in I$, where $\tilde{u}(t)$ is the continuous representative.

Remarks. In other words, there is an embedding (in the sense of representative)

$$\left\{u(t) \in L^p(I;X), \quad \frac{d}{dt}u(t) \in L^{p'}(I;X^*)\right\} \hookrightarrow C(I;X)$$

In view of Theorem 1.9, note $u(t)$ and $\frac{d}{dt}u(t)$ belong to mutually dual spaces.

Lemma 1.6. [Ehrling.] Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$. Then for any $a > 0$ there is $C > 0$ such that

$$\|u\|_X \le a\|u\|_Y + C\|u\|_Z \qquad \forall u \in Y$$

Theorem 1.13. [Aubin-Lions lemma.] Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$, where Y, Z are reflexive, separable. Let p, $q \in (1,\infty)$. Then for any sequence $u_n(t)$ bounded in $L^p(I;Y)$, with $\frac{d}{dt}u_n(t)$ bounded in $L^q(I;Z)$, there is a subsequence converging strongly in $L^p(I;X)$.

Remark. In other words, under the above assumptions, there is a compact embedding

$$\left\{u(t) \in L^p(I;Y), \quad \frac{d}{dt}u(t) \in L^q(I;Z)\right\} \hookrightarrow\hookrightarrow L^p(I;X)$$

2. Parabolic 2nd order equation

In this chapter we will consider a nonlinear second order parabolic equation

$$\partial_t u - \operatorname{div} a(\nabla u) + f(u) = h(t,x) \qquad (t,x) \in I \times \Omega \tag{P1}$$
$$u = u_0 \qquad t = 0,\ x \in \Omega \tag{P2}$$
$$u = 0 \qquad t \in I,\ x \in \partial\Omega \tag{P3}$$

Here $u = u(t,x)$ is the unknown solution. The right-hand side $h = h(t,x)$ and initial condition u_0 in (P2) are given data, and (P3) is the so-called Dirichlet boundary condition.

Assumptions. Throughout this chapter, we assume that:

(A0) $\Omega \subset \mathbb{R}^n$ is bounded regular (Lipschitz) boundary $\partial\Omega$

(A1) $a(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $a(0) = 0$ and for $\forall \xi_1$, $\xi_2 \in \mathbb{R}^n$

$$|a(\xi_1) - a(\xi_2)| \le \alpha_1|\xi_1 - \xi_2| \tag{2.1}$$
$$\big(a(\xi_1) - a(\xi_2)\big) \cdot (\xi_1 - \xi_2) \ge \alpha_0|\xi_1 - \xi_2|^2 \tag{2.2}$$

(A2) $f(z) : \mathbb{R} \to \mathbb{R}$ satisfies $\forall z_1$, $z_2 \in \mathbb{R}$

$$|f(z_1) - f(z_2)| \le \ell|z_1 - z_2|$$

Remark. Here $-\operatorname{div} a(\nabla u)$ is a nonlinear 2nd order elliptic operator. For $a(\xi) = \xi$ and $f \equiv 0$ we have a heat equation $\partial_t u - \Delta u = h(t, x)$ as a special case.

Recall. The spaces $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$ and $W^{-1,2} = (W_0^{1,2})^*$ are reflexive, separable; $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$. Poincaré inequality:

$$\|u\|_{L^2(\Omega)} \le c_p \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in W_0^{1,2}(\Omega)$$

implies that $\|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm in $W_0^{1,2}(\Omega)$.

Notation. We will write L^2, $W^{1,2}$ instead of $L^2(\Omega)$, $W^{1,2}(\Omega)$, etc, and $\|u\|_2$, $\|u\|_{1,2}$ will denote the norms in these spaces. Symbol $(\cdot,\cdot)$ is the scalar product in L^2

$$(f, g) = \int_\Omega f(x) \cdot g(x)\, dx$$

and $\langle \cdot, \cdot \rangle$ the duality of $W_0^{1,2}$ and $W^{-1,2}$.
We will work with the Gelfand triple $W_0^{1,2} \hookrightarrow L^2 \cong (L^2)^* \hookrightarrow W^{-1,2}$, and $\iota : W_0^{1,2} \to W^{-1,2}$ is the corresponding embedding, i.e., $\langle \iota u, v\rangle = (u, v)$ for all u, $v \in W_0^{1,2}$.

