18. Optimal control

$$x' = f(x, \underset{\text{control}}{u}) \; ; \; x(0) = x_0 \qquad f : \Omega \times U \to \mathbb{R}^n$$

(18.1)

$$\Omega \subset \mathbb{R}^n$$
$$U \subset \mathbb{R}^m \quad (m < n)$$

admissible controls $\mathcal{U} = \{ u(t) : [0,T] \to U, \text{ measurable} \}$

Problems. 1) $\exists\, u(\cdot) \in \mathcal{U}$ s.t. $x(t) = 0$ ("controllability")

2) — " — with minimal $\underline{t}$: ("time optimal control").

3) more generally: find $u(\cdot) \in \mathcal{U}$ s.t.

$$P[u(\cdot)] = g(x(T)) + \int_0^T r(x(t), u(t))\, dt$$

(variational problem). is maximal.

implicit: $u(\cdot) \in \mathcal{U}$, $x_0 \in \mathbb{R}^n$ given $\Rightarrow \exists!\, x(t)$ Car. sol. to (18.1)

(Examples).

18.I – Linear problem

$$x' = Ax + Bu \; ; \; x(0) = x_0$$

(18.2) $$u(\cdot) \in \mathcal{U} = L^\infty(0,T; \mathbb{R}^m)$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ given (constant) matr.

$x_0 \in \mathbb{R}^n$

Note. $u(\cdot) \in \mathcal{U}$ given $\Rightarrow \exists!\, x(t)$ Car. sol. given by (V.C.)

$$x(t) = e^{tA} x_0 + \int_0^t e^{(t-s)A} B u(s)\, ds. \quad \text{(V.C.)}$$

Def. $u(\cdot) \in \mathcal{U}$ takes x_0 to 0 in t, if $x(t) = 0$ for the corr. solution. Notation: $x_0 \xrightarrow[u(\cdot)]{t} 0$.

$$\mathcal{R}(t) = \{ x_0 \in \mathbb{R}^n ; \exists\, u(\cdot) \in \mathcal{U} \text{ s.t. } x_0 \xrightarrow[u(\cdot)]{t} 0 \}$$

… domain of controllability at time $\underline{t}$.

Observe: by v.c. formula $x_0 \xrightarrow[u(\cdot)]{t} 0$ iff $x_0 = -\int_0^{t} e^{-sA} B u(s)\,ds$.

Def. Kalman matrix of (18.2): $n \times mn$ matrix

$$\mathcal{K}(A,B) = \left(B, AB, A^2B, \dots, A^{n-1}B\right).$$

Theorem 18.1 Given (18.2), one has $\mathcal{R}(t) = \mathrm{Lin}\{g_1, \dots, g_{mn}\}$, $\{g_j\}$ ... columns of $\mathcal{K}(A,B)$, for any $t>0$ fixed.

Lemma 18.1 For any $\ell \geq 0$ integer: $A^\ell \in \underbrace{\mathrm{Lin}\{I, A, A^2, \dots, A^{n-1}\}}_{Z}$.

Pf. Caley-Hamilton thm $\Rightarrow$ $a(A) = 0$, $a(\lambda)$ -- char. poli of A

$$a(\lambda) = \det(\lambda I - A) = \lambda^n - \sum_{j=0}^{n-1} a_j \lambda^j$$

hence $\quad A^n = \sum_{j=0}^{n-1} a_j B^j \quad \Rightarrow \quad A^n \in Z$

induction: $\quad A^{n+1} = A\left(A^n\right) = \sum_{j=0}^{n-1} a_j B^{j+1}$ $\quad \leq n$, $\in Z$ by previous

$A^{n+\ell} \in Z \quad \forall \ell \geq 0$

$\ell = 0$ ... done

$$A^{n+\ell+1} = A\left(A^{n+\ell}\right) = \sum_{j=0}^{n-1} a_j B^{j+(\ell+1)} \quad \leq n+\ell$$

$\in Z$ by previous.

proof of Thm. 18.1: fix $t>0$: note $\mathcal{R}(t) \subset \mathbb{R}^n$ is a vector space!!

since $\mathcal{U}$ is vector space, $x_0 \xrightarrow[u(\cdot)]{t} 0$ is linear by (v.c.).

$$\left.\begin{array}{l} x_1 \xrightarrow[u_1(\cdot)]{t} 0 \\ x_2 \xrightarrow[u_2(\cdot)]{t} 0 \end{array}\right\} \Rightarrow \quad x_1 + x_2 \xrightarrow[u_1(\cdot)+u_2(\cdot)]{} 0.$$

enough to show: $\mathcal{R}(t)^\perp = \mathrm{Lin}\{g_1, \dots, g_{mn}\}^\perp$.

