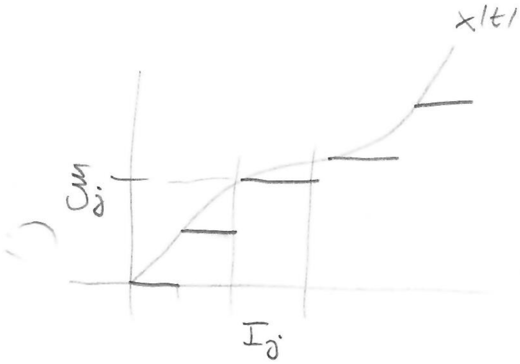
analysis — C^1 (19th century; 1st semester): Riemann, ~~Lebesgue~~

analysis — AC, L^1 .. Lebesgue, $\boxed{\text{a.e.}}$

Lemma 16·1 $f \in CAR(\Omega)$, $x : I \to \mathbb{R}^m$ cont., $\operatorname{graph} x \subset \Omega$

$\Rightarrow \quad t \mapsto f(t, x(t)) \in L^1_{loc}(I)$.

Pf. ? measurability: $x_m(t)$ .. piecewise constant approx. of $x(t)$



$f(t, x_m(t)) = f(t, \xi_j)$, $j \in I_j$

measurable (C1)

$f(t, x_m(t)) \to f(t, x(t))$ by (C2)

and $x_m(t) \overset{loc}{\rightrightarrows} x(t)$

? local integrability: fix $t_0 \in I$ .. $Q(t_0, x_0)$ .. cylinder

WLOG $\operatorname{graph} x|_{\mathcal{U}(t_0, \delta)} \subset Q(t_0, x_0)$.

$|f(t, x(t))| \le m(t) \in L^1(\mathcal{U}(t_0, \delta))$ ..

Lemma 16·2 $f \in CAR(\Omega)$, $x : I \to \mathbb{R}^m$ cont., $\operatorname{graph} x \subset \Omega$.

Then: x is a Car. sol. to (16·1) $\iff$ $x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s, x(s))\, ds$

for $\forall\, t_1, t_2 \in I$.

Pf. "$\Rightarrow$": $x'(t) = f(t, x(t))$ a.e. $\int_{t_1}^{t_2}$: LHS $x(t_2) - x(t_1)$ by Prop. 1

"$\Leftarrow$": fix $t_0 \in I$: $x(t) = x(t_0) + \underbrace{\int_{t_0}^{t} f(s, x(s))\, ds}_{\in L^1_{loc}(I) \text{ by L. 16·1}}$; $\forall t \in I$

hence: RHS $AC_{(loc)}(I)$ .. Prop 2

$x'(t) = f(t, x(t))$ a.e. ..

Thm 16.1 [Banach fixed point - generalized.] Λ, X .. metr. spaces, X complete, non-empty. $\Phi: \Lambda \times X \to X$ cont. w.r.t. $\lambda \in \Lambda$ for any $x \in X$ fixed.

key ass.: $\exists\, \kappa \in (0,1)$ s.t. $\|\Phi(\lambda,x) - \Phi(\lambda,y)\|_X \le \kappa \|x-y\|_X \quad \forall \lambda \in \Lambda,\ x,y \in X$

Then: (i) $\forall \lambda \in \Lambda\ \exists!\ x(\lambda) \in X$ s.t. $\Phi(\lambda, x(\lambda)) = x(\lambda)$

(ii) $\lambda \mapsto x(\lambda)$ is cont.

(iii) $\|y - x(\lambda)\| \le (1-\kappa)^{-1} \|y - \Phi(\lambda,y)\| \quad \forall \lambda \in \Lambda,\ y \in X$.

$\rho_X(x,y) = \|x-y\|$ -- notational cheating.

Pf. (i) define seq. of functions $x_n : \Lambda \to X$

$$x_0(\lambda) \equiv y, \quad y \in X \text{ fixed, arb.}$$

$$(*) \qquad x_{n+1}(\lambda) = \Phi(\lambda, x_n(\lambda))$$

induction: $\|x_n(\lambda) - x_{n-1}(\lambda)\| \le \kappa \|x_{n-1}(\lambda) - x_n(\lambda)\| \le \dots$

$$\le \kappa^{n-1} \|\underbrace{\phi(\lambda,y)}_{x_1(\lambda)} - \underbrace{y}_{x_0(\lambda)}\|$$

$\forall m > n$: $\|x_m(\lambda) - x_n(\lambda)\| \le \sum_{j=n+1}^{m} \|x_j(\lambda) - x_{j-1}(\lambda)\|$

Δ-ineq.

$$\le \sum_{j=n+1}^{m} \kappa^{j-1} \|\phi(\lambda,y) - y\|$$

$$= \frac{\kappa^n}{1-\kappa} \|\phi(\lambda,y) - y\|.$$

$\Rightarrow \{x_n(\lambda)\}_n$ .. Cauchy seq. in X (λ fixed)

$x(\lambda) := \lim_{n\to\infty} x_n(\lambda)$; $n \to \infty$ in $(*)$:

$$x(\lambda) = \Phi(\lambda, x(\lambda))$$

$x(\lambda)$ .. unique; ind. of the initial $y \in X$.