Recall. Let $h \in W^{-1,2}$. The function $u \in W_0^{1,2}$ is called weak solution of

$$\begin{aligned} -\operatorname{div} a(\nabla u) &= h \qquad && x \in \Omega \\ u &= 0 && x \in \partial\Omega \end{aligned}$$

provided that

$$(a(\nabla u), \nabla v) = \langle h, v\rangle \qquad \forall v \in W_0^{1,2}$$

Expanding the left-hand side this means

$$\int_\Omega a(\nabla u(x)) \cdot \nabla v(x)\, dx = \langle h, v\rangle \qquad \forall v \in W_0^{1,2}$$

By the (nonlinear) Lax-Milgram theorem, there exists unique such solution u. It is convenient to introduce the (nonlinear) operator $\mathcal{A} : W_0^{1,2} \to W^{-1,2}$, by the relation

$$\langle \mathcal{A}(u), v\rangle = (a(\nabla u), \nabla v)$$

Then the above problem is written simply as $\mathcal{A}(u) = h$, with $\mathcal{A} : W_0^{1,2} \to W^{-1,2}$ one-to-one continuous.

Assumption on the data. We will assume that the right-hand side of (P1) satisfies $h(t) \in L^2(I; W^{-1,2})$ and the initial condition $u_0 \in L^2$.

Definition. Function $u(t) \in L^2(I; W_0^{1,2})$ is called *weak solution* to (P1), provided that

$$\frac{d}{dt}(u(t), v) + (a(\nabla u(t)), \nabla v) + (f(u(t)), v) = \langle h(t), v\rangle$$

in the sense of distributions on $(0, T)$, for any $v \in W_0^{1,2}$ fixed.

Remarks. Expanding the definition of weak derivative $\frac{d}{dt}$, this means

$$-\int_I (u(t), v)\varphi'(t)\, dt + \int_I (a(\nabla u(t)), \nabla v)\varphi(t)\, dt + \int_I (f(u(t)), v)\varphi(t)\, dt = \int_I \langle h(t), v\rangle\, \varphi(t)\, dt$$

for any $v \in W_0^{1,2}$, $\varphi(t) \in \mathcal{D}(I)$. Expanding further the definition of $(\cdot,\cdot)$ yields

$$-\int\limits_{I\times\Omega} u(t,x)v(x)\varphi'(t)\,dtdx + \int\limits_{I\times\Omega} a(\nabla u(t))\cdot\nabla v(x)\varphi(t)\,dtdx + \int\limits_{I\times\Omega} f(u(t,x))v(t,x)\,dtdx = \int_I \langle h(t), v\rangle\,dt$$

The definition makes sense: the integrals converge due to assumptions (A1), (A2).

Lemma 2.1. Let $u(t) \in L^2(I; W_0^{1,2})$ be a weak solution. Then

1. $u(t)$ is weakly differentiable with
$$\frac{d}{dt}u(t) + \mathcal{A}(u(t)) + f(u(t)) = h(t)$$
in particular $\frac{d}{dt}u(t) \in L^2(I; W_0^{1,2})$.
2. $u(t) \in C(I; L^2)$ in the sense of representative
3. $t \mapsto \|u(t)\|_2^2$ is weakly differentiable, with
$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \langle\mathcal{A}(u(t)), u(t)\rangle + (f(u(t)), u(t)) = \langle h(t), u(t)\rangle$$
for a.e. $t \in I$.

Remark. We agree to always use the continuous representative. (Hence, $t \mapsto \|u(t)\|_2^2$ is AC function.) Now it makes sense to speak of the value $u(t)$ for *all* $t \in I$, in particular, the initial condition $u(0)$.

Lemma 2.2. [Gronwall lemma.] Let $y(t)$, $g(t)$ be nonnegative (scalar) functions, $y(t)$ continuous and $g(t)$ integrable, such that

$$y(t) \le K + \int_0^t g(s)y(s)\,ds \qquad \forall t \in I$$

Then

$$y(t) \le K \exp\left(\int_0^t g(s)\,ds\right) \qquad \forall t \in I$$

Theorem 2.1. Weak solution is unique.

Recall. Let $\mathscr{X}$ be a Banach space. Operator $\mathcal{A}: \mathscr{X} \to \mathscr{X}^*$ is *monotone*, if $\langle\mathcal{A}(u) - \mathcal{A}(v), u - v\rangle_{\mathscr{X}^*,\mathscr{X}} \ge 0$ for all u, $v \in \mathscr{X}$. It is *hemicontinuous*, if the function $t \mapsto \mathcal{A}(u + tv)$ is continuous (from $\mathbb{R}$ to $\mathscr{X}^*$) for any u, $v \in \mathscr{X}$ fixed.

