18·III – Linear problem – optimal time

(18.4) $x' = Ax + Bu$ $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ given

$x(0) = x_0$ $\mathcal{U} = \{u : [0,T] \to [-1,1]^m$, measurable$\}$

Problem: find $u(\cdot) \in \mathcal{U}$ s.t. $x_0 \xrightarrow[u(\cdot)]{t} 0$; t minimal.

Proposition 1 [Banach–Alaoglu] $\mathcal{U}$ is $*$-weakly seq. compact in i.e. $L^\infty(0,T;\mathbb{R}^m)$, i.e. given a seq. $\{u_n(\cdot)\} \subset \mathcal{U}$ there ex. a subseq. $\{\tilde{u}_n(\cdot)\}$ and $u(\cdot) \in \mathcal{U}$ s.t. $u_n \xrightarrow{*} u$, i.e.

$$\int_0^T \Pi(t)\cdot\tilde{u}_n(t)\,dt \to \int_0^T \Pi(t)\cdot u(t)\,dt\,; \quad \forall \Pi(t) \in L^1(0,T;\mathbb{R}^m) \text{ fixed.}$$

Remark: (v.c.) $x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} B u(s)\,ds$

implies) $u_n \xrightarrow{*} u \Rightarrow x_n(t) \to x(t)$ pointwise

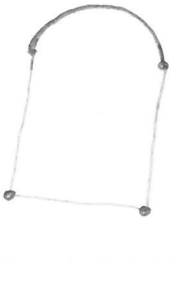
$\left(\text{take } \Pi = \chi_{(0,t)} e^{(t-s)A} B \in L^1\right)$

Proposition 2 [Krein–Milman] X.. loc. convex top. space; $K \subset X$ non-empty, convex, compact. $\Rightarrow K = \overline{\mathrm{co}}(\mathrm{ext}\,K)$, where ext K are extremal points

In particular: K has at least one extremal point.

Def. $a \in K$ is extremal :: $\nexists\, x, y \in K$, $x \neq y$ s.t. $a = \frac{x+y}{2}$

Equivalently: $a \in K$ extremal iff $K \setminus \{a\}$ is convex.



clearly $\mathrm{ext}\,K \subset \partial K$

Theorem 18.6 Given (18.4), the set $\mathcal{R}(t)$ is convex, symmetric, closed. Also: $t_1 < t_2 \Rightarrow \mathcal{R}(t_1) \subset \mathcal{R}(t_2)$.

Pf. recall: $x_0 \in \mathcal{R}(t) \iff x_0 = -\int_0^t e^{-sA} B u(s)\, ds$ (v.c.)

for some $u(\cdot) \in \mathcal{U}$

$x_0 \xrightarrow[u(\cdot)]{t} 0$ notation

? convex: $x_i \in \mathcal{R}(t)$, $x_i \xrightarrow[u_i(\cdot)]{t} 0$, $i = 1,2$,

$x_0 = \lambda x_1 + (1-\lambda) x_2$; $\lambda \in (0,1)$

(v.c.) $\Rightarrow$ $x_0 \xrightarrow[u_0(\cdot)]{t} 0$; $u_0(t) = \lambda u_1(t) + (1-\lambda) u_2(t)$

symmetric. H.W.

? $\mathcal{R}(t_1) \subset \mathcal{R}(t_2)$; $t_1 < t_2$: if $x_0 \xrightarrow[u_1(\cdot)]{t_1} 0$.

set: $u_2(s) = \begin{cases} u_1(s); & s \in [0,t_1]; \\ 0 & s \in (t_1, t_2]. \end{cases}$

? closed: need Banach-Alaoglu here: ; $t > 0$ fixed

$x_n \in \mathcal{R}(t)$, $x_n \to x_0 \overset{?}{\in} \mathcal{R}(t)$

$\exists\, u_n(\cdot) \in \mathcal{U}$; $x_n \xrightarrow[u_n(\cdot)]{t} 0$; i.e. $x_n = -\int_0^t e^{-sA} B u_n(s)\, ds$

