13. Dynamical systems

Def. Dynamical system (d.s.) :: (φ, Ω), where $\Omega \subset \mathbb{R}^n$, and $\varphi = \varphi(t,x) : \mathbb{R} \times \Omega \to \Omega$

(i) $\varphi(0,x) = x$ for $\forall x \in \Omega$

(ii) $\varphi(s, \varphi(t,x)) = \varphi(s+t, x)$ for $\forall t,s \in \mathbb{R}, x \in \Omega$

(iii) $(t,x) \longmapsto \varphi(t,x)$ is continuous

Ex. (1) $x' = f(x)$; $f : \Omega \to \mathbb{R}^n$ ... solution operator
$x(0) = x_0$ regular enough (e.g. locally Lipschitz)

$\varphi(t,x_0) := x(t)$ ← solution of (1)

↑ is dynamical system (canonical example) – different vocabulary

in particular: $x' = Ax$
$x(0) = x_0$

... $\varphi(t,x) = e^{tA} x_0$

Def. Let (φ, Ω) be d.s. The set $M \subset \Omega$ is called

- positively invariant :: $\varphi(t,x) \in M$ for $\forall t \geq 0, x \in M$
- negatively invariant :: $\varphi(t,x) \in M$ for $\forall t \leq 0, x \in M$
- (fully) invariant :: $\varphi(t,x) \in M$ for $\forall t \in \mathbb{R}, x \in M$

given $x_0 \in \Omega$, we define:

- positive orbit $\gamma^+(x_0) = \{\varphi(t,x_0), t \geq 0\}$
- negative orbit $\gamma^-(x_0) = \{\varphi(t,x_0), t \leq 0\}$
- (full) orbit $\gamma(x_0) = \{\varphi(t,x_0), t \in \mathbb{R}\}$

Rem. orbit (pos./neg./full) is invariant (pos./neg./fully)
M is inv. $\Leftrightarrow \forall x_0 \in M : \gamma(x_0) \subset M$.

Proof: CHW

Rem. orbit : $\{\varphi(t,x_0), t \in I\} \subset \Omega$ is a set

Def. Let (φ, Ω) be d.s. The ω-limit set of $x_0 \in \Omega$ is defined $\omega(x_0) := \{y \in \Omega\,;\ \exists\, t_n \to \infty \text{ s.t. } \varphi(t_n, x_0) \to y\}$.

Rem. Easy to see $y \in \omega(x_0)$ iff $\forall \varepsilon > 0\ \forall T > 0\ \exists t > T$ s.t. $|y - \varphi(t, x_0)| < \varepsilon$

Proof: Heine (HW).

$y \in \Omega \setminus \omega(x_0)$ iff $\exists \varepsilon > 0\ \exists T > 0\ \forall t \geq T : |y - \varphi(t, x_0)| \geq \varepsilon$.

$\omega(x_0)$ .. all points of $\gamma^+(x_0)$, relevant for large t.

Lemma 13.1 $\omega(x_0) = \bigcap_{\tau > 0} \overline{\gamma^+(\varphi(\tau, x_0))}$.

Pf. „$\subseteq$" $y \in \omega(x_0)$ given; $\tau > 0$ arbitrary

note $\gamma^+(\varphi(\tau, x_0)) = \{\underbrace{\varphi(t, \varphi(\tau, x_0))}_{\varphi(t+\tau, x_0)}\,;\ t \geq 0\}$

$t_k > \tau : \varphi(t_k, x_0) \in \gamma^+(\varphi(\tau, x_0))$

$\Rightarrow y \in \overline{\gamma^+(\varphi(\tau, x_0))}$

$\tau > 0$ arbitrary: $y \in$ R.H.S

„$\supseteq$" $y \in$ R.H.S : hence $y \in \overline{\gamma^+(\varphi(k, x_0))}\ \forall k \in \mathbb{N}$

i.e. $\exists\, z_j \in \gamma^+(\varphi(k, x_0))$, $z_j \to y$

in part. $\boxed{\exists\, z_k \in \gamma^+(\varphi(k, x_0))}$

$|z_k - y| < \frac{1}{k}$

$z_k = \varphi(\tau_k, \varphi(k, x_0)) = \varphi(\underbrace{\tau_k + k}_{t_k}, x_0)$

