Dalibor Pražák, Josef Žabenský
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Problem Set 7

7.1. Solve the equations:

(a) $x' = x^2/y$, $y' = x$ $(x, y > 0)$

(b) $x' = y^2$, $y' = yz$, $z' = -z^2$ $(y, z > 0)$

7.2. Find a first integral for $x' = x + y$, $y' = x^2 - y^2$.

Hint: Begin with computing $y = y(x)$.

7.3. Consider the equation $x'' + \sin x = 0$.

(a) Find a nontrivial function $V(x, x')$ that is constant along each solution.

(b) Write the equation as a system of 2 first order equations and draw the phase portrait for this system.

(c) Give a formula for the period of periodic solutions as a function of a given amplitude a.

7.4. Let $(x(t), y(t))$ solve a 2-dimensional autonomous system

$$\begin{aligned} x'(t) &= f(x(t), y(t)) \\ y'(t) &= g(x(t), y(t)), \end{aligned}$$

in the neighbourhood of t_0 for continuous f and g. Let $f(x(t_0), y(t_0)) \neq 0$ and denote $\tau(x)$ the inverse of $x(t)$ in the neighbourhood of $x(t_0)$. Then function $\hat{y}(x) := y(\tau(x))$ solves

$$\frac{d\hat{y}}{dx}(x) = \frac{g(x, \hat{y}(x))}{f(x, \hat{y}(x))},$$

in the neighbourhood of $x(t_0)$.

7.5. Derive Taylor's formula with the integral form of the remainder

$$x(t) = \sum_{j=0}^{n} \frac{x^{(j)}(t_0)}{j!}(t - t_0)^j + \frac{1}{n!}\int_{t_0}^{t} x^{(n+1)}(s)(t - s)^n \, ds$$

for $x \in C^{n+1}$ from the variation of constants for an autonomous equation of n-th order.

Hint: $x^{(n+1)} = x^{(n+1)}$.

7.6. (*easy and completely optional*) Consider a *companion matrix*

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ -c_0 & -c_1 & \cdots & \cdots & -c_{n-1} \end{pmatrix}.$$

(a) Show that $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$ is the characteristic polynomial of A.

(b) Show that the geometric multiplicity of every eigenvalue of A is one. What does it tell you about its Jordan canonical form? (The geometric multiplicity is dimension of the corresponding eigenspace. It is always smaller than or equal to the algebraic multiplicity.)

7.7. **Food for thought:** Let x be a C^2 function and $x''(t) + x'(t) + x(t) \to 0$ for $t \to \infty$. Show that then $x(t) \to 0$ for $t \to \infty$.

Hint: Use the foxy hint from 7.5 for inspiration.

Problem 7 solutions

7.1 a) $x' = \dfrac{x^2}{y}$

$y' = x$ $\quad , x, y > 0$

Solution: $x \neq 0 \Rightarrow$ we can divide the equations to obtain

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\ln|y(x)| = \ln|x| + c \quad , c \in \mathbb{R}$$

$$x, y > 0 \Rightarrow y(x) = cx, \quad c > 0$$

$$\Rightarrow x' = \frac{x^2}{y} = \frac{x}{c} \Rightarrow \boxed{\begin{array}{ll} x = \tilde{c}e^{\frac{t}{c}} & \tilde{c} \in \mathbb{R} \\ y = \tilde{c}c\,e^{\frac{t}{c}}, & c > 0 \\ & t \in \mathbb{R} \end{array}}$$

b) $x' = y^2$

$y' = yz$ $\quad , y, z > 0$

$z' = -z^2$

Solution: $z \neq 0 \Rightarrow \dfrac{dy}{dz} = -\dfrac{y}{z}$

$$\ln(y(z)) = -\ln(z) + c \quad , c \in \mathbb{R}$$

$$y, z > 0 \Rightarrow y(z) = \frac{c}{z}, \quad c > 0 \qquad (*)$$

$$z' = -z^2 \Rightarrow -\frac{z'}{z^2} = 1$$

$$\frac{1}{z(t)} = t + \tilde{c}, \quad \boxed{\tilde{c} \in \mathbb{R}, \; t > -\tilde{c}}$$

$$\Rightarrow \boxed{z(t) = \frac{1}{t + \tilde{c}}, \quad y(t) = c\,(t + \tilde{c})}$$

$$x' = c(t+\tilde{c})^2 = ct^2 + 2c\tilde{c}t + c\tilde{c}^2$$

$$\Rightarrow \boxed{x(t) = \frac{c}{3}t^3 + c\tilde{c}t^2 + c\tilde{c}t + \hat{c} \, , \quad \hat{c} \in \mathbb{R}}$$