"$\supseteq$": let $p \in \mathbb{R}^n$, $p \perp g_j \ \forall j$ -- $p \underset{\uparrow}{\cdot} A^k b_i \ \forall k = 0, \dots, n-1$

scalar product $\quad b_i$ col's of B

by L. 18.1 $\quad p \cdot A^\ell b_i \ \forall \ell \geq 0$

let $x_0 \in \mathcal{R}(t) \overset{?}{\Longrightarrow} p \cdot x_0 = 0$.

by v.c. $\quad p \cdot x_0 = -p \cdot \int_0^t e^{-sA} B u(s)\, ds$; where $u(\cdot) \in \mathcal{U}$ $\quad$ to $\xrightarrow[u(\cdot)]{t} 0$.

$$= -\int_0^t p \cdot (\underbrace{e^{-sA}}_{\sum_{\ell=0}^{\infty} \frac{(-s)^\ell}{\ell!} A^\ell} B u(s))\, ds = -\int_0^\infty \sum_{\ell=0}^{\infty} \sum_i \frac{(-s)^\ell}{\ell!} (p \cdot A^\ell b_i) u_i(s)$$

$$Bu = \sum_{i=1}^m b_i u_i$$

u_i ← components of u

b_i ← columns of B

$= 0$ by the above

"$\subseteq$" $p \in \mathcal{R}(t)^\perp$ -- take $u(\cdot) \in \mathcal{U}$ s.t.

$$u_i(s) = \begin{cases} \phi(s); & i = j \\ 0 & i \neq j \end{cases}$$

where $\phi(s) \in L^\infty(0, t; \mathbb{R})$, j -- are fixed, arbitrary

by v.c.

$$x_0 = -\int_0^t e^{-sA} B u(s)\, ds = -\int_0^t e^{-sA} b_j \phi(s)\, ds \in \mathcal{R}(t)$$

$$0 = p \cdot x_0 = -\int_0^t p \cdot e^{-sA} b_j \phi(s)\, ds; \quad \phi(s) \text{ arbitrary}$$

$$\Rightarrow p \cdot e^{-sA} b_j \equiv 0 \text{ in } [0, t]$$

$\left(\frac{d}{ds}\right)^k$ — $p \cdot (-A)^k e^{-sA} b_j \equiv 0$

set $s=0$ $\quad p \cdot A^k b_j \equiv 0 \; ; \; \forall k, j$

$\Rightarrow p \perp \mathcal{K}(A,B)$, q.e.d.

Corollary. (18.2) is globally controllable (i.e. $R(t) = \mathbb{R}^n$ for all $t>0$) $\Longleftrightarrow$ rank $\mathcal{K}(A,B) = n$.

Def. The problem
$$x' = Ax, \quad x(0) = x_0 \qquad (18.3)$$
$$y = Bx$$
is observable via $y = Bx$, if there holds: given $x_1(t), x_2(t)$ two solutions s.t. $Bx_1 \equiv Bx_2$ on some (non-trivial) interval $[0,\tau]$, then $x_1(0) = x_2(0)$. ($\Leftrightarrow x_1 \equiv x_2$ for all times).

Theorem 18.2 The following are equivalent:

1. problem (18.3) is observable via $y = Bx$
2. problem $x' = A^T x + B^T u$ is globally controllable
3. rank $\mathcal{K}(A^T, B^T) = n$.

Pf. 2. $\Leftrightarrow$ 3. ... Cor. of Thm 18.1

$\neg 1. \Rightarrow \neg 3.$ assume $x_1 \not\equiv x_2$, i.e. $x_1(0) \neq x_2(0)$, yet $Bx_1 \equiv Bx_2$.

set $x(t) = x_1(t) - x_2(t)$; i.e. $x(t) = e^{tA} x_0$; $x_0 = x_1(0) - x_2(0) \neq 0$

$B e^{tA} x_0 \equiv 0 \; ; \left(\frac{d}{dt}\right)^k \qquad Bx(t) \equiv 0$

$B A^k e^{tA} x_0 \equiv 0 \; ; \quad t=0$

$B A^k x_0 = 0 \quad \forall k \geq 0 \ldots$

transpose : $x_0^T (A^T)^k B^T = 0$; $\forall k = 0, \dots m-1$

$x_0 \neq 0$

$\Rightarrow \mathcal{K}(A^T, B^T) = (B^T, A^T B^T, \dots (A^T)^{n-1} B^T)$

has rank $< n$.

