Problem Set 6

6.1. Using the polar coordinates, determine if the origin is a stable or asymptotically stable equilibrium of

$$\begin{aligned} x' &= -2y + ax\sqrt{x^2+y^2} \\ y' &= 2x + ay\sqrt{x^2+y^2} \end{aligned}$$

depending on $a \in \mathbb{R}$.

6.2. (a) Decide about stability of solutions (not only stationary points) to

$$\begin{aligned} x' &= -(x^2+y^2)y \\ y' &= (x^2+y^2)x. \end{aligned}$$

Note: Recall that a solution $\bar{x}$ is called *(Lyapunov) stable* if for each $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that if $x(t)$ is a solution and $|x(t_0) - \bar{x}(t_0)| < \delta$ then $|x(t_0) - \bar{x}(t_0)| < \varepsilon$ for all $t \geq t_0$.

(b) If the solutions are not stable, can you define your own notion of stability here so that the previously unstable solutions become stable in this new sense of yours?

Note: Unfortunately for you, this notion already has a name – *orbital stability.*

6.3. Draw the phase portrait of the following equation given in polar coordinates:

$$\begin{aligned} r' &= r(1-r) \\ \varphi' &= \sin^2(\varphi/2). \end{aligned}$$

Are the stationary solutions stable or locally attractive? Can you find a connection to Vinogradov's example?

6.4. Sketch the phase portrait of the linearized system in the neighbourhood of equilibria:

$$\begin{aligned} x' &= 2x + y^2 - 1 \\ y' &= \sin(x) - y^2 + 1. \end{aligned}$$

What phase portraits do you expect from the original system around these points?

6.5. Food for thought: Three gods A, B and C are called, in no particular order, True, False and Random. True always speaks truly, False always speaks falsely, but whether Random speaks truly or falsely is a completely random matter every time he speaks. Your task is to determine the identities of A, B and C by asking three yes-no questions; each question must be put to exactly one god. A single god may be asked more than one question and questions are permitted to depend on the answers to earlier questions. The gods understand Czech, but will answer all questions in their own language, in which the words for *yes* and *no* are *brrr* and *grrr*, in some order. You do not know which word means which.

Problem set 6 solutions

6.1 $x(t) = r(t)\cos\varphi(t)$
$y(t) = r(t)\sin\varphi(t)$

$$\Rightarrow \quad x' = r'\cos\varphi - r\varphi'\sin\varphi = -2r\sin\varphi + a r^2\cos\varphi$$
$$y' = r'\sin\varphi + r\varphi'\cos\varphi = 2r\cos\varphi + a r^2\sin\varphi$$

$$\Rightarrow \quad r' = a r^2$$

$\Rightarrow$ I. $a<0 \Rightarrow r \to 0,\ t\to\infty \Rightarrow$ origin is asymptotically stable

II. $a>0 \Rightarrow r\to\infty,\ t\to\infty \Rightarrow$ origin is unstable

III. $a=0 \Rightarrow r \equiv r_0 \Rightarrow$ origin is stable

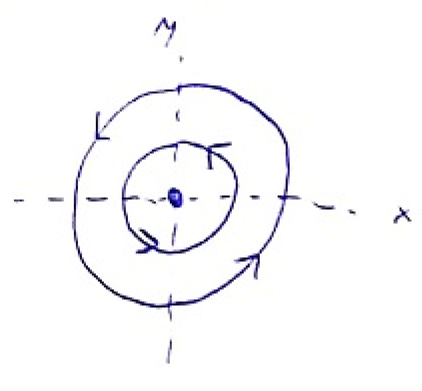
6.2a) polar coordinates again:

$$r'\cos\varphi - r\varphi'\sin\varphi = -r^3\sin\varphi$$
$$r'\sin\varphi + r\varphi'\cos\varphi = r^3\cos\varphi$$

$$\Rightarrow \left.\begin{aligned} r' &= 0 \\ r\varphi' &= r^3 \end{aligned}\right\} \overset{\text{WLOG } r\neq 0}{\Longleftrightarrow} \begin{aligned} r' &= 0 \\ \varphi' &= r^2 \end{aligned}$$

(((WLOG = BÚNO)))

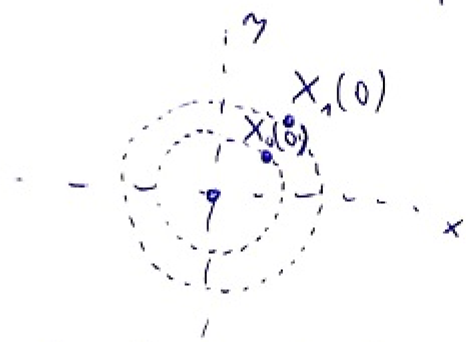
$\Rightarrow$



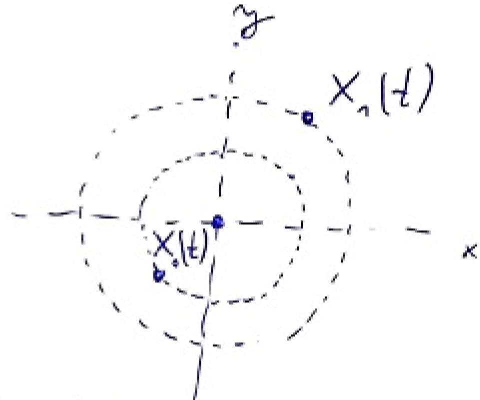
the trajectories of solutions keep a constant distance between them but the solutions cruise them at different angular velocities