Lemma 2.3. [Minty's trick.] Let $\mathscr{X}$ be reflexive, let $\mathcal{A}: \mathscr{X} \to \mathscr{X}^*$ be monotone, hemicontinuous. Let $u_n \rightharpoonup u$ in $\mathscr{X}$, $A(u_n) \rightharpoonup \alpha$ in $\mathscr{X}^*$, and let moreover

$$\limsup_{n\to\infty}\langle\mathcal{A}(u_n), u_n\rangle_{\mathscr{X}^*,\mathscr{X}} \le \langle\alpha, u\rangle_{\mathscr{X}^*,\mathscr{X}}$$

Then $\mathcal{A}(u) = \alpha$, i.e. $A(u_n) \rightharpoonup A(u)$.

Theorem 2.2. [Compactness of w.s.] Let $u_n(t) \in L^2(I; W_0^{1,2})$ be weak solutions such that $u_n(0) \to u_0$ in L^2. Then there is a subsequence $\tilde{u}_n(t)$ converging weakly to some $u(t) \in L^2(I; W_0^{1,2})$, where $u(t)$ is again a weak solution, and $u(0) = u_0$.

Recall. The Dirichlet laplacian eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \qquad & x \in \Omega \\ u &= 0 \qquad & x \in \partial\Omega \end{aligned}$$

has a sequence $(w_j, \lambda_j) \in W_0^{1,2} \times (0, +\infty)$ of eigenfunctions and eigenvalues such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to +\infty$, and w_j form a complete ON basis of L^2, but also a complete OG basis of $W_0^{1,2}$, where the latter space is equipped with the scalar product $(\!(u, v)\!) = \int_\Omega \nabla u \cdot \nabla v$. The above problem in the weak form can thus be written as

$$(\!(u, v)\!) = \lambda(u, v) \qquad \forall v \in W_0^{1,2}$$

By P_n we will denote a projection of L^2 onto the finite-dimensional space $\mathrm{span}\{w_1, \dots, w_n\}$. Clearly $\|P_n\| = 1$. But an important fact is that P_n is also an ON projection w.r.t. $W_0^{1,2}$ with the above scalar product $(\!(\cdot, \cdot)\!)$, and $\|P_n\| = 1$ also in this space.

Theorem 2.3. [Existence of w.s.] Let $u_0 \in L^2$ and $h(t) \in L^2(I; W^{-1,2})$ be given. Then there exists $u(t) \in L^2(I; W_0^{1,2})$ a weak solution to (P1)–(P3) such that $u(0) = u_0$.

Lemma 2.4. [Chain rule for weak derivative.] Let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth function with ψ' and ψ'' bounded. Then:

1. If $u \in W_0^{1,2}$, then $\psi(u) \in W_0^{1,2}$, with $\nabla\psi(u) = \psi'(u)\nabla u$ in the weak sense. Moreover, the operator $u \mapsto \psi(u)$ is continuous $W_0^{1,2} \to W_0^{1,2}$.
2. If $u(t) \in L^2(I; W_0^{1,2})$ with $\frac{d}{dt}u(t) \in L^2(I; W^{-1,2})$, then $t \mapsto \int_\Omega \psi(u(t))\, dx$ is weakly differentiable with
$$\frac{d}{dt}\int_\Omega \psi(u(t))\, dx = \left\langle \frac{d}{dt}u(t), \psi'(u(t)) \right\rangle$$
for a.e. $t \in I$.

Definition. If $v \in L^2$ or $W^{1,2}$, then $v \geq 0$ (resp. $v \leq 0$) means: $v(x) \geq 0$ (resp. $v(x) \leq 0$) for a.e. $x \in \Omega$. If $f \in W^{-1,2}$, then $f \geq 0$ (resp. $f \leq 0$) means: $\langle f, v\rangle \geq 0$ (resp. $\langle f, v\rangle \leq 0$) for all $v \in W_0^{1,2}$ such that $v \geq 0$.

Theorem 2.4. [Maximum principle for w.s.] Let $u(t)$ be a weak solution. Let $f(\cdot) \geq 0$, $u(0) \leq 0$ and $h(t) \leq 0$ for a.e. $t \in I$. Then $u(t) \leq 0$ for a.e. $t \in I$.

Remarks. Minimum principle: if $f(\cdot) \leq 0$, $u(0) \geq 0$ and $h(t) \geq 0$ for a.e. $t \in I$, then $u(t) \geq 0$ for a.e. $t \in I$. Generalization: if $u(0) \leq M$ and $u \leq M$ on $\partial\Omega$ (in the sense of traces), and $f(\cdot) \geq 0$, $h(t) \leq 0$, then $u(t) \leq M$ for a.e. $t \in I$.

