Ordinary Differential Equations ZS 17/18

Dalibor Pražák, Josef Žabenský 20. listopadu 2017

Problem Set 5

5.1. Draw the phase portraits of linear system $x' = Ax$, where

(a) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, with $a > 0$ and $b \lesseqgtr 0$, respectively. Differentiate additionally the cases $|b| \lesseqgtr |a|$, i.e. you should produce 7 pictures in total (minus the cases covered by me).

(b) $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, with $b > 0$ and $a \lesseqgtr 0$, respectively.

Hint: Solve the equation first.

5.2. Find stationary points and decide about their stability:

$$\begin{aligned} x' &= xy - 2x - y + 2, \\ y' &= xy + yz + xz, \\ z' &= 2y(z + 1). \end{aligned}$$

5.3. Show that for large μ the system

$$\begin{aligned} x' &= \sin(x) + \cos(y) - \exp(\mu y), \\ y' &= -\sin(2y) + \frac{x}{1 + y^2}, \end{aligned}$$

has a stable equilibrium at the origin. For which μ will be the origin unstable?

5.4. For $A \in \mathbb{R}^{2\times 2}$ characterize $\operatorname{Re} \sigma(A) < 0$ by means of $\det(A)$ and $\operatorname{tr}(A)$. Do not use the Hurwitz theorem for this task.

Hint: When do the roots of $x^2 + px + q$ have a strictly negative real part?

5.5. Consider a polynomial of degree 4 with a positive lead coefficient. What conditions on its coefficients guarantee that all its roots have a strictly negative real part?

5.6. Food for thought: An evil wizard has imprisoned 100 dwarves. Each dwarf wears a hat and the wizard colours it randomly with one of three colours. Each dwarf can see everybody else's hat except for his own. The wizard asks each dwarf his hat colour and should the answer be incorrect, the unlucky dwarf dies. Find a way to save at least 99 dwarves.

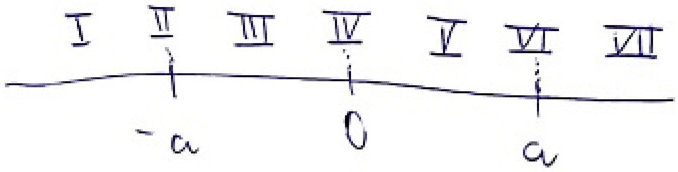
A note regarding the rules: The dwarves are smart and, perhaps by a divine inspiration, they had foreseen the ordeal could discuss the strategy in advance. Once imprisoned, however, they can communicate no longer and their answer to the wizard consists purely of the hat colour. Still, they can see everybody else's hats and hear everybody elses's answers.

Problem set 5 solution

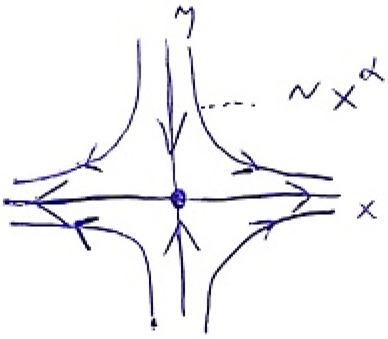
5.1a)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{at} x_0 \\ e^{bt} y_0 \end{pmatrix} \Rightarrow y(t) \overset{x_0 \neq 0}{=} \left(x(t)\right)^{\frac{b}{a}} y_0 \, x_0^{-\frac{b}{a}}$$

cases in terms of location of b:

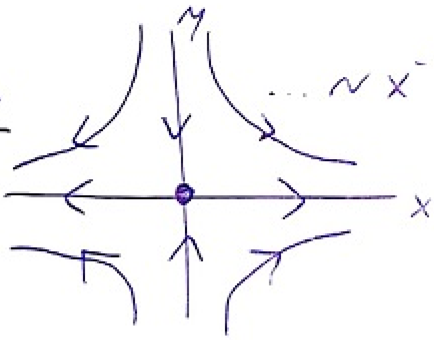


I.



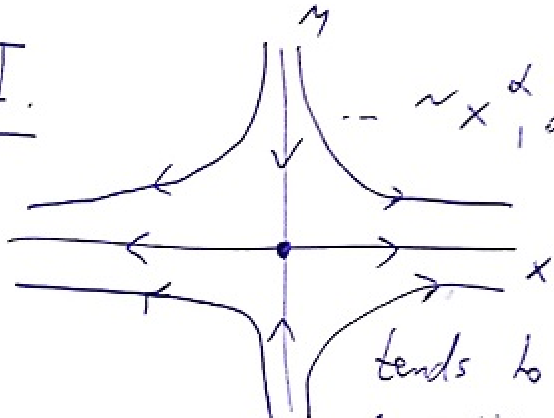
$\sim x^{\alpha}$, $\alpha < -1$ $\Rightarrow$ tends to the x-axis quicker then x^{-1} for $x > 1$

II.



$\sim x^{-1}$

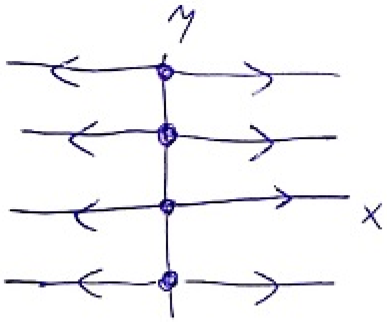
III.