$r^2 = \varphi' =$ angular velocity $\Rightarrow$ the corresponding solution completes it's orbit in $\frac{2\pi}{r^2}$ time units.

$\Rightarrow$ suppose we have two points $X_0(0), X_1(0)$



then in finite time t it will happen that $X_0(t)$ is in the 1^{st} quadrant, while $X_1(t)$ is in the 3^{rd} quad.



$\Rightarrow$ a non-zero solution is not stable since if we start near it, in finite time we lose the initial closeness

i.e. $\exists\, \varepsilon > 0 \ \exists\, t_0 \in \mathbb{R} \ \forall \delta > 0 \ \exists\, t \geq t_0$:

$$|x(t_0) - \bar{x}(t_0)| < \delta \wedge |x(t) - \bar{x}(t)| \geq \varepsilon$$

zero solution, on the other hand, stable.

6.2 b)

We want to capture the situation in which the trajectories are close to each other from certain moment on:

$\bar{x}(t)$ is orbitally stable $\equiv$

$$\forall t_0, t_1 \in \mathbb{R} \ \ \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall t \geq t_1$$

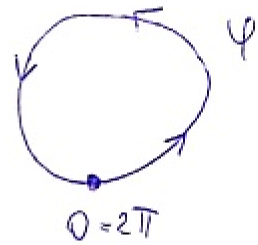
$$|x(t_1) - \bar{x}(t_0)| < \delta \Rightarrow x(t) \in \underbrace{\bigcup_{\bar{t} \geq t_0} B(\bar{x}(\bar{t}), \varepsilon)}_{\varepsilon\text{-neighbourhood of } \bar{x}(t) \text{ since } t_0}$$

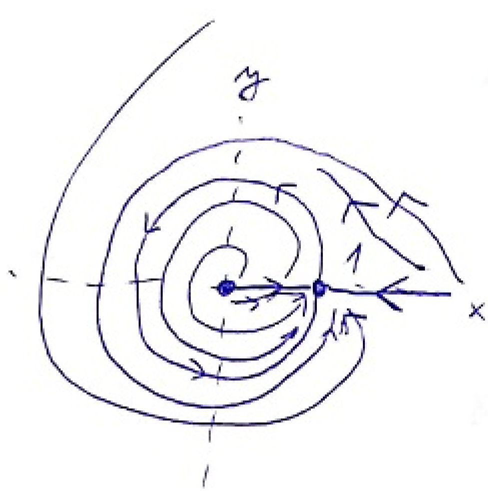
6.3

$r' = r(1-r)$

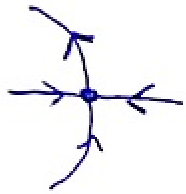
! bounded solutions of autonomous equations converge always to stationary points.

$\Rightarrow$ 

$\varphi' = \sin^2(\varphi/2)$ $\Rightarrow$ 

$\Rightarrow$ 

Connection to Vinogradov: solution $(x,y) \equiv (1,0)$ is not stable but it is a local attractor nonetheless!



it would have been global, but the point $(0,0)$ stays still

6.4

equilibria: $0 = 2x + y^2 - 1$, $0 = \sin(x) - y^2 + 1$ $\iff (x,y) \in \{(0,1),(0,-1)\} =: \{X_0, X_1\}$

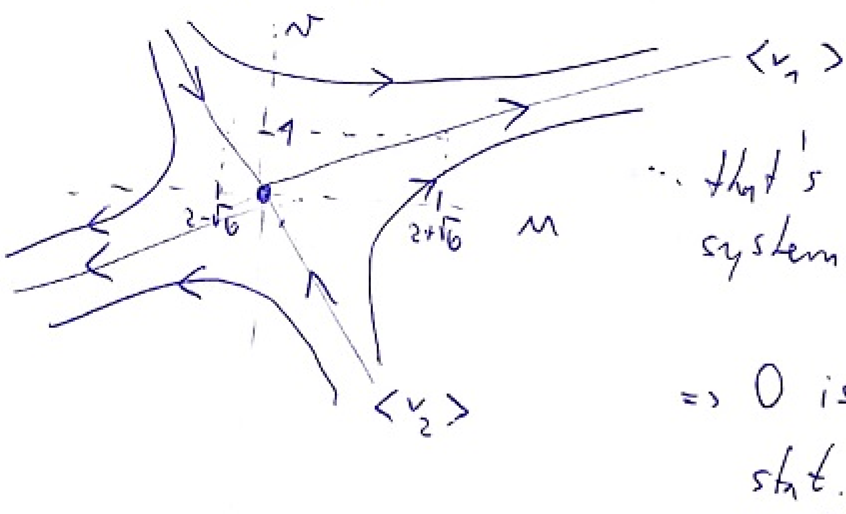
I: X_0: $Df(X_0) = \begin{pmatrix} 2 & 2y \\ \cos(x) & -2y \end{pmatrix}\Bigg|_{(x,y)=(0,1)} = \begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}$

