Ordinary Differential Equations

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ZS 17/18
8. ledna 2018

Problem Set 9

9.1. Let I be an interval, q be a continuous function defined in this interval and let y be a non-trivial solution of $y'' + q(t)y = 0$ in I. Consider a number $a > 0$. Show that

(a) if $q(t) \geq a$ in I then the adjacent zero points of y in I are placed at most $\pi/\sqrt{a}$ apart.

(b) if $q(t) \leq a$ in I then the adjacent zero points of y in I are placed at least $\pi/\sqrt{a}$ apart.

9.2. Verify that every solution of $y'' + \dfrac{y}{\sqrt{t}} = 0$ has infinitely many zero points in $(0, \infty)$.

9.3. Let $y'' + p(t)y' + q(t)y = 0$ with continuous p, p' and q. Find a function $u(t)$ such that the substitution $z(t) = u(t)y(t)$ transforms the equation into $z'' + r(t)z = 0$. Express $r(t)$ by means of $p(t)$ and $q(t)$.

9.4. Prove that any non-trivial solution of

(a) $y'' + \sin(t)\, y = 0$ has at most 2 zero points in $[-\pi, \pi]$.

(b) $y'' + 2ty' + 4ty = 0$ has at most 4 zero points in $\mathbb{R}$.

Farewell Gift

Putnam 2/6 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Putnam 2/6 Functions f, g, h are differentiable on some open interval around 0 and satisfy

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad & f(0) &= 1, \\ g' &= fg^2h + \frac{4}{fh}, \quad & g(0) &= 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad & h(0) &= 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Putnam 5/6 Show that there is no strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x.

Putnam 5/6 Let $f : (1, \infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2 + 1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x \to \infty} f(x) = \infty$.

Problem set 9 solutions

9.1 a)

Let t_1, t_2 be 2 adjacent zero points, $t_1 < t_2$

$$t_2 - t_1 > \frac{\pi}{\sqrt{a}} \quad \Rightarrow \exists\, t_0 : t_1 < t_0 < \frac{\pi}{\sqrt{a}} + t_0 < t_2$$

Let $z'' + az = 0$.

$\Rightarrow z(t) = \sin[\sqrt{a}(t - t_0)]$ is a solution

and $z(t_0) = z(t_0 + \frac{\pi}{\sqrt{a}}) = 0$

But $q \geq a \Rightarrow$ there must be a zero point of y

in $[t_0, t_0 + \frac{\pi}{\sqrt{a}}]$ $\lightning$

9.1 b) analogously: $t_1 < t_2$, $t_0 : t_0 < t_1 < t_2 < t_0 + \frac{\pi}{\sqrt{a}}$

$z'' + az = 0$ & $z(t) = \sin[\sqrt{a}(t - t_0)]$

$\lightning$

9.2 $\quad (1,\infty) = \bigcup_{n\in\mathbb{N}} I_n \quad$ for $I_n = (2^{2n-2}, 2^{2n}]$

in I_n we have $\frac{1}{\sqrt{t}} \geq \frac{1}{2^n}$

9.1a)
$\Rightarrow$ adjacent zero points of y in I_n are at most $2^{\frac{n}{2}}\pi$ apart. But $|I_n| = \frac{3}{4} 2^{2n} > 2^{\frac{n}{2}}\pi \quad \forall n > 1$

$\Rightarrow$ There is a zero point of y in each I_n

9.3

$$z'' + rz = 0 \overset{z=uy}{\Rightarrow} (uy)'' + ruy = 0$$

$$\Rightarrow uy'' + 2u'y' + (u'' + ru)y = 0$$

$$\& \quad e^{\frac{1}{2}\int p} y'' + p e^{\frac{1}{2}\int p} y' + q e^{\frac{1}{2}\int p} y = 0$$

$$\Rightarrow \text{let } \underline{\underline{u(t) := e^{\frac{1}{2}\int_0^t p(s)ds}}}$$

$$\Rightarrow 2u' = p e^{\frac{1}{2}\int p}$$

$$q e^{\frac{1}{2}\int p} = u'' + ru$$

$$= \frac{1}{2} p' e^{\frac{1}{2}\int p} + \frac{1}{4} p^2 e^{\frac{1}{2}\int p} + r e^{\frac{1}{2}\int p}$$

$$\Rightarrow \underline{\underline{r(t) = q(t) - \frac{1}{2}p'(t) - \frac{1}{4}p^2(t)}}$$

9.4 a) $\sin(t) \leq 1 \overset{9.1\,b)}{\Longrightarrow}$ the zero points are placed at least π apart

$\Rightarrow$ The only possibility for there being more than 2 zero points would be if

$y(-\pi) = y(0) = y(\pi) = 0 \quad (*)$

But we know $q(t) \leq 0 \Rightarrow$ there is at most 1 zero point

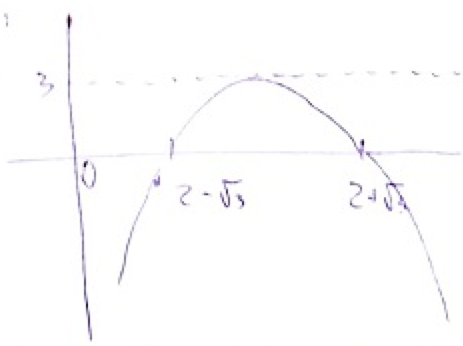
$\Rightarrow$ in $[-\pi, 0]$ there can be at most 1 zero point

$\Rightarrow (*)$ impossible!

9.4 b) Using 9.3: $y'' + 2ty' + 4ty = 0 \rightsquigarrow z'' - (t^2 - 4t + 1)z = 0$

with $z(t) = 0 \Leftrightarrow y(t) = 0$

$-t^2 + 4t - 1$:



$\mathbb{R} = (-\infty, 2-\sqrt{3}] \cup (2-\sqrt{3}, 2+\sqrt{3}) \cup [2+\sqrt{3}, \infty)$

- $(-\infty, 2-\sqrt{3}]$: $q \leq 0 \Rightarrow$ at most 1 zero point
- $[2+\sqrt{3}, \infty)$: $q \leq 0 \Rightarrow$ at most 1 zero point
- $(2-\sqrt{3}, 2+\sqrt{3})$: $q \leq 3 \overset{9.1\,b)}{\Longrightarrow}$ at most 2 zero points

since there couldn't squeeze in more than 2 zero points at least $\frac{\pi}{\sqrt{3}}$ apart in $(2-\sqrt{3}, 2+\sqrt{3})$