Recall. Regularity of the laplace equation: if $u \in W_0^{1,2}$ is a weak solution to $-\Delta u = \iota h$, where $h \in L^2$ and we moreover have $\partial\Omega \in C^2$, then $u \in W^{2,2}$ and there holds $\|u\|_{2,2} \leq c_R\|f\|_2$, where c_R only depends on Ω.

Theorem 2.5. [Strong solution.] Let $u(t) \in L^2(I; W_0^{1,2})$ be a weak solution of the heat equation

$$\partial_t u - \Delta u + f(u) = h(t, x)$$

and let $u_0 \in W_0^{1,2}$, $h(t) \in L^2(I; L^2)$ and $\partial\Omega \in C^2$. Then

$$u(t) \in L^\infty(I; W_0^{1,2}) \cap L^2(I; W^{2,2})$$
$$\frac{d}{dt} u(t) \in L^2(I; L^2)$$

Remarks. The equation now holds a.e. in $I \times \Omega$. Note that $u(t) \in L^2(I; W^{1,2})$ and $\frac{d}{dt} u(t) \in L^2(I; L^2)$ implies (in fact is equivalent to) $u(t, x) \in W^{1,2}(I \times \Omega)$.
Note also that even if $u(0) \in L^2$ only, we have $u(t) \in W_0^{1,2}$ for a.e. $t \in I$, hence the regularity of Theorem 2.5 holds at least *locally*, i.e. on $[\tau, T]$ for arbitrary $\tau > 0$. This is a general principle: parabolic equations *regularize* in time.
The regularity can be further improved – as far as the data permit.

3. Hyperbolic 2nd order equation

In this chapter we will consider a semilinear second order hyperbolic equation

$$\partial_{tt} u - \Delta u + \alpha u_t + f(u) = h(t, x) \qquad (t, x) \in I \times \Omega \tag{H1}$$
$$u = u_0 \qquad t = 0,\ x \in \Omega \tag{H2}$$
$$\partial_t u = u_1 \qquad t = 0,\ x \in \Omega \tag{H3}$$
$$u = 0 \qquad t \in I,\ x \in \partial\Omega \tag{H4}$$

Again, $u = u(t, x)$ is the unknown solution. The right-hand side $h = h(t, x)$ and initial conditions u_0, u_1 in (H2), (H3) are given. Dirichlet boundary condition is imposed in (H4).

Assumptions. We will assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, $f(z) : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, and $\alpha \in \mathbb{R}$. Concerning the data, we assume $h(t) \in L^2(I; L^2)$, $u_0 \in W_0^{1,2}$ and $u_1 \in L^2$.

Definition. Function $u(t) \in L^\infty(I; W_0^{1,2})$ with $\frac{d}{dt} u(t) \in L^\infty(I; L^2)$ is called weak solution to (H1), provided that

$$\frac{d^2}{dt^2}(u(t), v) + (\nabla u(t), \nabla v) + \alpha(\frac{d}{dt} u(t), v) + (f(u(t)), v) = (h(t), v)$$

in the sense of distributions on $(0, T)$, for any $v \in W_0^{1,2}$ fixed.

Remarks. Expanding the definition, weak solution means that

$$\int_I (u(t), w)\varphi''(t)\, dt + \int_I (\nabla u(t), \nabla w)\varphi(t)\, dt + \alpha \int_I (\frac{d}{dt} u(t), w)\varphi(t)\, dt$$
$$+ \int_I (f(u(t)), w)\varphi(t)\, dt = \int_I (h(t), w)\varphi(t)\, dt$$

for any $v \in W_0^{1,2}$, $\varphi(t) \in \mathcal{D}(I)$. On the other hand, if define $A : W_0^{1,2} \to W^{-1,2}$ as $\langle Au, v\rangle = (\nabla u, \nabla v)$ for $v \in W_0^{1,2}$ (so essentially $A = -\Delta$ weakly), then the weak formulation can be written more succintly as

$$\frac{d^2}{dt^2} \iota u(t) + Au(t) + \alpha\iota \frac{d}{dt} u(t) + \iota f(u(t)) = \iota h(t)$$

as equation in $W^{-1,2}$. Or equivalently, denoting $v(t) = \frac{d}{dt}u(t)$

$$\frac{d}{dt}\iota v(t) + Au(t) + \alpha\iota v(t) + \iota f(u(t)) = \iota h(t)$$

It can be shown that if $u(t)$ is weak solution, then $u(t) : I \to W_0^{1,2}$ and $\frac{d}{dt}u(t) : I \to L^2$ are weakly continuous (in the sense of representative). Hence the initial conditions (H2), (H3) are meaningful.