$\varphi(t_k, x_0) = z_k \to y$, $k \to \infty$ $\qquad t_k \to +\infty$

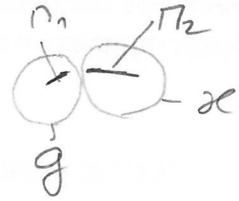
hence $y \in \omega(x_0) =$ L.H.S.

Recall. $M \subset \mathbb{R}^n$ is connected :: $\nexists\ \mathcal{G}, \mathcal{H}$ open, disjoint s.t.

$$M \subset \mathcal{G} \cup \mathcal{H}, \quad M \cap \mathcal{G} \neq \emptyset$$
$$M \cap \mathcal{H} \neq \emptyset$$

M not connected :: $M = M_1 \cup M_2$ s.t. $M_1, M_2 \neq \emptyset$

$\exists\ \mathcal{G}, \mathcal{H}$ open disjoint $M_1 \subset \mathcal{G}$, $M_2 \in \mathcal{H}$



- $M \subset \mathbb{R}$ connected $\Leftrightarrow$ M is an interval
- $M \subset \mathbb{R}^n$ connected, $f: M \to \mathbb{R}^k$ cont. $\Rightarrow f(M) \subset \mathbb{R}^k$ connected.

Theorem 13.1 [Properties of $\omega(x_0)$]. Let (φ, Ω) be d.s., $x_0 \in \Omega$.

1. $\omega(x_0)$ is closed, fully invariant
2. $\gamma^+(x_0)$ rel. compact $\Rightarrow \omega(x_0) \neq \emptyset$, compact, connected.

Pf. 1. closed $\Leftarrow$ L. 13.1 (intersection of closed sets)

invariant: $\underbrace{y \in \omega(x_0)}$, $t \in \mathbb{R}$ given $\overset{?}{\Rightarrow} \varphi(t, y) \in \omega(x_0)$

$\exists\ t_k \to \infty$, s.t. $\varphi(t_k, x_0) \to y$

Let $\varphi(\underbrace{t + t_k}_{\substack{\| \\ t_k' \to \infty}}, x_0) = \varphi(t, \underbrace{\varphi(t_k, x_0)}_{\downarrow\, y}) \to \varphi(t, y)$

$\uparrow$ continuity of d.s.

hence $\varphi(t, y) \in \omega(x_0)$ by definition

2. recall: $\gamma^+(x_0)$ relat. compact: $\overline{\gamma^+(x_0)}$ compact in Ω

$t_k \to \infty$ arb: $\varphi(t_k, x_0) \subset K$, hence

$\exists$ subseq. t_k' (still $t_k' \to \infty$)

s.t. $\varphi(t_k', x_0) \to y \in K$. hence $\omega(x_0) \neq \emptyset$

? $\omega(x_0)$ compact: by L. 13.1, $\omega(x_0)$ is closed, $\subseteq \overline{\gamma^+(x_0)}$

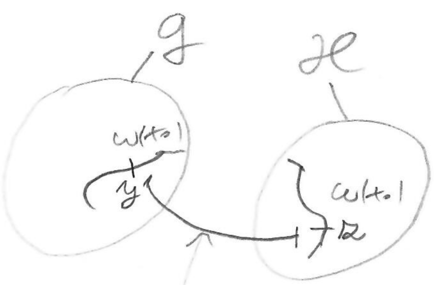
↑ compact

? $\omega(x_0)$ connected: $\boxed{??}$ $\exists\, \mathcal{G}, \mathcal{H}$ open s.t. $\omega(x_0) \subset \mathcal{G} \cup \mathcal{H}$

↑ proof by contradiction (assume NOT)