$\neg 3. \Rightarrow \neg 1.$ -- similar: rank $\mathcal{K}(A^T, B^T) < n$

$\Rightarrow \exists x_0 \in \mathbb{R}^n,\ x_0 \neq 0$ s.t. $x_0^T \mathcal{K}(A^T, B^T) = 0$

i.e. $x_0^T (A^T)^k B^T = 0$; $\forall k < n$

by Lemma 18.1. $\forall k \geq 0$.

transpose $B A^k x_0 = 0$; $\forall k \geq 0$

hence $B \underbrace{e^{tA} x_0}_{} \equiv 0$; $\forall t \geq 0$

non-zero solution with a zero observation

-- $\neg 1$.

Thm. 18.3 [Local controllability.]

Assume: $f(x, u)$ is C^1 close to $(0,0) \in \mathbb{R}^{n+m}$, $f(0,0) = 0$

$\mathcal{U}$ (values of adm. contr.) contains neigh. of $0 \in \mathbb{R}^m$

linearized problem (i.e. (18.2) with $A = D_x f(0,0)$, $B = D_u f(0,0)$)

is globally controllable

$\Rightarrow$ (18.1) $x' = f(x, u)$, $x(0) = x_0$ is locally controllable;

i.e. $R(t)$ contains a neigh. of $0 \in \mathbb{R}^n$ for $\forall t > 0$.

Pf. fix $t>0$; $y^{(1)}, \dots, y^{(m)} \in \mathbb{R}^m$, basis.

by assumption: $u^{(j)} : (0,t) \to \mathbb{R}^m$ bdd., measurable

s.t. $y^{(j)} \xrightarrow[u^{(j)}(\cdot)]{t} 0$ for linearized problem,

i.e. $\begin{aligned} z' &= Az + Bu^{(j)} \\ z(0) &= y^{(j)} \end{aligned} \Rightarrow z(t) = 0.$

define map: $\Psi : \mathbb{R}^m \to \mathbb{R}^m$

$\lambda \mapsto x(0)$

- $x(t)$ solution to $x' = f(x, u_\lambda)$

$x(t) = 0$

where $u_\lambda = \sum_{j=1}^m \lambda_j u^{(j)}$; $\lambda = (\lambda_1, \dots, \lambda_m)$.

in other words: $\Psi(\lambda) \in \mathbb{R}^m$ is the init. condition

s.t. $\Psi(\lambda) \xrightarrow[u_\lambda(\cdot)]{t} 0$ for the nonlinear problem.

in other words $\operatorname{rng} \Psi \subset \mathcal{R}(t)$.

by IFT we will show: $\operatorname{rng} \Psi$ contains a neigh. of zero.

- we are done.

- $\Psi(0) = 0$, since $x \equiv 0$ is (the only) solution to $x' = f(x, 0)$, $x(t) = 0$
- $\Psi \in C^1$, defined on some $\mathcal{U}(0, \delta)$

Carath. theory - dep. on parameter & solution operator ...

- $D\Psi(0)$ ... regular ?!

$\frac{\partial \Psi}{\partial \lambda_i}(0) = R(0)$, where $R(t)$ solves "eq. in vars"

$$R' = D_x f(x, u_\lambda) R + D_u f(x, u_\lambda) \frac{\partial u_\lambda}{\partial \lambda_i}$$

$$R(t) = 0$$

; for $\lambda = 0$: $x \equiv 0$

$u_\lambda \equiv 0$

$\frac{\partial u_\lambda}{\partial \lambda_i} = u^{(i)}$

$$R' = AR + Bu^{(i)}$$

$$R(t) = 0 \quad \Rightarrow \quad R(0) = y^{(i)}$$

$$D\Psi(0) = \left(y^{(1)} \middle| \dots \middle| y^{(m)} \right) \dots \text{ regular !!}$$

18.II.- Stabilizability

$x' = f(x, u)$ ... ? automatic control ... feedback

$x(0) = x_0$

$u = F(x)$ s.t. $x(t) = 0$ ... not possible (uniqueness)

but .. $x(t) \to 0$, $t \to \infty$

Lemma 18.2 Matrix $A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \beta_0 & \beta_1 & \dots & \beta_{m-1} \end{pmatrix}$ has a char. poly.

$$\chi(\lambda) = \lambda^m - \sum_{j=0}^{m-1} \beta_j \lambda^j.$$

In particular, by choosing β_j properly, $\sigma(A) \subset \mathbb{C}$ can be arbitrary.