Ban.-Al. $\Rightarrow \exists$ subseq. $\tilde{u}_n \xrightarrow{*} u_0 \in \mathcal{U}$

but: (LHS): $\int_0^t e^{-sA} B \tilde{u}_n(s) \to \int_0^t e^{-sA} B u_0(s)\, ds$

take $\eta(s) = e^{-sA} B \cdot \chi_{(0,t)}(s) \in L^1$

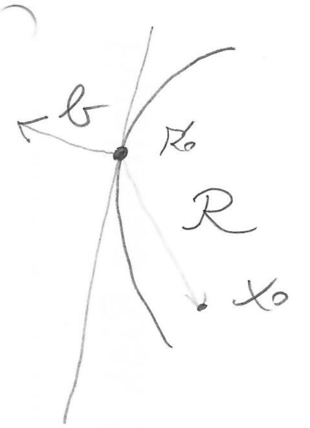
$\Rightarrow x_0 \xrightarrow[u_0(\cdot)]{t} 0$; q.e.d.

Corollary Set of "global controllability" $\mathcal{R}_\infty = \bigcup_{t>0} \mathcal{R}(t)$ is convex, symmetric.

Theorem 18.7 Let rank $\mathcal{K}(A,B)=n$, let $\mathrm{Re}\lambda \le 0$ for $\forall \lambda \in \sigma(A)$. Then $\bigcup_{t>0} \mathcal{R}(t) = \mathbb{R}^n$.

Rem.: if even $\mathrm{Re}\lambda < 0 \ \forall \lambda \in \sigma(A)$: easy by Thm. 18.3 and decay if $u \equiv 0$.

Pf. ?? $\mathcal{R} = \bigcup_{t>0} \mathcal{R}(t) \neq \mathbb{R}^n$ .. $\exists\ z_0 \in \partial\mathcal{R}$



$\exists$ tangent hyperplane

i.e. $\exists\ b \neq 0$ s.t.

$b \cdot (x_0 - z_0) \le 0 \quad \forall x_0 \in \mathcal{R}$

$$\boxed{\Rightarrow b \cdot x_0 \le c \quad \forall x_0 \in \mathcal{R} \quad (i)} := b \cdot z_0$$

^ constant

— bring to the contradiction...

Notation: $x_0, b, \dots$ column vectors $\in \mathbb{R}^{n \times 1}$

$b \cdot x_0 \doteq b^T x_0$; b^T .. row vector —

no symbol: matrix multiplication.

by v.c. $b \cdot x_0 = -\int_0^t \underbrace{b^T e^{-sA} B}_{} u(s)\, ds$ if $x_0 \xrightarrow[u(\cdot)]{t} 0$

!!

$v(s) \in \mathbb{R}^{1 \times m}$ row

will show that $\boxed{\int_0^\infty |v(s)|\, ds = +\infty \quad (ii)}$

Set: $\tilde{u}(s) := \begin{cases} 0 ; & v(s)=0 \\ -\dfrac{v^T(s)}{|v(s)|} ; & v(s) \neq 0 \end{cases}$

admissible:

by (v.c.) : $\tilde{x}_0 \xrightarrow[\tilde{u}(\cdot)]{t} 0$; where $\tilde{x}_0 = -\int_0^t e^{-sA} B \tilde{u}(s)\, ds$

$$\Rightarrow b \cdot \tilde{x}_0 = b^T \tilde{x}_0 = -\int_0^t \underbrace{b^T e^{-sA} B}_{v(s)} \left(-\frac{v^T(s)}{|v(s)|} \right) ds$$

$$= \int_0^t |v(s)|\, ds > c$$ for t large if (ii) holds contradicting (i).