$\omega(x_0) \cap \mathcal{H} \neq \phi$, $\mathcal{H} \cap \mathcal{G} = \phi$

$\omega(x_0) \cap \mathcal{G} \neq \phi$



$\exists\, t_k \to \infty : \varphi(t_k, x_0) \to y \in \mathcal{G}$

$\exists\, s_k \to \infty : \varphi(s_k, x_0) \to z \in \mathcal{H}$

WLOG: $t_k < s_k < t_{k+1} < s_{k+1}$

(without loss of generality) $\varphi(t_k, x_0) \in \mathcal{G}$

$O_k = \{\varphi(t, x_0) ; t \in (t_k, s_k]\}$. $\varphi(s_k, x_0) \in \mathcal{H}$

observe: O_k connected (cont. image of interval $[t_k, s_k]$)

$\Rightarrow \exists\, w_k \in O_k$ s.t. $w_k \notin \mathcal{G} \cup \mathcal{H}$.

$\qquad ||$

$\varphi(\tau_k, x_0)$; $\tau_k \in [t_k, s_k]$.

compactness: $\exists$ subseq. τ_k' s.t. $\varphi(\tau_k, x_0) \to w$

now $w \in \omega(x_0)$ by def.,

but $w \notin \mathcal{G} \cup \mathcal{H}$ ... ↯

(contradiction)

Rem: compactness assumption: necessary.
also based on compactness.

Theorem 13.2 (φ, Ω) d.s., $K \subset \Omega$ compact. Then $\omega(x_0) = K \Leftrightarrow \varphi(t, x_0) \to K$ (distance), for $t \to \infty$

In particular: $\omega(x_0) = \{z\} \Leftrightarrow \varphi(t, x_0) \to z$, $t \to \infty$.

Def. D.s. (φ, Ω), (ψ, Θ) are <u>topologically conjugate</u> :: $\exists$ homeomorphism $h: \Omega \to \Theta$ s.t. $h(\varphi(t,x)) = \psi(t, h(x))$ for $\forall t \in \mathbb{R}, x \in \Omega$. Equivalently: $\varphi(t,\cdot) = h^{-1}(\psi(t, h(\cdot)))$ for $\forall t \in \mathbb{R}$.

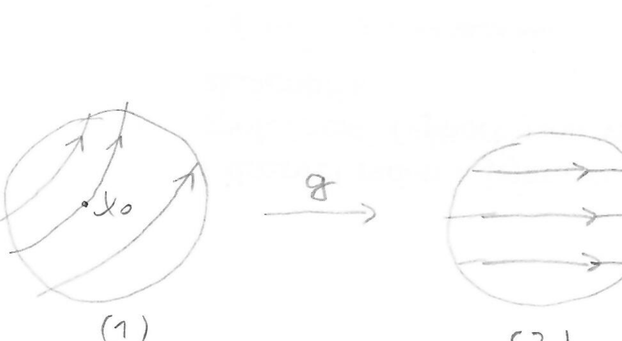
Rem. natural notion of equivalence.
(stationary points, stability, periodic orbits, ... ω-limit sets.)

Rem. (φ, Ω) .. d.s. :: x_0 stat. point :: $\varphi(t, x_0) = x_0 \ \forall t_0 \in \mathbb{R}$
(dictionary)
stable: $\forall \varepsilon > 0 \ \exists \delta > 0 \ |x_0 - x| < \delta \Rightarrow |\varphi(t,x) - x_0| < \varepsilon \ \forall t > 0$

asympt. stable: stable and moreover: $\exists \eta > 0 \ \forall |x - x_0| < \eta: \varphi(t,x) \to 0$, $t \to \infty$

$\Gamma \subset \Omega$ periodic orbit: $\exists x_0 \in \Gamma \ \exists T > 0$ s.t. $\varphi(T, x_0) = x_0$.