Pf. $\chi(\lambda) = \det(\lambda I - A)$ ... $\underbrace{\text{by}}_{\text{expand}}$ last row :

$$\begin{vmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \\ -\beta_0, & -\beta_1, & \cdots & \lambda-\beta_{m-1}\end{vmatrix} = (\lambda-\beta_{m-1})\cdot \underbrace{\begin{vmatrix}\lambda & -1 & \\ & \ddots & -1\\ & & \lambda\end{vmatrix}}_{\lambda^{m-1}} +$$

$$+ \sum_{j=0}^{m-2}(-\beta_j)\cdot(-1)^{m+j+1}\cdot \underbrace{\begin{vmatrix}\begin{matrix}\lambda & -1 & \\ & \ddots & -1\\ & & \lambda\end{matrix} & \\ \underbrace{\qquad\qquad}_{j} & \begin{matrix}-1 & & \\ \lambda & \ddots & \\ & \ddots\ \lambda & -1\end{matrix}\end{vmatrix}}_{\lambda^{j}(-1)^{m-1-j}}$$

Thm. 18.4. Assume $x' = Ax + Bu$ glob. controllable.

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$ given

$\Rightarrow \exists\, F \in \mathbb{R}^{m\times n}$ s.t. $\sigma(A+BF) = \{\lambda_1, \dots, \lambda_n\}$

In particular: as. stability via linear feedback $u = Fx$ is possible.

Pf. Def. v .. generator of cyclic basis w.r.t. A if ("g.c.b.")

$\{v, Av, \dots A^{n-1}v\}$ is a basis.

1. Let $m=1$; i.e. (i) $x' = Ax + bu$; $b \in \mathbb{R}^{n\times 1}$

glob. controllable by Thm. 18.1 $\Leftrightarrow$ rank $\mathcal{K}(A,b) = n$ — $u: [0,T] \to \mathbb{R}$ scalar

$\mathcal{K}(A,b) = (b, Ab, \dots, A^{n-1}b) = (v_1, \dots, v_n)$

i.e. b is a g.c.b.

change to this basis: (i) $\to$ (ii) $x' = \tilde{A}x + \tilde{b}u$

$$A \rightsquigarrow \tilde{A} = \begin{pmatrix} 0 & & & \alpha_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \vdots \\ & & 1 & \alpha_{m-1} \end{pmatrix} \quad \text{where } A^m b = \sum_{j=0}^{m-1} \alpha_j \cdot A^j b$$

$$A: v_1 \to v_2$$
$$\vdots$$
$$v_{m-1} \to v_m$$
$$v_m \to A^m b =$$

$$b \rightsquigarrow \tilde{b} = \begin{pmatrix} 1 \\ 0 \\ | \\ 0 \end{pmatrix}$$

auxiliary system $x' = \hat{A}x + \hat{b}u$ (iii)
special

where $\hat{A} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \alpha_0, & \alpha_1, & \dots & \alpha_{m-1} \end{pmatrix}$; $\hat{b} = \begin{pmatrix} 0 \\ | \\ 0 \\ 1 \end{pmatrix}$

observe: $\mathcal{K}(\hat{A}, \hat{b}) = (\hat{b}, \hat{A}\hat{b}, \dots (\hat{A})^{m-1}\hat{b})$

$$= \begin{pmatrix} 0 & & 1 \\ & 1 & \dots \\ 1 & \alpha_{m-1} & \dots \end{pmatrix} \quad \text{regular.}$$

Special case: $m = 1$, glob. control.
can be transformed to (ii).

