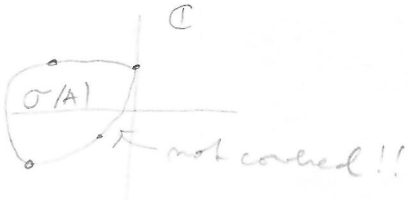
14. La Salle's invariance principle

Recall. (1) $x' = f(x)$; $x_0 \dots$ stationary ; $A = Df(x_0)$

$f \in C^1(\mathcal{U}(x_0))$

principle of linearized (in)stability: (i) $\operatorname{Re}\lambda < 0 \;\; \forall \lambda \in \sigma(A) \Rightarrow x_0$ asympt. stable

(ii) $\exists \lambda \in \sigma(A),\ \operatorname{Re}\lambda > 0 \Rightarrow x_0$ unstable



Lyapunov functions: WLOG $x_0 = 0$; $\mathcal{U} \dots$ neigh. of x_0.

(i) $V(0) = 0$; $V(x) > 0 \;\; \forall x \in \mathcal{U} \setminus \{0\}$ ("positively definite")

(ii) $t \mapsto V(x(t))$ non-increasing for $\forall x(t)$ solutions of (1) in $\mathcal{U}$

orbital derivative of V w.r.t. (1):

$$\dot V_f(x) = \nabla V(x) \cdot f(x) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_j}(x)\, f_j(x)$$

Lemma. $\frac{d}{dt} V(x(t)) = \dot V_f(x(t))$; $\forall t$; $\forall x(t)$ sol. of (1)

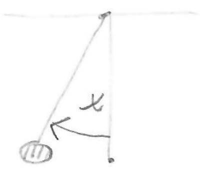
Lyapunov theorem: $\exists$ Lyap. fun. $\Rightarrow$ stability

if moreover $\dot V(x) < 0$ in $\mathcal{U} \setminus \{0\}$

$\Rightarrow$ asym. stability.

Ex. damped (mech.) oscillator:

$x'' + g(x') + \sin x = 0$; $x = x(t) \dots$ displacement angle



$x'' = -\sin x - g(x') \dots\ g(y): \mathbb{R} \to \mathbb{R}$

$g(0) = 0,\ g(y)\,y \geq 0$

system: $X = \binom{x}{y}$; $y = x'$

$$X' = F(X) = \begin{pmatrix} y \\ -\sin x - g(y) \end{pmatrix}$$

... ? stability of $X_0 = \binom{0}{0}$.

$\nabla F = \begin{pmatrix} 0 , & 1 \\ -1 , & -g'(y) \end{pmatrix}$; $A = \nabla F(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix}$

$a = g'(0) \geq 0$

eigenvalues: $\chi(\lambda) = \begin{vmatrix} \lambda , & -1 \\ 1 , & \lambda + a \end{vmatrix} = \lambda(\lambda + a) + 1 = \lambda^2 + a\lambda + 1$

$D = a^2 - 4$

$\lambda_{1,2} = \dfrac{-a \pm \sqrt{a^2-4}}{2}$

$a > 0$: asymp. stability, since $\sqrt{D} < a$ or $\in \mathbb{C}$.

$a > 0$: instability ; analogous argument

$\boxed{a = 0}$ — we do not know

Lyapunov funct: $V = 1 - \cos x + \frac{1}{2} y^2$

$\dot{V} = \frac{d}{dt}\left(1 - \cos x + \frac{1}{2} y^2\right) = \sin x \cdot \overset{y}{x'} + y \cdot \underset{(-\sin x - g(y))}{y'}$

$= -g(y)\, y \leq 0$

$\Rightarrow$ stability by Lyap. 1.

? asym. stab: we do not have $\dot{V} < 0 \quad \forall (x,y) \neq (0,0)$

almost: we can have $\dot{V} < 0 \quad y \neq 0$

can this made be done? yes... La Salle !!

Theorem 14.1 (La Salle's inv. principle)

Assume: (φ, Ω) d.s. for (1) $x' = f(x)$, $f: \Omega \to \mathbb{R}^n$.

$\exists\, V(x): \Omega \to \mathbb{R}$, C^1, bdd from below

$\exists\, l \in \mathbb{R}$ s.t. $\Omega_l = \{x \in \Omega;\ V(x) < l\}$ is bdd.

$\dot{V}_f(x) \le 0$ in Ω_l.

Denote: $R = \{x \in \Omega_l;\ \dot{V}_f = 0\}$

$M = \{y \in R,\ \gamma(y) \subset R\}$.

Then: $\omega(x_0) \subset M$ for all $x_0 \in \Omega_l$.

Rem. M .. largest invariant subset of R.

$\omega(x_0) \subset M$ means: $\varphi(t, x_0) \to M$ distance (cf. Thm. 13.1)

in particular: if $M = \{\bar{x}\}$, then: asym. stability

Pf. fix $x_0 \in \Omega_l$, $y \in \omega(x_0)$ ... ? $y \in M$.

set $x(t) = \varphi(t, x_0)$ .. solution starting from x_0.

$\frac{d}{dt} V(x(t)) = \dot{V}_f(x(t)) \le 0$; i.e. $t \mapsto V(x(t))$ non-increasing, bdd fr. below

$\exists\, c = \lim_{t\to\infty} V(x(t))$; $\in \mathbb{R}$.

claim: $V \equiv c$ on $\omega(x_0)$.

$y \in \omega(x_0)$ arbitrary: $\exists\, t_k \to \infty$, $x(t_k) \to y$

$V(x(t_k)) \to V(y)$

$\downarrow$

c

$\omega(x_0)$ invariant (Th. 13.1)

$\Rightarrow \varphi(t,y) \in \omega(x_0)$

$V(\varphi(t,y)) \equiv c \qquad \Big| \ \frac{d}{dt}$

$\frac{d}{dt} V(\varphi(t,y)) = \dot{V}_g(\varphi(t,y)) = 0$

i.e. $\varphi(t,y) \in R \ \ \forall t \in \mathbb{R}$

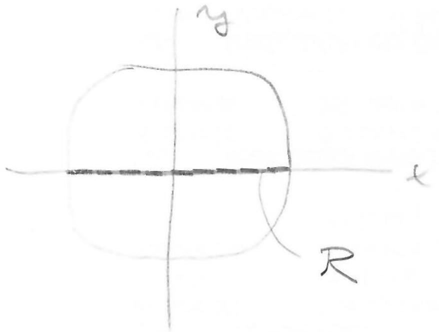
i.e. $\gamma(y) \subset R$ ...

Ex. conclusion:

$V = \underbrace{1-\cos x}_{\approx \frac{x^2}{2};\ x\sim 0} + \frac{y^2}{2}$ $\qquad \ell > 0$ small s.t.

$V_\ell = \{(x,y);\ V < \ell\}$ - neigh. of $(0,0)$

$R = \{(x,y);\ \dot{V}_F = 0\} = \{(x,y) \in V_\ell;\ y = 0\}$



claim: $M = \{(0,0)\}$.

Pf. $(x(0), y(0)) = (x_0, 0)$; $x_0 \neq 0$

$y'(0) = x_0 > 0$; i.e.

$y(t) \neq 0$ for $t > 0$,

$(x(t), y(t)) \notin R$

hence: Thm. 13.2 $\xRightarrow{\text{not n}} (x(t), y(t)) \longrightarrow (0,0)$

$\forall (x_0, y_0) \in V_\ell$