---

to show (ii) in two steps: iia) $v \not\equiv 0$

iib) $v(s)$ .. lin. comb. of $s^j e^{\mu s}$; $\mathrm{Re}\,\mu \geq 0$

iia) ?! $v(s) = b^T e^{-sA} B \equiv 0 \quad | \left(\frac{d}{ds}\right)^k$

$b^T (-A)^k e^{-sA} B \equiv 0$; $s=0$

$b^T A^k B \equiv 0$; $\forall k = 0, \dots n-1$

$\Rightarrow b \perp$ columns of $\mathcal{K}(A,B)$

$\overset{\shortparallel}{0}$ .. $\lightning$ since $\mathcal{K}(A,B)$ has rank n

iib) clear as e^{-sA} .. elements: $s^j e^{-\lambda s}$

$\lambda \in \sigma(A)$; ie. $\mathrm{Re}\,\lambda \leq 0$

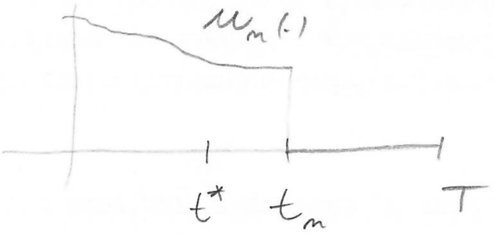
Theorem 18.8 [$\exists$ of time optimal control.]. Let $x_0 \in \bigcup_{t>0} R(t)$. Then $\exists\, t^*>0$, $u^*(\cdot)\in\mathcal{U}$ s.t. $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$, with $t^*>0$ the least possible.

Pf. Set $t^* := \inf\{t>0;\ x_0\in R(t)\}$

choose $t_m \searrow t^*$, $u_m(\cdot)\in\mathcal{U}$ s.t. $x_0 \xrightarrow[u_m(\cdot)]{t_m} 0$

wLoG: $t^* < t_m < T$ .. fixed ; $u_m(s) = 0$; $s > t_m$.

Ban.-Al. (wLoG) $\Rightarrow$ $u_m(\cdot) \xrightarrow{*} u^*(\cdot)$ in $L^\infty(0,T;L^\infty)$



will show: $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$ .. done. (inf = min)

by (v.c.) $$x_0 = -\int_0^{t_m} e^{-sA} B u_m(s)\,ds \underset{\uparrow}{=} -\int_0^{T} e^{-sA} B u_m(s)\,ds$$

since $u_m \equiv 0$ for $s > t_m$

$$= -\int_0^{t^*}(\quad) - \int_{t^*}^{t_m}(\quad) = -I_1 - I_2$$

but: $-I_1 \to -\int_0^{t^*} e^{-sA} B u^*(s)\,ds$; $|I_2| \le c\cdot|t_m - t^*| \downarrow 0$

$*$-weak convergence

hence: $x_0 = -\int_0^{t^*} e^{-sA} B u^*(s)\,ds$; q.e.d.

Def. We say that $u(\cdot) \in \mathcal{U}$ is "bang-bang" control,

if $u_i(t) = \pm 1$ for a.e. t, $\forall i = 1, \dots m$.

In other words; $u(t) \in \text{ext}\,[-1,1]^m$ for a.e. t.

Theorem 18.9 [Bang-bang principle.] Let $x_0 \in \mathcal{R}(t)$. Then there is a bang-bang $\tilde{u}(\cdot) \in \mathcal{U}$ s.t. $x_0 \xrightarrow[\tilde{u}(\cdot)]{t} 0$.

Pf. Set $\mathcal{K} = \{u(\cdot) \in \mathcal{U};\ x_0 \xrightarrow[u(\cdot)]{t} 0\}$.

$\mathcal{K} \neq \phi$, convex; compact (Ban. Al.) in $\underbrace{L^\infty(0,t;\mathbb{R}^m)}_{\overset{\shortparallel}{X}}$.

Krein-Milman $\Rightarrow \exists \tilde{u}(\cdot) \in \mathcal{K}$ extremal.

---

claim: $\tilde{u}(\cdot)$ is bang-bang

?? $\exists i \in \{1, \dots, m\}$, $\exists F \subset (0,t)$, $|F| > 0$

s.t. $|\tilde{u}_i(s)| < 1$ for $s \in F$.

in fact we can assume $|u_i(s)| \leq 1 - \varepsilon$ $\forall s \in F$

by Lebesgue theory where $\varepsilon > 0$.

