15. Poincaré–Bendixson theory

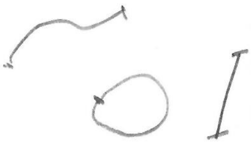
problem of periodic solutions in $\mathbb{R}^{2}$ ← important!!

Assume. (1) $x' = f(x)$; $f:\Omega\to\mathbb{R}^2$, $\Omega\subset\mathbb{R}^2$ domain (open, connected)

C^1 ... $\varphi(t,x)$ ... d.s. of (1), C^1

defined for all $t\ge 0,\ x\in\Omega$

Def. $\gamma\subset\mathbb{R}^n$ ... curve :: $\gamma=\psi([a,b])$; $\psi:[a,b]\to\mathbb{R}^n$ inj, continuous



... Jordan curve :: $\gamma=\psi([a,b])$; $\psi:[a,b]\to\mathbb{R}^n$ cont; inj. on $[a,b)$, $\psi(a)=\psi(b)$.

... (line) segment :: curve s.t. ψ affine.

Note. orbit (per. orbit) is curve (Jordan curve).

Jordan theorem. If $\gamma\subset\mathbb{R}^2$ is a Jordan curve, then $\mathbb{R}^2=\Omega_1\cup\gamma\cup\Omega_2$, where $\Omega_{1,2}$ are domains, one bdd ("interior" of γ), one unbounded ("exterior" of γ).

Pf. see google/web.

Theorem 15.1 [Poincaré–Bendixson.] Let $p\in\Omega$ be s.t. $\gamma^+(p)$ is rel. comp., and $\omega(p)$ does not contain a stationary point. Then $\omega(p)=\Gamma$ is a non-trivial periodic orbit.

Def. A set $\Sigma\subset\Omega$ is called transversal if Σ is a segment s.t. Σ is not parallel to $f(p)$ for $\forall p\in\Sigma$.

Rem. $\Sigma=\psi([a,b])$; $\psi(t)=at+b$; $a\ne\alpha f(p)$ $\forall p\in\Sigma$, $\alpha\in\mathbb{R}$

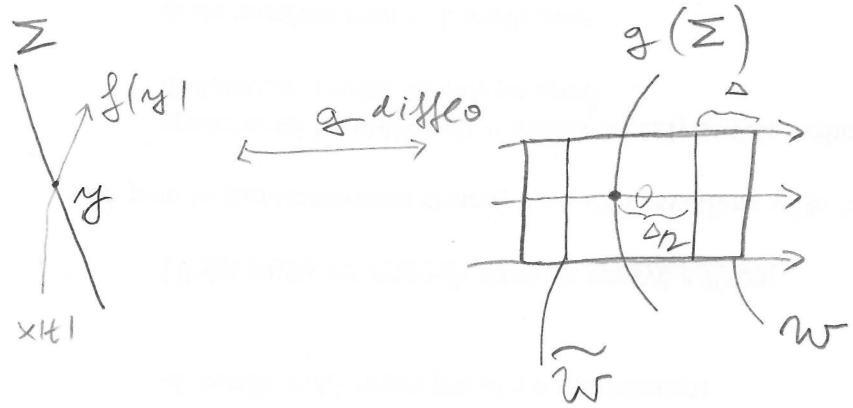
Easy to see: solutions cross Σ with a nonzero speed (and the same direction)

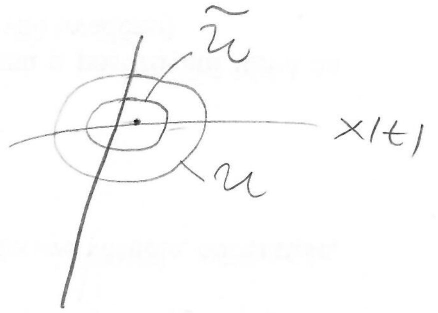
$\forall x_0\in\Omega$, $f(x_0)\ne 0 \Rightarrow \exists$ trans. Σ s.t. $x_0\in\Sigma$.

Lemma 15.1 $\Sigma \subset \Omega$ transversal, $y \in \Sigma \Rightarrow \exists\, \mathcal{U} \supset \tilde{\mathcal{U}}$ neigh. of y, and $\Delta > 0$ s.t. $x_0 \in \tilde{\mathcal{U}}$ implies (i) $x(t) = \varphi(t, x_0) \in \mathcal{U} \;\forall |t| < \Delta$

$x(t) = \varphi(t, x_0)$ (ii) $\exists |\tilde{t}| < \frac{\Delta}{2}$ s.t. $x(\tilde{t}) \in \Sigma \cap \tilde{\mathcal{U}}$.