<u>7.2</u> $x' = x + y$

$y' = x^2 - y^2$

<u>Solution:</u>

$x = -y \Rightarrow$ stat. point

Let $x + y \neq 0 \Rightarrow \frac{dy}{dx} = x - y$

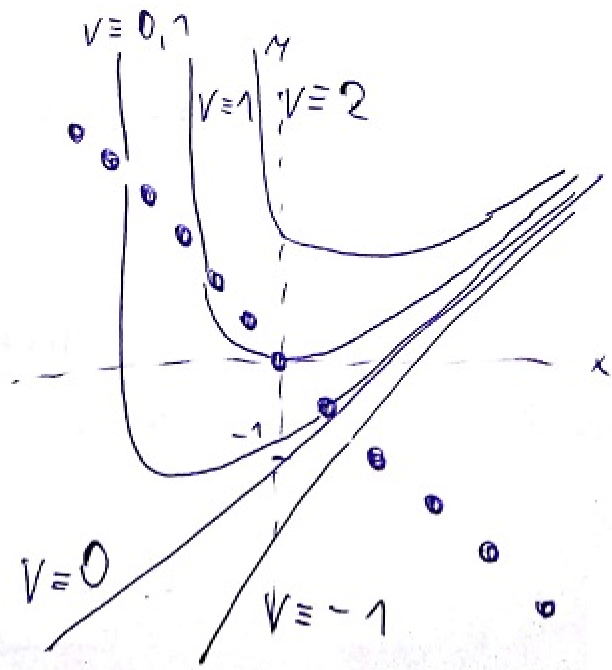
$$\Rightarrow \frac{d}{dx}\left(y(x)e^x\right) = e^x \cdot x$$