---

choose $\phi(s)$ s.t. $|\phi(s)| \leq \varepsilon$ on F, $\phi(s) \equiv 0$ elsewhere

set $v(s) = (0 — 0, \phi(s), 0 — 0)$

$\uparrow$ i-th component

we can assume $\int_F e^{-sA} b_i \phi(s)\,ds = 0$; yet $\phi(s) \not\equiv 0$.

essentially because $L^2(F, \mathbb{R}^m)$ has ∞-dimension

---

set $u_1 = \tilde{u} + v$ $\Rightarrow u_{1,2} \in \mathcal{K}$, but

$u_2 = \tilde{u} - v$

$\tilde{u} = \frac{1}{2}(u_1 + u_2)$; $u_1 \neq u_2$

$\Rightarrow \tilde{u}$ not extremal $\lightning$

Theorem 18.10 [Pontryagin maximum principle.]

Let $t^*>0$, $u^*(\cdot)\in\mathcal{U}$ be s.t. $x_0 \xrightarrow[u^*(\cdot)]{t^*} 0$, with $t^*>0$ minimal.

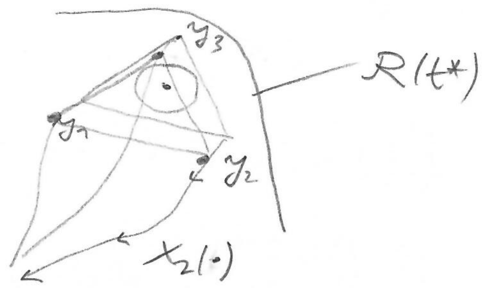
Then $\exists\, h\in\mathbb{R}^m\setminus\{0\}$ s.t.

$$(18.5)\quad h\cdot e^{-tA}Bu^*(t) = \max_{y\in[-1,1]^m}\{h\cdot e^{-tA}By]$$

for a.e. $t\in[0,t^*]$.

Pf. key observation: $x_0\in\partial R(t^*)$.

?? $x_0\in\mathcal{U}(x_0,\varepsilon)\subset R(t^*)$.



wLOG: $\mathcal{U}(x_0,\varepsilon)\subseteq \mathrm{co}\{y_1,\dots,y_{m+1}\}$

s.t. $y_j \xrightarrow[u^{(j)}(\cdot)]{t^*} 0$; $u^j(\cdot)\in\mathcal{U}$.

$x_j(\cdot)$.. corresp. solutions:

$\tilde{y}_j = x_j(\delta)$; $\delta>0$ small s.t.

$$x_0\in\mathrm{co}\{\tilde{y}_1,\dots,\tilde{y}_{m+1}\}.$$

but: $\tilde{y}_j\in R(t^*-\delta)$; hence is x_0 by convexity ... ↯.

$x_0\in\partial R(t^*)$; $R(t^*)$ convex $\Rightarrow\exists$ tangent hyperplane

i.e. $\exists\, h\in\mathbb{R}^m\setminus\{0\}$ s.t. $h\cdot(x_1-x_0)\le 0\ \forall x_1\in R(t^*)$

but: $x_0 = -\int_0^{t^*} e^{-sA}Bu^*(s)\,ds$

$x_1 = -\int_0^{t^*} e^{-sA}Bu(s)\,ds$; $u(\cdot)\in\mathcal{U}$ arbitrary

$$\Rightarrow \int_0^{t^*} h^T e^{-sA}B[u^*(s)-u(s)]\,ds \ge 0 \ ;\ \forall u(\cdot)\in\mathcal{U}$$

claim: (18.5) follows.. ?? not: $\exists\, E \subset (0,t^*);\ |E|>0$

s.t. for $\forall s \in E\ \exists\, \eta_s \in [-1,1]$

s.t. $h^T e^{-sA} B u^*(s) < h^T e^{-sA} B \eta_s$

set $\hat{u}(s) := \begin{cases} u^*(s) & , s \in [0,t^*] \setminus E \\ \eta_s & ; s \in E \end{cases}$

$$\Rightarrow \int_0^t h^T e^{-sA} B[u^*(s) - \hat{u}(s)]\, ds = \int\limits_{\underset{\text{pos. measure}}{\underset{\uparrow}{E}}} \underbrace{h^T e^{-sA} B(u^*(s) - \eta_s)}_{<0} ds$$

< 0 .. ↯

Rem.: problem of measurable selection..