Theorem 3.1. Weak solution is unique.

Notation. The energy is defined

$$E[u] = \frac{1}{2}\left(\|\frac{d}{dt}u\|_2^2 + \|\nabla u\|_2^2\right)$$

Multiplying (H1) with $\partial_t u$ and integrating over Ω, the first two terms give $\frac{d}{dt}E[u]$; unfortunately, this is not justified in the class of weak solutions. We however have the following:

Lemma 3.1. Let $u(t) \in L^2(I; W_0^{1,2})$ with $\frac{d}{dt}u(t) \in L^2(I; L^2)$ be such that $\frac{d}{dt}\iota u(t) + Au(t) = \iota H(t)$ weakly in I, where $H(t) \in L^2(I; L^2)$. Then $t \mapsto E[u(t)]$ is weakly differentiable and

$$\frac{d}{dt}E[u(t)] = (H(t), \frac{d}{dt}u(t))$$

for a.e. $t \in I$.

Theorem 3.2. Let $h(t) \in L^2(I; L^2)$, $u_0 \in W_0^{1,2}$ and $u_1 \in L^2$ be given. Then there exists $u(t) \in L^\infty(I; W_0^{1,2})$ with $\frac{d}{dt}u(t) \in L^\infty(I; L^2)$ a weak solution to (H1)–(H4) such that $u(0) = u_0$, $\frac{d}{dt}u(0) = u_1$ in the sense of representatives.

Theorem 3.3. [Strong solution.] Let $u(t) \in L^\infty(I; W_0^{1,2})$ with $\frac{d}{dt}u(t) \in L^\infty(I; L^2)$ be a weak solution to the wave equation

$$\frac{d^2}{dt^2}u - \Delta u + \alpha\frac{d}{dt}u = h(t)$$

Let the data satisfy $h(t) \in W^{1,2}(I; L^2)$, $u_1 \in W_0^{1,2}$, $u_0 \in W_0^{1,2}$, $h(0) + \Delta u_0 \in L^2$ and let $\partial\Omega$ be C^2. Then

$$u(t) \in L^\infty(I; W^{2,2}), \quad \frac{d}{dt}u(t) \in L^\infty(I; W_0^{1,2})$$
$$\frac{d^2}{dt^2}u(t) \in L^\infty(I; L^2)$$

Remark. For parabolic equation, we have seen that any weak solution immediately becomes more regular (i.e. strong) even if $u(0) \in L^2$ only.
For the wave equation this cannot be true: note that if we reverse time, only the sign of α is changed.

Notation. For any weak solution $u = u(t, x)$ we define

$$e(t, x) = \frac{1}{2}|\frac{d}{dt}u(t, x)|^2 + \frac{1}{2}|\nabla u(t, x)|^2$$

Note that $e(t, x) \in L^1(I \times \Omega)$ at least, and is related to the energy via

$$E[u(t)] = \int_\Omega e(t, x)\, dx$$

We will also write $B(x_0,r) = \{x \in \mathbb{R}^n;\ |x - x_0| \le r\}$.

Theorem 3.4. [Wave principle.] Let $u(t)$ be weak solution to the damped wave equation

$$\frac{d^2}{dt^2}u - \Delta u + \alpha \frac{d}{dt}u = 0$$

where $\alpha \ge 0$. Let $x_0 \in \Omega$, $\tau \in I$ be such that $B(x_0,\tau) \subset \Omega$. Then

$$\int\limits_{B(x_0,\tau-t)} e(t,x)\,dx \le \int\limits_{B(x_0,\tau)} e(0,x)\,dx$$

for all $t \in [0,\tau]$.

Remark. The above theorem uses a formula, which holds for any smooth $e = e(t,x)$:

$$\frac{d}{dt}\int\limits_{B(x_0,\tau-t)} e(t,x)\,dx = \int\limits_{B(x_0,\tau-t)} \partial_t e(t,x)\,dx - \int\limits_{\partial B(x_0,\tau-t)} e(t,x)\,dS(x)$$

Corollary. [Finite speed of propagation.] Let $u(t)$ be weak solution to damped wave equation; let $u(0) = 0$, $\frac{d}{dt}u(0) = 0$ in some $B(x_0,\tau) \subset \Omega$. Then $u \equiv 0$ a.e. in the cone

$$\{(t,x);\ |x - x_0| \le \tau - t\}$$

4. Theory of semigroups

4.1. Homogeneous problem

Our first aim in this chapter is to develop a theory of abstract equations of the form

$$\frac{d}{dt}x = Ax \tag{4.1}$$

where $x(t) : I \to X$ and $A : X \to X$ is linear, but not necessarily bounded operator. In other words, we will want to define e^{tA} in some generalized sense.