Theorem 13.3 [Rectification lemma.] Let $f: \Omega \to \mathbb{R}^n$ be C^1 close to x_0, let $f(x_0) \neq 0$. Then $\exists V, W$ neigh. of x_0, 0 resp. and diffeomorphism $g: V \to W$ s.t. $x(t)$ solves (1) $x' = f(x)$ in V iff $y(t) = g(x(t))$ solves (2) $y' = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ in W.
Moreover: $f \in C^r \Rightarrow g \in C^r$, $\forall r \geq 2$.
In other words: x_0 not stationary $\Rightarrow$ d.s. given by (1) is topologically conj. to d.s. given by (2). on V, W.



(1) (2)

Pf. STEP 1. WLOG $x_0 = 0$, $f(0) = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$; $\alpha \neq 0$

define $G : W \to \mathbb{R}^n$ — l.o. (solution oper.) to (1).

$$(y_1, \dots, y_n) \longmapsto \varphi(y_1; (0, y_2, \dots, y_n))$$

W .. small enough neigh. of $0 \in \mathbb{R}^n$

$\Rightarrow$ G well-defined, C^r .. see ODE 1, Ch 1-4.

STEP 2. G invertible ... $DG(0) = ?$; note $G(0) = 0$

$$\frac{\partial G}{\partial y_1}(0) = \frac{\partial}{\partial t}\varphi(t, y)\Big|_{\substack{t=0\\ y=0}} = f(\varphi(t,y))\Big|_{\substack{t=0\\ y=0}} = f(0) = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\frac{\partial G}{\partial y_r}(0) = \frac{\partial}{\partial y_r}\varphi(0; \underbrace{(0, y_2, \dots y_n)}_{\begin{pmatrix} 0 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}})\Big|_{y_2, \dots, r = 0} = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ r\text{-th row} \\ \\ \\ \end{array}$$

$r = 2, \dots n$

hence: $DG(0) = \begin{pmatrix} \alpha & ? & ? & \cdots & ? \\ 0 & 1 & & & \\ | & & \ddots & & 0 \\ | & 0 & & & \\ 0 & & & & 1 \end{pmatrix}$ .. regular matrix

by IFT (inverse / implicit function theorem)

W small enough $\Rightarrow$ $G : W \to U$ .. neigh of $G(0) = 0$

injective; $G^{-1} : U \to W$ is C^r.

STEP 3: $g := G^{-1}$ is the sought-for diffe ($\sim C^r$).

$x(t)$ solves (1) in U $\Leftrightarrow$ $y(t) = g(x(t))$ solves (2) in W

$\Leftarrow$ $y(t)$ solves (2) in W; i.e. $y_1' = 1$

$y_k' = 0$; $k = 2, \dots, n$

i.e. $y_1 = t + c_1$

$y_k = c_k$, $k = 2, \dots, n$ $c_j \in \mathbb{R}$ constants

$y(t) = g(x(t)) \Leftrightarrow x(t) = G(y(t))$

$= \varphi(t + c, (0, c_2, \dots, c_n))$

$= \tilde{x}(t + c)$; where $x(t)$ solves

(1) for initial cond.

$t \in I \subset \mathbb{R}$ interval

$\Rightarrow$ $x(t)$ solves (1) in V; set $y(t) = g(x(t)) = G_{-1}(x(t))$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

by def. of G_{-1}: $(t, y_2, \dots, y_n)$

s.t.

$x(t) = \varphi(t, (0, y_2, \dots, y_n))$

$y_1(t) = t$

$y_k(t) \equiv \text{const}$, $k = 2, \dots, n$

__Recall__. x_0 ... hyperbolic stationary point of (1) :: $f(x_0) = 0$

$\mathrm{Re}\,\lambda \neq 0$ $\forall \lambda \in \sigma(A)$

where $A = Df(x_0)$.

__Theorem 13.4__ (Hartman–Grobman.) Let x_0 be hyp. stat. point of (1). Then $\exists$ V, W neigh. of x_0, 0 resp. s.t. dyn. syst. given by (1) and (3) $y' = Ay$ are top. conj. on V, W.

__Rem.__ interesting dynamics: only close to non-hyp. stat. points

[illegible] ODE?