Problem of Bolza. $x' = f(x,u)$, $u(\cdot) \in \mathcal{U} = \{u : [0,T] \to U \text{ meas.}\}$

(T>0 fixed) $x(0) = x_0$ (18.6)

... max $P[u(\cdot)] = g(x(T)) + \int_0^T r(x(s),u(s))\,ds$

Theorem 18.11 [Pontryagin principle – problem of Bolza.]

Assume: $\bar{u}^*(\cdot) \in \mathcal{U}$ .. local maximum for (18.6)

$x^*(\cdot) \in AC([0,T])$ .. corr. solution ("response")

$f = f(x,u)$, $r = r(x,u)$, $g = g(x)$ .. C^1 close to graphs of $u^*(\cdot), x^*(\cdot)$.

Define: Hamiltonian $H(x,\lambda,u) = \lambda^T f(x,u) + r(x,u)$ (18.8)

Then: $H(x^*(t),\lambda^*(t),u^*(t)) = \max\limits_{\eta \in U} H(x^*(t),\lambda^*(t),\eta)$ for a.e. $t \in [0,T]$

where $\lambda^*(t)$ solves the adjoint problem

$$(\lambda^T)' = -\nabla_x H(x^*(t),\lambda,u^*(t)) \qquad (18.9)$$

$$\lambda^T(T) = \nabla_x g(x^*(T)) \qquad (18.10)$$

Rem. $$\lambda_i' = -\sum_{j=1}^{m} \lambda_j \frac{\partial f_j}{\partial x_i}(x^*(t),u^*(t)) - \frac{\partial r}{\partial x_i}(x^*(t),u^*(t))$$

$$H(x,\lambda,u) = \sum_{j=1}^{m} \lambda_j f_j(x,u) + r(x,u)$$

given $x^*(t), u^*(t)$ .. $\exists!\ \overset{*}{\lambda}(t)$ sol. to (18.9), (18.10).
(linear problem in $\lambda(t)$).

Lemma 18.3 Let $R' = A(t)R$, $(\lambda^T)' = -\lambda^T A(t)$, where $A(t) \in L^1(0,T)$. Then $t \mapsto \lambda(t)\cdot R(t)$ is constant.

Pf. $(\lambda \cdot R)' = (\lambda^T R)' = (\lambda^T)'R + \lambda^T R'$

$= -\lambda^T A(t) R + \lambda^T A(t) R = 0$

Pf. (of Thm 18.11) STEP 1: $r \equiv 0$; omit $*$.

... $u(\cdot) \in \mathcal{U}$ lok. max; -- $P[u(\cdot)] = g(x(T))$; $H = \lambda^T f$

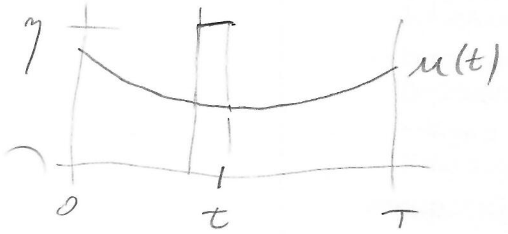
$x(\cdot)$ .. solution

$\lambda(\cdot)$ .. adjoint; i.e. $(\lambda^T)' = -\lambda^T D_x f(x(t), u(t))$

$$\lambda^T(T) = D_x g(x(T))$$

---

fix $\eta \in \mathcal{U}$, $\tau \in (0,T)$, set $u_\varepsilon(t) = \begin{cases} \eta, & t \in (\tau-\varepsilon, \tau) \\ u(t), & \text{elsewhere} \end{cases}$



$x_\varepsilon(t)$ .. corr. sol. "needle variation"

---

denote: $D_{0+} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0+}$: $P[u_\varepsilon(\cdot)] = g(x_\varepsilon(T)) \leq g(x(T))$

$$\boxed{\nabla g(x(T)) \cdot D_{0+} x_\varepsilon(T) \leq 0 \quad (i)}$$

auxiliary problem: $D_{0+} x_\varepsilon(t) = ?$, $t \in (0,T)$.