→ enough to consider systems of form (iii)

→ we can handle!!

take $\hat{F} = (\beta_0 - \alpha_0, \beta_1 - \alpha_1, \dots, \beta_{m-1} - \alpha_{m-1})$

$$\hat{A} + \hat{b}\hat{F} = \hat{A} + \begin{pmatrix} & 0 & \\ \beta_0 - \alpha_0, & \dots & \beta_{m-1} - \alpha_{m-1} \end{pmatrix}$$

= matrix from Lem. 18.2
arbitr. spectrum by choosing β_j properly

2. $m > 1$ ... find "generalized" cyclic basis

$\{v_1, \dots, v_n\}$ s.t. $v_1 = B u_0$

$v_{i+1} = A v_i + B u_i \quad i < n$

where $u_0, \dots u_{n-1} \in \mathbb{R}^m$

$\exists \Leftarrow$ rank $\mathcal{K}(A, B) = n$ (proof a bit later).

---

define $\tilde{F}: \mathbb{R}^n \to \mathbb{R}^m$ by linearity and

$\tilde{F} v_i = u_i \; ; \; i = 1, \dots n-1$

$\tilde{F} v_n$ arb.

note $(A + B\tilde{F}) v_1 = A v_1 + B \underbrace{\tilde{F} v_1}_{u_1} = v_2$

$(A + B\tilde{F}) v_i = v_{i+1} \, , \; \forall i < n$

$\Rightarrow \mathcal{K}(A + B\tilde{F}, v_1)$ ... regular, in other words

$x' = (A + B\tilde{F}) x + v_1 w$ -- glob. controll.

by case $m = 1$: $\exists f \in \mathbb{R}^{1 \times n}$ s.t.

$A + B\tilde{F} + v_1 f$ has the spectrum I want.

but: $v_1 = B u_0$; hence

$$A + B \underbrace{(\tilde{F} + u_0 f)}_{\substack{\| \\ F \in \mathbb{R}^{m \times m}}}$$

has the spectrum I want.

$\exists$xistence of generalized cyclic basis, i.e.

$\{v_1, \dots, v_n\}$ s.t. $v_1 = Bu_0$

$v_{i+1} = Av_i + Bu_i \quad i < n$

where $u_0, \dots u_{n-1} \in \mathbb{R}^m$ are suitable vectors

- take u_0 s.t. $v_1 = Bu_0 \neq 0$ ..
  .. possible, as $\operatorname{rank} \mathcal{K}(A,B) = n \Rightarrow B \neq 0$.

- by induction: $v_1, \dots v_k$, $k<n$ already $u_0, \dots u_{k-1}$ constructed

$\boxed{\text{claim:}}$ $\exists\, u_k \in \mathbb{R}^m$ s.t. $v_{k+1} = Av_k + Bu_k \notin W$

where $W = \operatorname{Lin}\{v_1, \dots, v_k\} \Rightarrow$ induction goes on..

by contradiction: ?? $Av_k + Bu \in W \quad \forall u \in \mathbb{R}^m$

$\Rightarrow$ in particular $(u=0)$: $Av_k \in W$, hence $Bu \in W \quad \forall u \in \mathbb{R}^m$

$\Rightarrow Av_i \in W \quad \forall i = 1, \dots k$, since $v_{i+1} = Av_i + Bu_i \quad i \leq k-1$

and $v_1 = Bu_0 \in W$ by above

---

thus finally $AW \subset W$, $B\mathbb{R}^m \subset W$

$\Rightarrow \operatorname{rank} \mathcal{K}(A,B) = \operatorname{rank}(B, AB, \dots A^{n-1}B)$

$\leq \operatorname{rank} W = k < n$

↯

Thm 18.5 Assume: $f(x,u) \in C^1$ close to $(0,0) \in \mathbb{R}^{n+m}$

$f(0,0) = 0$, $\mathcal{U} \supset \mathcal{U}(0,\delta)$ in $\mathbb{R}^m$

linearized problem ((18.2) with $A = D_x f(0,0)$,
$B = D_u f(0,0)$)

is globally controllable.

$\Rightarrow \exists F \in \mathbb{R}^{m \times n}$ s.t. $x' = f(x, Fx)$ is asympt. stable.

Pf. set $h(x) = f(x, Fx)$; $F \in \mathbb{R}^{m \times n}$

$h(0) = 0$

$D_x h(0) = D_x f(0,0) + D_u f(0,0) F = A + BF$

by Thm 18.4 : $\sigma(A+BF) \subset \{\mathrm{Re} < 0\}$

$\Rightarrow x' = h(x)$ $x = 0$ as. stable (see Ch 7 of ODE 1)