Notation. In this chapter X is a Banach space with norm $\|\cdot\|$, $\mathcal{L}(X)$ is the space of linear continuous operators $L : X \to X$ with the usual norm. By unbounded operator we mean a couple $(A, \mathcal{D}(A))$, where $\mathcal{D}(A) \subset X$ is a linear subspace and $A : \mathcal{D}(A) \to X$ is linear (but not necessarily bounded) operator.

Definition. Function $S(t) : [0,\infty) \to \mathcal{L}(X)$ is called a *semigroup* of operators in X, if
(i) $S(0) = I$
(ii) $S(t)S(s) = S(t+s)$ for all t, $s \ge 0$
If moreover
(iii) $S(t)x \to x$ as $t \to 0+$ for any $x \in X$ fixed, we say that $S(t)$ form a c_0*-semigroup* in X.

Remarks. Replacing (iii) by a stronger condition (iii)' $S(t) \to I$ in $\mathcal{L}(X)$ for $t \to 0+$, we obtain the so-called *uniformly continuous semigroup*. In such a case one already has $S(t) = e^{tA}$ for some $A \in \mathcal{L}(X)$.

Lemma 4.1. Let $S(t)$ be a c_0-semigroup in X. Then

1. $\exists M \ge 1$, $\omega \ge 0$ such that $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$ for all $t \ge 0$

2. the map $t \mapsto S(t)x$ is continuous $[0,\infty) \to X$ for any $x \in X$ fixed

Definition. By a *generator* of semigroup $S(t)$ we mean an unbounded operator $(A, \mathcal{D}(A))$, defined by

$$Ax = \lim_{h\to 0+} \frac{1}{h}\big(S(h)x - x\big) \qquad \text{for } x \in \mathcal{D}(A)$$

where we set

$$\mathcal{D}(A) = \Big\{x \in X;\ \lim_{h\to 0+} \frac{1}{h}(S(h)x - x) \text{ exists in } X\Big\}$$

Remark. It is easy to verify that in the above definition $\mathcal{D}(A) \subset X$ is a linear subspace, and $A : \mathcal{D}(A) \to X$ is a linear operator.

Theorem 4.1. [Basic properties of generator.] Let $S(t)$ be a c_0-semigroup in X, let $(A, \mathcal{D}(A))$ be its generator. Then

1. $x \in \mathcal{D}(A) \implies S(t)x \in \mathcal{D}(A)$ for all $t \geq 0$
2. $x \in \mathcal{D}(A) \implies AS(t)x = S(t)Ax = \frac{d}{dt}S(t)x$ for any $t \geq 0$ (derivative at $t = 0$ is only from the right)
3. for any $x \in X$ and $t \geq 0$, one has $\int_0^t S(s)x\,ds \in \mathcal{D}(A)$ and $A\big(\int_0^t S(s)x\,ds\big) = S(t)x - x$

Remark. The above theorem says that $\mathcal{D}(A)$ is invariant w.r.t. $S(t)$, that A and $S(t)$ commute on $\mathcal{D}(A)$ and most importantly, if $x_0 \in \mathcal{D}(A)$, then $x(t) = S(t)x_0$ is a classical solution to (4.1) with initial condition $x(0) = x_0$.

Definition. We say that the unbounded operator $(A, \mathcal{D}(A))$ is *closed*, if $u_n \in \mathcal{D}(A)$, $u_n \to u$ and $Au_n \to v$ imply $u \in \mathcal{D}(A)$ and $Au = v$.

Remark. It can be shown that $(A, \mathcal{D}(A))$ is closed if and only if $\mathcal{D}(A)$ is complete (i.e. Banach) space w.r.t. the norm $\|u\| + \|Au\|$. In this situation $A : \mathcal{D}(A) \to X$ is *continuous.*

Theorem 4.2. Let $(A, \mathcal{D}(A))$ be a generator of some c_0-semigroup in X. Then $\mathcal{D}(A)$ is dense in X and $(A, \mathcal{D}(A))$ is closed.

Lemma 4.2. [Unicity of semigroup.] Let $S(t)$, $\tilde{S}(t)$ be c_0-semigroups that have the same generator. Then $S(t) = \tilde{S}(t)$ for all $t \geq 0$.