$\det(Df(x_0) - \lambda I) = \lambda^2 - 6 \Rightarrow \lambda_1 = \sqrt{6}$, $\lambda_2 = -\sqrt{6}$

$\Rightarrow X_0$ is a hyp. stat. point and we can use H-G theorem to deduce its stability from $U' = Df(X_0)U$, $U = (u,v)$

$v_1 = \begin{pmatrix} 2+\sqrt{6} \\ 1 \end{pmatrix}$ ... unstable subspace $(\lambda_1 > 0)$

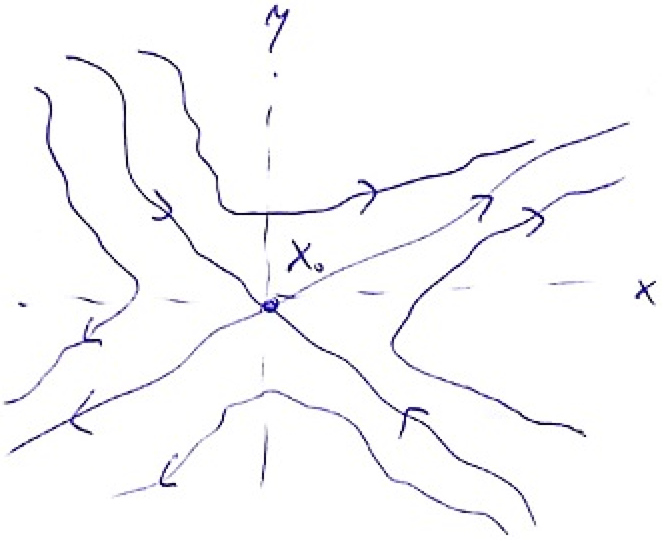
$v_2 = \begin{pmatrix} 2-\sqrt{6} \\ 1 \end{pmatrix}$ ... stable subspace $(\lambda_2 < 0)$



... that's how the linearized system looks

$\Rightarrow 0$ is an unstable stat. point of the linearized system

H-G theorem $\Rightarrow$ X_0 is unstable for the original system and we can expect roughly



Note that $X_0 + \langle v_1 \rangle$ and $X_0 + \langle v_2 \rangle$ are tangents of the respective "curly" invariant spaces at X_0

II. X_1: $\quad Df(X_1) = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}$

$\lambda_1 = 2 + i\sqrt{2} \qquad v_1 = \begin{pmatrix} -2 \\ i\sqrt{2} \end{pmatrix}$

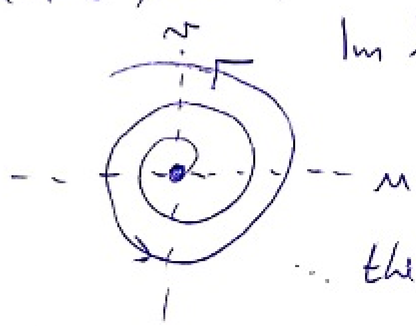
$\lambda_2 = 2 - i\sqrt{2} \qquad v_2 = \begin{pmatrix} -2 \\ -i\sqrt{2} \end{pmatrix}$

$\Rightarrow$ X_1 is also a hyp stat point

$\Rightarrow$ we can investigate behavior of the linearized system near 0 to deduce behavior of the original system near X_1

$U' = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} U$, $\quad \operatorname{Re} \lambda_{1,2} > 0 \Rightarrow 0$ is unstable

$\operatorname{Im} \lambda_{1,2} \neq 0 \Rightarrow$ see 5.1 b (a whirlpool!)



... the linearized system near 0

⌈ Recall that if $A \in \mathbb{R}^{2\times 2}$, $\operatorname{Im} \sigma(A) \neq 0 \Rightarrow \lambda_1 = \overline{\lambda_2}$

and $A \sim \begin{pmatrix} \operatorname{Re} \lambda_1 & -\operatorname{Im} \lambda_1 \\ \operatorname{Im} \lambda_1 & \operatorname{Re} \lambda_1 \end{pmatrix}$ ⌋

=> X_1 is an unstable stat. point of the original system and although it can be shown it also looks whirlpool-like, all that we can say with our tools is that it probably looks like