Pf. apply Thm 13.3 (rectification lemma)



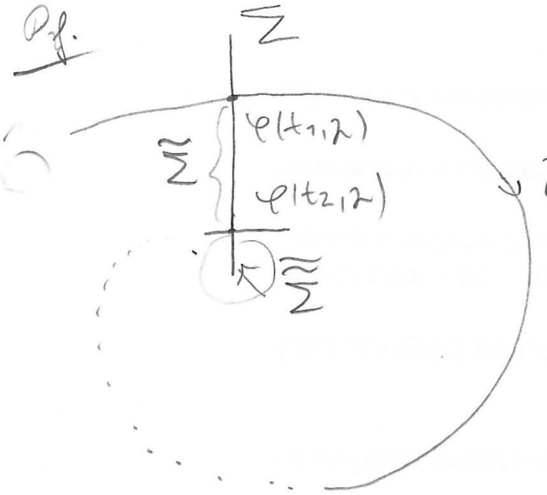


set $\mathcal{U} = g^{-1}(\mathcal{W})$

$\tilde{\mathcal{U}} = g^{-1}(\tilde{\mathcal{W}})$

Lemma 15.2 $\Sigma \subset \Omega$ transversal, $x \in \Omega \Rightarrow \gamma^+(x) \cap \Sigma$ is a monotone sequence. More precisely: if $t_1 < t_2 < t_3$ s.t. $\varphi(t_i, x) \in \Sigma$ then either (i) $\varphi(t_1, x) = \varphi(t_2, x) = \varphi(t_3, x)$, or (ii) $\varphi(t_2, x)$ lies (strictly) between $\varphi(t_1, x)$, $\varphi(t_3, x)$.

Pf.



Key observations:

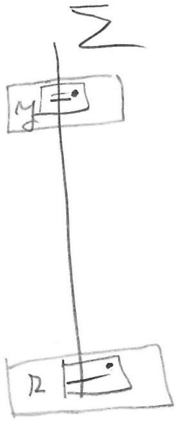
(i) $\gamma = \varphi([t_1, t_2], x) \cup \tilde{\Sigma}$ is a Jordan curve.

(ii) $\varphi((t_2, \infty), x)$, $\tilde{\tilde{\Sigma}}$ belong to the same component of $\mathbb{R}^2 \setminus \gamma$

$\Rightarrow \gamma^+(\varphi(t_2, x))$ can only intersect Σ in $\tilde{\tilde{\Sigma}}$, q.e.d.

Lemma 15.3 $\Sigma \subset \Omega$ transversal, $x \in \Omega \Rightarrow \omega(x) \cap \Sigma$ consists of at most one point.

Pf. ?? $\exists\, y \neq z$, $y, z \in \omega(x) \cap \Sigma$



$\ldots \exists\, t_n, s_n \to \infty$ s.t. $\varphi(t_n, x) \to y$

WLOG $t_n < s_n < t_{n+1}$ $\quad \varphi(s_n, x) \to z$

$\mathcal{U}, \tilde{\mathcal{U}}$ and $\mathcal{V}, \tilde{\mathcal{V}}$ -- neighb. of y, z

$\mathcal{U} \cap \mathcal{V} \neq \emptyset$ from L. 15.1

WLOG -- n large s.t. $\varphi(t_n, x) \in \tilde{\mathcal{U}}$

$t_n < s_n < t_{n+1} < s_{n+1}$ $\quad \varphi(s_n, x) \in \tilde{\mathcal{V}}$

$\exists\, \tilde{t}_n$ s.t. $\varphi(\tilde{t}_n, x) \in \tilde{\mathcal{U}} \cap \Sigma$; $|\tilde{t}_n - t_n| < \frac{\Delta}{2}$

$\tilde{s}_n$ $\quad \varphi(\tilde{s}_n, x) \in \tilde{\mathcal{V}} \cap \Sigma$; $|\tilde{s}_n - s_n| < \frac{\Delta}{2}$

but: $|t_n - s_n| > 2\Delta$ ..

hence $\tilde{t}_n < \tilde{s}_n < \tilde{t}_{n+1} < \tilde{s}_{n+1} < \ldots$

$\Rightarrow$ non-monotone intersections $\gamma^+(x) \cap \Sigma$

↯ (L. 15.2)

Proof of Thm. 15.1 $\quad$ fix arbitrary $q \in \omega(x) \neq \emptyset$ (Thm. 13.1)