Definition. Let $(A, \mathcal{D}(A))$ be an unbounded operator in X. We define

- the *resolvent set* $\rho(A) = \{\lambda \in \mathbb{C},\ \lambda I - A : \mathcal{D}(A) \to X \text{ is one-to-one }\}$
- the *resolvent* $R(\lambda, A) = (\lambda I - A)_{-1} : X \to \mathcal{D}(A)$, defined if $\lambda \in \rho(A)$
- the *spectrum* $\sigma(A) = \mathbb{C} \setminus \rho(A)$

Remarks. The following are easy to show:

- if $(A, \mathcal{D}(A))$ is closed, then $R(\lambda, A)$, once it is defined, must be already continuous, i.e. $R(\lambda, A) \in \mathcal{L}(X)$; moreover $\rho(A) \subset \mathbb{C}$ is open, and the mapping $\lambda \mapsto R(\lambda, A)$ is analytic $\mathbb{C} \to \mathcal{L}(X)$

- one has the resolvent indentity:

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A), \qquad \text{for any } \lambda,\, \mu \in \rho(A)$$

- one also has

$$\begin{aligned} AR(\lambda, A)x &= \lambda R(\lambda, A)x - x, \quad x \in X, \\ R(\lambda, A)Ax &= \lambda R(\lambda, A)x - x, \quad x \in \mathcal{D}(A); \end{aligned}$$

  in particular

$$AR(\lambda, A) = R(\lambda, A)A \qquad \text{on } \mathcal{D}(A)$$

Lemma 4.3. Let $(A, \mathcal{D}(A))$ be a generator of c_0-semigroup $S(t)$ on X; and let $\|S(t)\|_{\mathcal{L}(X)} \le Me^{\omega t}$ for all $t \ge 0$. Then for every $\lambda > \omega$ one has $\lambda \in \rho(A)$, the resolvent can be computed as

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} S(t)x \, dt$$

and moreover, $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{\lambda - \omega}$.

Remark. If $S(t)$ is a c_0-semigroup with generator $(A, \mathcal{D}(A))$, then $\tilde{S}(t) = e^{-\omega t} S(t)$ is a c_0-semigroup with generator $\tilde{A} = A - \omega I$, and $\mathcal{D}(\tilde{A}) = \mathcal{D}(A)$. Clearly also $\lambda \in \rho(\tilde{A})$ if and only if $\lambda + \omega \in \rho(A)$, and $R(\lambda, \tilde{A}) = R(\lambda + \omega, A)$.

Definition. We call $S(t)$ a *contraction c_0-semigroup*, provided that $\|S(t)\|_{\mathcal{L}(X)} \le 1$ for all $t \ge 0$.

Theorem 4.3. [Hille-Yosida.] Let $(A, \mathcal{D}(A))$ be an unbounded operator on X. Then the following are equivalent:

1. there exists a c_0-semigroup $S(t)$ of contractions on X such that $(A, \mathcal{D}(A))$ is its generator
2. $(A, \mathcal{D}(A))$ is closed, $\mathcal{D}(A)$ is dense in X and for every $\lambda > 0$ one has $\lambda \in \rho(A)$, $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{1}{\lambda}$

Remarks. The proof $2 \implies 1$ uses the so-called Yosida approximation $A_n x \to Ax$, where $A_n = n^2 R(n, A) - nI$ are bounded. Other possible ,,exponential formula" reads

$$S(t)x = \lim_{n\to\infty} \left(I - \frac{tA}{n} \right)^{-n} x$$

where $\left(I - \frac{tA}{n}\right)^{-1} = \frac{n}{t} R(\frac{n}{t}, A)$. Cf. the well-known formula $e^a = \lim_{n\to\infty}(1 - a/n)^{-n}$.
A general version of Hille-Yosida theorem says: $(A, \mathcal{D}(A))$ is a generator of c_0-semigroup satisfying $\|S(t)\|_{\mathcal{L}(X)} \le Me^{t\omega}$, if and only if $(A, \mathcal{D}(A))$ is closed, $\mathcal{D}(A)$ is dense in X and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and $\|R(\lambda, A)\|_{\mathcal{L}(X)} \le \frac{M}{\lambda - \omega}$.
A somewhat related is the Lumer-Phillips theorem: $(A, \mathcal{D}(A))$ is a generator of c_0-semigroup of contractions, if and only if it is closed, densely defined, and for every $\lambda > 0$ one has $\|\lambda x - Ax\|_X \ge \lambda \|x\|_X$, $x \in \mathcal{D}(A)$ and moreover, $\lambda_0 I - A : \mathcal{D}(A) \to X$ is onto for some $\lambda_0 > 0$.