$$\Rightarrow y(x) = e^{-x}\left(xe^x - e^x + c\right)$$

$$= x - 1 + c \cdot e^{-x} \, , \quad c \in \mathbb{R}$$

$$\Rightarrow (y - x + 1)\, e^x = c =: V(x,y)$$

<u>Note:</u>



7.3 $x'' + \sin x = 0$

Solution

a) $\cdot x' \Rightarrow \quad x''x' + x'\sin x = 0$

$$\left(\frac{(x')^2}{2} - \cos x\right)' = 0$$

$$\Rightarrow V(x, x') = \frac{(x')^2}{2} - \cos x$$

b) $x' = y$

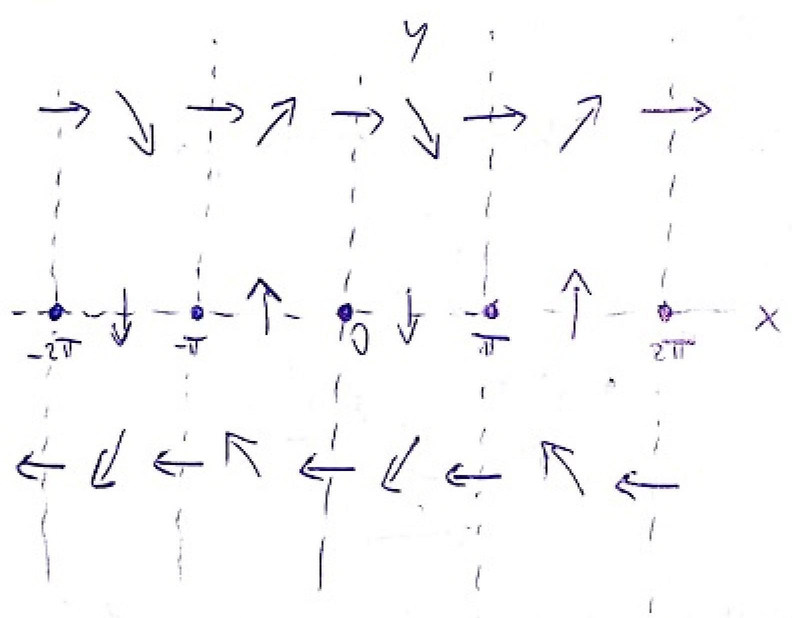
$y' = -\sin x$

$\Rightarrow V(x,y) = \frac{y^2}{2} - \cos x = c$ along solutions

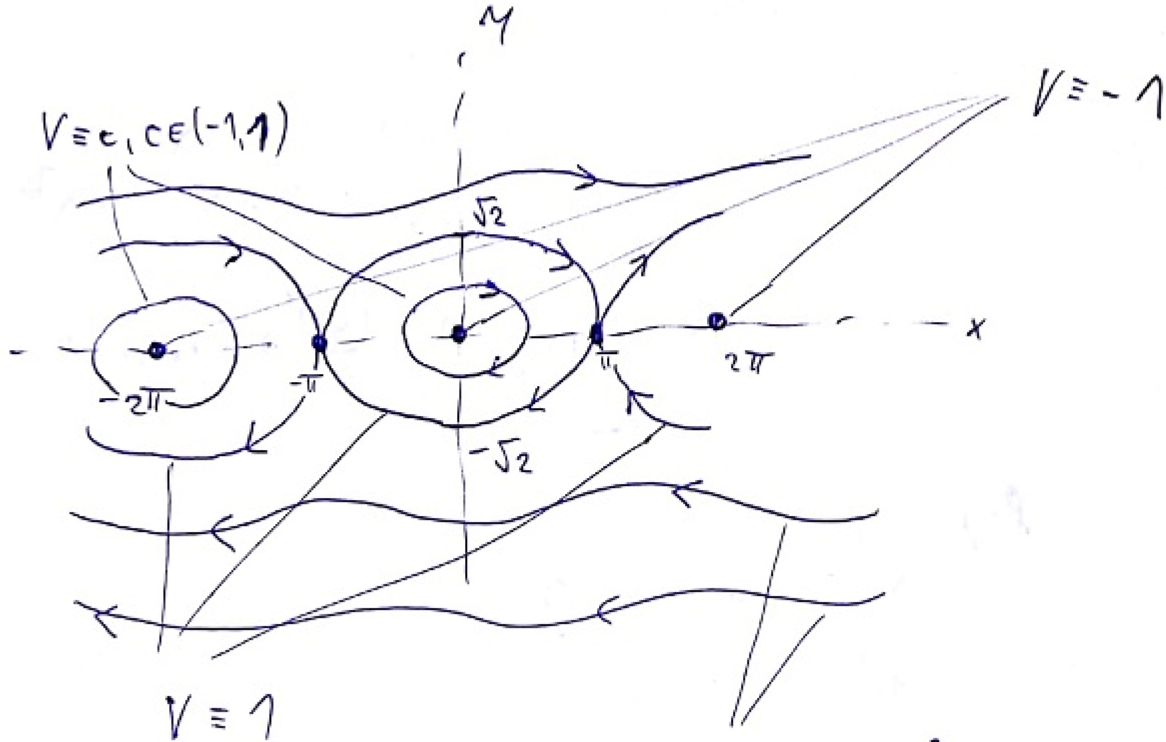
$c \geq -1$

$\Rightarrow y^2 = 2c + 2\cos x$

directions only:

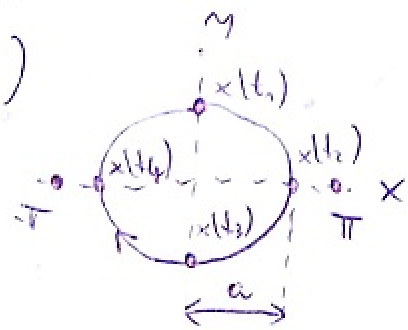


entire trajectories



c) from ↗ we can see that periodic solutions are those, for which $V\equiv c,\quad c\in(-1,1)$
(if $V\equiv 1$, the corresponding solutions cannot pass x-axis)

$\Rightarrow$ the amplitude $a\in(0,\pi)$



Recall Barrow's formula to measure time:
Between t_1 and t_2 we have

$$x' = y = \sqrt{2c+2\cos x} = \sqrt{2}\sqrt{\cos x-\cos a}$$

$c=-\cos a$

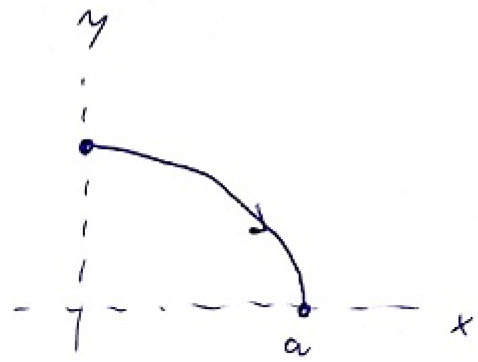
$$\Rightarrow \frac{x'}{\sqrt{2}\sqrt{\cos x-\cos a}} = 1 \quad \Big|\int_{t_1}^{t_2} dt$$