- $t < \tau$: $x_\varepsilon(t) = x(t)$, ε small $\Rightarrow$ $Dx_\varepsilon(t) = 0$
- $t = \tau$: later
- $t > \tau$: $u_\varepsilon(t) = u(t)$; hence

  $x_\varepsilon(t) = y_\varepsilon(t)$, where $y_\varepsilon(t)$ solves

$$y' = f(y, u(t))$$
$$y(\tau) = x_\varepsilon(\tau)$$

$\Rightarrow$ $Dx_\varepsilon(t) =: r(t)$ from eq. in vars.

$$\boxed{\begin{aligned} r' &= D_x f(x,u) r \\ r(\tau) &= v \quad (ii) \end{aligned}}$$

$$v = Dx_\varepsilon(\tau).$$

? $Dx_\varepsilon(\tau) = ?$

$$x(\tau) = x_0 + \int_0^\tau f(x(s), u(s))\,ds$$

$$\qquad\qquad \text{''} \qquad \text{''} \quad \text{for } s < \tau - \varepsilon$$

$$x_\varepsilon(\tau) = x_0 + \int_0^\tau f(x_\varepsilon(s), u_\varepsilon(s))\,ds$$

$$x_\varepsilon(\tau) - x(\tau) = \int_{\tau-\varepsilon}^{\tau} f(x_\varepsilon(s), \eta) - f(x(s), u(s))\,ds$$

$$\frac{1}{\varepsilon}\left[x_\varepsilon(\tau) - x(\tau)\right] = \frac{1}{\varepsilon}\int_{\tau-\varepsilon}^{\tau}\left[f(x_\varepsilon(s), \eta) - f(x(s), u(s))\right]ds$$

$\varepsilon \to 0+$: $\quad \to f(x(\tau), \eta) - f(x(\tau), u(\tau))$.

for: wLOG: τ - Leb. point of $s \mapsto f(x(s), u(s))$

and $x_\varepsilon(s) \overset{loc}{\rightrightarrows} x(s)$, $\varepsilon \to 0+$.

$$\Rightarrow \boxed{(iii)\ v = Dx_\varepsilon(\tau) = f(x(\tau), \eta) - f(x(\tau), u(\tau)) \quad \text{a.e. } \tau.}$$

$$0 \geq (D_x g)(x(T)) \cdot Dx_\varepsilon(T) = p(T) \cdot R(T)$$ by L. 18.3

$$= \lambda(\tau) \cdot R(\tau)$$

$\lambda(t), R(t)$ adjoint systems!

$$= \lambda^T(\tau)\, v$$

$$= \lambda^T(\tau)\left[f(x(\tau), \eta) - f(x(\tau), u(\tau))\right]$$

$$\lambda^T(\tau) f(x(\tau), \eta) \geq \lambda^T(\tau) f(x(\tau), u(\tau))$$

max. of Hamiltonian $(r \equiv 0)$.

STEP2: $r \neq 0$ in general.. reduce to STEP1.

by adding a variable $\boxed{x_{n+1}}$.

$$\underset{\sim}{x}' = \underset{\sim}{f}(\underset{\sim}{x}, u) \; ; \; x(0) = x_0$$

$$x_{n+1}' = r(x, u) \; ; \; x_{n+1}(0) = 0 \; ; \qquad X = \begin{pmatrix} \underset{\sim}{x} \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$$

$$F = \begin{pmatrix} f(\underset{\sim}{x}, u) \\ r(x, u) \end{pmatrix}$$

$$P[u(\cdot)] = G(X(T)) \; ; \quad G(X) = g(x) + x_{n+1}$$

$$= g(x(T)) + \underbrace{\int_0^T r(x(s), u(s))\, ds}_{x_{n+1}(T)} .$$

easy to see: $\lambda_i' = -\sum_{j=1}^{n} \lambda_j \frac{\partial f_i}{\partial x_j}(\dots) - \lambda_{j+1} \frac{\partial r}{\partial x_i}(\dots)$

$$\lambda_{n+1}' = 0$$

$$p(T) = D_x G(X(T)) = \begin{pmatrix} D_x g(x(T)) \\ 1 \end{pmatrix}$$

$$\Rightarrow \lambda_{n+1} \equiv 1 , \dots$$