STEP 1: $q \in \Gamma$, where Γ is a periodic orbit

let $x_0 \in \omega(q)$ be arbitrary; $\gamma^+(q) \in \omega(x)$ (invariance of ω)

hence $\omega(q) \subset \omega(x)$ non-empty, compact;

x_0 not stationary $\Rightarrow \exists$ transversal Σ s.t. $x_0 \in \Sigma$

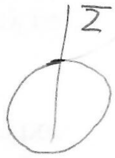
$\exists\, t_n \to \infty$; $\varphi(t_n, q) \to x_0$

L.15.1. $\exists\, |\tilde{t}_n - t_n| < \frac{\Delta}{2}$; $\varphi(\tilde{t}_n, q) \in \Sigma \cap \tilde{\mathcal{U}}$ — neigh. of x_0.

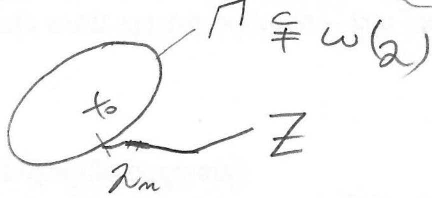
but: $r_n = \varphi(\tilde{t}_n, q) \in \Sigma \cap \gamma^+(q) \cap \omega(q)$ ... singleton!! (L.15.3)

i.e. $x_0 = r_n = \varphi(\tilde{t}_n, q)$ for all n large enough.

$\Rightarrow \gamma(q) =: \Gamma$ is a closed (periodic) orbit.



STEP 2: $\omega(q) \subset \Gamma = \gamma(q)$.



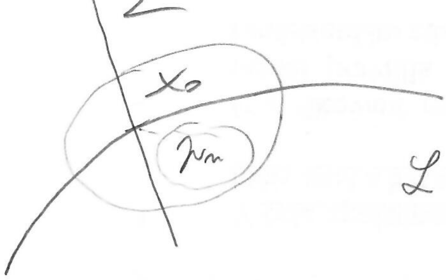
?? $Z := \omega(q) \setminus \Gamma \neq \emptyset$

connected; hence Z, Γ are not separated, i.e.

$\exists\, z_n \in Z$ s.t. $z_n \to \Gamma$ distance

WLOG $z_n \to x_0 \in \Gamma$

take Σ transversal s.t. $x_0 \in \Sigma$



$\Gamma = \gamma(q) = \gamma(x_0)$.

L.15.1: $\exists\, \mathcal{U}$ neigh. of x_0

s.t. $\underbrace{\gamma(z_n)}_{\subset\, \omega(q)} \cap \Sigma \cap \mathcal{U} \neq \emptyset$ hence all the intersections equal to $x_0 \in \omega(q)$

$\Rightarrow z_n \in \Gamma$ ... ↯

Theorem 15.2 [Bendixson–Dulac.] Let $\Omega \subset \mathbb{R}^2$ be simply connected, let $\exists\, B : \Omega \to \mathbb{R}$ a C^1 function s.t. $\operatorname{div}(Bf) > 0$ a.e. in Ω.

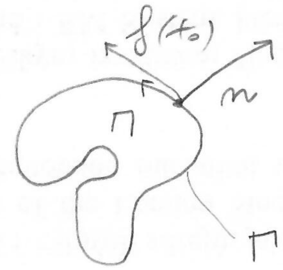
Then $\nexists$ periodic orbit in Ω.

Rem. $\Omega \subset \mathbb{R}^2$ simply connected :: $\forall \gamma \subset \Omega$ Jordan curve can be (continuously) shrinked to a point.

$\iff$ int $\gamma \subset \Omega$.

$\Pi =$ int Γ

Pf. ?? $\exists\, \Gamma \subset \Omega$ .. per. orbit .. Gauss thm:



$$\int_\Pi \operatorname{div}(Bf)\, d\lambda_2 = \int_\Gamma (B(f \cdot n))\, dS$$

scalar

outer normal

LHS > 0 since integrand > 0 a.e.

RHS $= 0$, since $\underline{f \cdot n = 0}$ $\qquad$ $f(x_0)$ .. tangential direction of Γ; orbit of $x' = f(x)$

↑ key observation

Thm. (φ, Ω) .. d.s., C^1, $\Omega \subset \mathbb{R}^2$ bdd, domain;

$\gamma^+(x)$ rel. compact $\Rightarrow$ there is

either (i) $\omega(x) = \{x_0\}$, x_0 .. stationary point

or (ii) $\omega(x) = \Gamma$ .. periodic orbit

or (iii) $\omega(x)$ is a finite union of stat. points and connecting orbits.