$x(t_1)=0$
$x(t_2)=a$

$$\Rightarrow \int_0^a \frac{ds}{\sqrt{2}\sqrt{\cos s-\cos a}} = t_2-t_1$$

$\Rightarrow$ time needed for travelling

is $\displaystyle\int_0^a \frac{ds}{\sqrt{2}\sqrt{\cos s - \cos a}}$



for other 3 parts of the orbit, we obtain the same result

T := period

$$\Rightarrow \quad T = 4 \int_0^a \frac{ds}{\sqrt{2}\sqrt{\cos s - \cos a}}$$

7.4

$x' = f(x(t_0), y(t_0)) \neq 0 \Rightarrow \exists\, \tau(x) \in C^1(\mathcal{U}(x(t_0)))$

$$\& \quad \frac{d\tau}{dx}(x) = \frac{1}{x'(\tau(x))} = \frac{1}{f(x(\tau(x)), y(\tau(x)))}$$
$$= \frac{1}{f(x, \hat{y}(x))}$$

derivative of composition $y \circ \tau$

$$\frac{dy}{dx}(\tau(x)) = \frac{dy}{d\tau}(\tau(x)) \cdot \frac{d\tau}{dx}(x) = \frac{g(x, \hat{y}(x))}{f(x, \hat{y}(x))}$$

7.5 Let $y(t)$ be an unknown function solving

$$y^{(m+1)} = f := x^{(m+1)}$$

$$y(t_0)=x(t_0),\ y'(t_0)=x'(t_0), \dots, y^{(m)}(t_0) = x^{(m)}(t_0)$$

1) the equation for y has a unique solution (theory for this kind of eq's) and $y \equiv x$ clearly is a solution $\Rightarrow$ $\underline{y(t) \equiv x(t)}$

2) by the variation of constants formula, we know (WLOG $t_0 = 0$)

$$y(t) = \underbrace{y_h}(t) + \int_0^t u(t-s) f(s)\,ds,$$

- where $\leftarrow$ solves the hom. eq with the same init. conditions

$\Rightarrow$ $y_h^{(m+1)} = 0$ $\Rightarrow$ y_h is a polynomial of at most m-th order

& $y_h(0) = x(0), \dots, y_h^{(m)}(0) = x^{(m)}(0)$

$$\Rightarrow y_h(t) = \sum_{j=0}^{m} \frac{x^{(j)}(0)}{j!} t^j$$

- and $u(t)$ solves the $\underbrace{\text{hom eq.}}_{u^{(m+1)}=0}$ with $u(0) = \dots = u^{(j-1)}(0) = 0,\ u^{(j)}(0) = 1$

$$\Rightarrow u(t) = \frac{t^m}{m!}$$

- and $f(t) = x^{(m+1)}(t)$ $\Rightarrow$ $x(t) = y(t) = \sum_{j=0}^{m} \frac{x^{(j)}(0)}{j!} t^j + \int_0^t \frac{(t-s)^m}{m!} x^{(m+1)}(s)\,ds$

7.6

a) compute $\det(A-\lambda I)$ by expanding it with respect to the last row

$$\Rightarrow \chi_A(\lambda) = c_0(-1)^{n+2} + c_1\lambda(-1)^{n+4} + \dots + \lambda^n(-1)^{n+2n}$$

$$= p(\lambda)\cdot(-1)^n \qquad \text{(modulo sign :))}$$

b) let $\lambda \in \sigma(A)$

geom multiplicity = $\dim \ker(A-\lambda I) = n - \operatorname{rank}(A-\lambda I)$

1) $\lambda \in \sigma(A) \Rightarrow \operatorname{rank}(A-\lambda I) \leq n-1$

2) $\operatorname{rank}(A-\lambda I) \geq n-1$:

$$A-\lambda I = \begin{pmatrix} -\lambda & 1 & & & 0 \\ & -\lambda & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ & & & -\lambda & 1 \\ -c_0 & -c_1 & \cdots & -c_{n-2} & -c_{n-1}-\lambda \end{pmatrix}$$

consider $(n-1)\times(n-1)$ matrix created by deleting 1'st column and n'th row of $A-\lambda I$

$$\Rightarrow \begin{pmatrix} 1 & & 0 \\ -\lambda & 1 & \\ & \ddots & \ddots \\ & & -\lambda \; 1 \end{pmatrix}. \quad \text{This matrix has } \det = 1$$

$\Rightarrow \operatorname{rank}(A-\lambda I) \geq n-1$

$\Rightarrow$ in the Jordan canonical form of A there is 1 Jordan block for λ whose size is at least 1 $\Rightarrow$ there is exactly one Jordan block for λ and its size must be equal to the alg. multiplicity