4.2. Nonhomogeneous problem

We will now consider a general (nonhomogeneous) Cauchy problem

$$\frac{d}{dt}u = Au + f(t), \qquad u(0) = u_0 \tag{4.2}$$

where $u_0 \in X$ and $f(t) : I \to X$ are given, $I = [0, T]$. We will assume that $(A, \mathcal{D}(A))$ is an unbounded operator, which generates a c_0-semigroup $S(t)$, and that $f(t) \in L^1(I; X)$ at least.

Definition. Function $u(t)$ is called *classical solution* to (4.2), if $u(t) \in C^1(I; X) \cap C(I; \mathcal{D}(A))$, and (4.2) holds for every $t \in I$.
Function $u(t)$ is called *strong solution* to (4.2), provided that $u(t) \in W^{1,1}(I; X) \cap L^1(I; \mathcal{D}(A))$ and (4.2) holds for a.e. $t \in I$.

Remarks. Classical implies strong, and strong (in view of Lemma 1.3) is equivalent to

$$u(t) = u_0 + \int_0^t Au(s) + f(s)\, ds$$

for a.e. $t \in I$, where the right-hand side is the absolutely continuous representative of $u(t)$.
Note that $u(t) \in C(I; \mathcal{D}(A))$ (where $\mathcal{D}(A)$ is considered with the graph norm) is equivalent to: $u(t) \in C(I; X)$, $u(t) \in \mathcal{D}(A)$ for every $t \in I$ and moreover, $t \mapsto Au(t)$ is continuous $I \to X$.
Analogous assertion is concerning integrability is:

Lemma 4.4. [Hille's theorem.] Let $(A, \mathcal{D}(A))$ be closed operator. Then $u(t) \in L^1(I; \mathcal{D}(A))$ if and only if $u(t) \in L^1(I; X)$, $u(t) \in \mathcal{D}(A)$ for a.e. $t \in I$ and $Au(t) \in L^1(I; X)$.
In this situation one also has $\int_I u(t)\, dt \in \mathcal{D}(A)$ and

$$A\left(\int_I u(t)\, dt\right) = \int_I Au(t)\, dt$$

Definition. Function $u(t) : I \to X$ is called *mild solution* to (4.2), provided that

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(s)\, ds, \qquad t \in I$$

Remarks. Recalling that $S(t)$ is a semigroup generated by A, hence „$S(t) = e^{tA}$“, the definition is motivated by the „variation of constants“ formula. The integral is finite in fact, $u(t) \in C(I; X)$, see lemma below.
Note that trivially, one has existence and uniqueness of mild solution.

Lemma 4.5. [Abstract convolution.] Let $S(t)$ be a c_0-semigroup on X, let $v(t) = \int_0^t S(t-s)f(s)\, ds$, $t \in I$. Then

1. $f(t) \in L^1(I; X) \implies v(t) \in C(I; X)$
2. $f(t) \in C^{0,1}(I; X) \implies v(t) \in C^{0,1}(I; X)$
3. $f(t) \in C^1(I; X) \implies v(t) \in C^1(I; X)$, and $v'(t) = S(t)f(0) + \int_0^t S(t-s)f'(s)\, ds$, $t \in X$.

Lemma 4.6. Function $u(t)$ is mild solution to (4.2), if and only if $u(t) \in C(I;X)$, for every $t \in I$ one has $\int_0^t u(s)\,ds \in \mathcal{D}(A)$ and

$$u(t) = u_0 + A\left(\int_0^t u(s)\,ds\right) + \int_0^t f(s)\,ds \qquad \forall t \in I$$

Corollary. Strong solution is also mild. Classical and strong solutions, whenever they exist, are unique.

Remark. Mild solution, in general, is not strong: assume that $x \in X$ is such that $S(t)x \notin \mathcal{D}(A)$ for any $t \geq 0$. Then $u(t) = tS(t)x$ is mild solution of (4.2) with $f(t) = S(t)x$, $u_0 = 0$, but $u(t) \notin \mathcal{D}(A)$ for any $t > 0$.
However, every mild solution is a uniform limit of classical solutions. For $h > 0$, we set $u_h(t) = 1/h \int_t^{t+h} u(s)\,ds$ and verify (with help of Lemma 4.6.) that $u_h(t)$ is classical solution of (4.2) with initial condition $u_h(0) \in \mathcal{D}(A)$, and right-hand side $f_h(t)$. Clearly $u_h(t) \rightrightarrows u(t)$ thanks to properties of convolution.

Theorem 4.4. [Regularity of mild solution.]

1. Let $u_0 \in \mathcal{D}(A)$ and $f(t) \in C^1(I;X)$. Then mild solution is classical.
2. Let $u_0 \in \mathcal{D}(A)$ and $f(t) \in C^{0,1}(I;X)$; let moreover X be reflexive. Then mild solution is strong.