(iii) $\|x_m(\lambda) - x_n(\lambda)\| \le \dfrac{K^n}{1-K} \|\Phi(\lambda, y) - y\|$ $\quad K^n = 1$

$\quad x(\lambda) \qquad y$

take $n = 0$, $m \to \infty$

(iii) choose $y = x(\lambda_0)$, $\lambda = \lambda_n$ in (ii):

$$\|x(\lambda_0) - x(\lambda_n)\| \le \frac{1}{1-K} \|x(\lambda_0) - \Phi(\lambda_n, x(\lambda_0))\|$$

$$= \frac{1}{1-K} \|\Phi(\lambda_0, x(\lambda_0)) - \Phi(\lambda_n, x(\lambda_0))\|$$

$\lambda_0 \to \lambda_n$: RHS $\to 0$,

i.e. $x(\lambda_n) \to x(\lambda_0)$. continuity of $\lambda \mapsto x(\lambda)$.

Thm 16.2 [Picard-generalized]. $I = [0, T]$, Π .. metr. space.

$f = f(t, x, \lambda) : I \times \mathbb{R}^n \times \Pi \to \mathbb{R}^n$ s.t.

1. $f(\cdot, \cdot, \lambda) \in CAR(I \times \mathbb{R}^n)$ $\forall \lambda \in \Pi$ fixed.
2. $\exists\, \ell \in L^1(I)$ s.t. $\|f(t, x, \lambda) - f(t, y, \lambda)\| \le \ell(t) |x - y|$ for a.e. $t \in I$, for $\forall x, y \in \mathbb{R}^n$, $\lambda \in \Pi$
3. $\lambda \mapsto \int_0^t f(s, x(s), \lambda)\, ds$ is cont. $\Pi \to C(I)$ for any $x = x(t) \in C(I)$ fixed.

Then: $\forall x_0 \in \mathbb{R}^n$ $\lambda \in \Pi$ $\exists!\, x \in AC(I)$ .. Car. sol. to $x' = f(t, x, \lambda)$, $x(0) = x_0$

moreover: x depends cont. on x_0, λ:

$$x_{0n} \to x_0,\ \lambda_n \to \lambda_0 \implies x_n \rightrightarrows x \text{ in } I$$

where x_n sol. cor. to $\lambda = \lambda_n$, $x_0 = x_{0n}$.

Pf: apply Thm 16.1 $X = C([0,T]; \mathbb{R}^n)$

$$\|x\| = \sup_{t\in[0,T]} |x(t)| e^{-Lt} \quad ; \quad L>0 \text{ will be chosen later (TRICK)}$$

$$\Lambda = \mathbb{R}^n \times \Pi \ni (x_0, \lambda)$$

operator : $\Phi : (x_0, \lambda_0, x(\cdot)) \longmapsto x_0 + \int_0^t f(s, x(s), \lambda_0)\, ds$

$t \in [0,T]$

everything fits nicely: fixed point = solution we seek (by Lemma 16.2)

continuity assump. & conclusion : Thm 16.1, (ii)

only to verify: uniform contraction :

fix $x_0, \lambda_0, x(\cdot), y(\cdot)$, set $\hat{x} = \Phi(x_0, \lambda_0, x(\cdot))$

$t \in [0,T]$ $\qquad \hat{y} = \Phi(x_0, \lambda_0, y(\cdot))$

$$|\hat{x}(t) - \hat{y}(t)| = \left| \int_0^t f(s, x(s), \lambda_0) - f(s, y(s), \lambda_0)\, ds \right| \quad \text{by assump. 2.}$$

$$\leq \int_0^t \ell(s) \underbrace{|x(s) - y(s)|}_{\leq e^{Ls} \|x-y\|_X} ds \leq \int_0^t \ell(s) e^{Ls} |x(s) - y(s)|\, ds$$

$$\Rightarrow \|\hat{x} - \hat{y}\|_X \leq K \|x - y\|_X \; ; \text{ where } K = \sup_{t\in[0,T]} \int_0^t \ell(s) e^{L(s-t)}\, ds$$

$K < 1$? ... $\ell \in L^1$, hence $\ell = \ell_1 + \ell_2$; $\int_0^T \ell_1 \leq \frac{1}{4}$, $|\ell_2| \leq M$ (large)

$$\int_0^t \ell(s) e^{L(s-t)}\, ds = \int_0^t \ell_1(s) e^{L(s-t)}\, ds + \int_0^t \ell_2(s) e^{L(s-t)}\, ds = K_1 + K_2$$

clearly : $K_1 \leq \frac{1}{4}$

$$K_2 \leq M \int_0^t e^{L(s-t)}\, ds = M \int_0^t e^{-L\sigma}\, d\sigma < \frac{M}{L} \leq \frac{1}{4} \; ;$$

by taking $L \geq 2M$