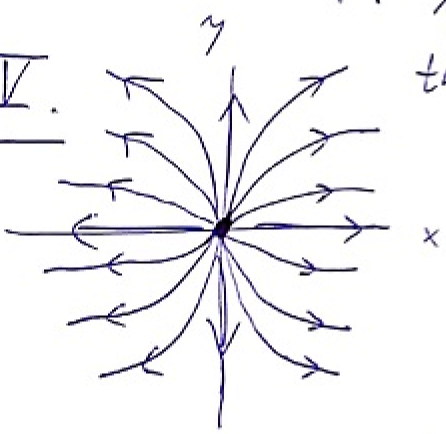
$\sim x^{\alpha}$, $\alpha \in (-1, 0)$

tends to the y-axis quicker then x^{-1} for $x < 1$

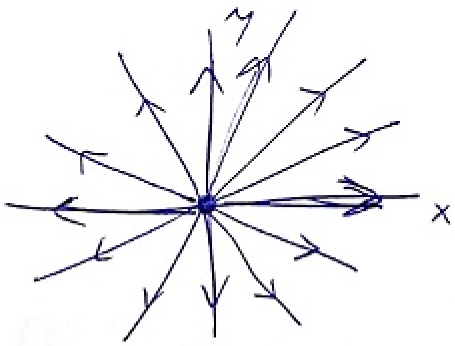
IV.



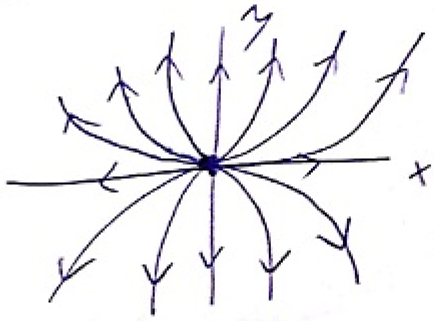
V.



VI.



VII.



5.1 b)

$$X(t) = e^{At} X_0$$

$$A = a \underbrace{I}_{} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{} \quad , \quad B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

commuting matrices

$$\Rightarrow e^{At} = e^{aIt} \cdot e^{bBt} = e^{at} \cdot e^{bBt}$$

$$B^2 = -I \Rightarrow e^{bBt} = \sum_{n=0}^{\infty} \frac{B^n b^n t^n}{n!} =$$

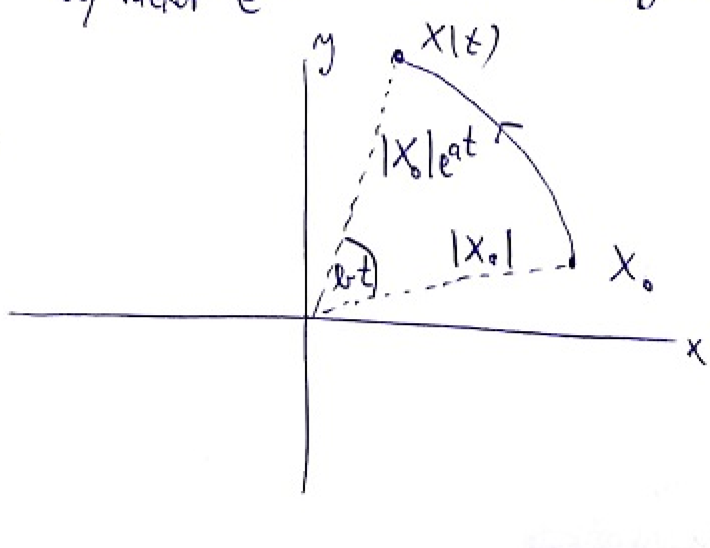
$$= \sum_{k=0}^{\infty} \frac{(-I)^k b^{2k} t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k b^{2k+1} t^{2k+1}}{(2k+1)!} B$$

$$= \cos(bt) I + \sin(bt) B$$

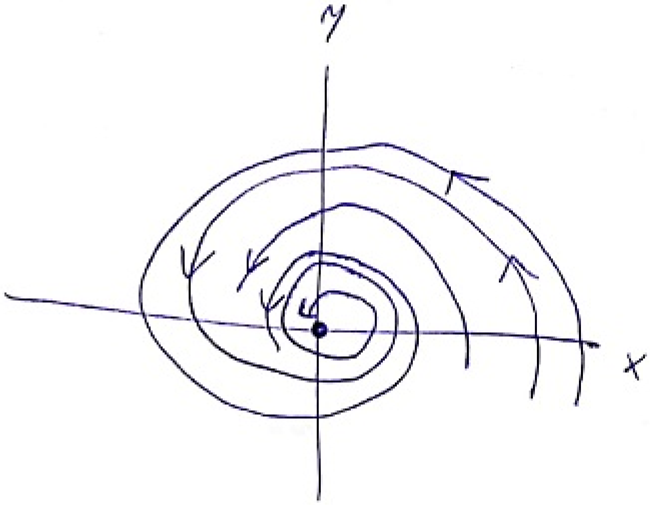
$$= \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$$

$$\Rightarrow X(t) = \underbrace{e^{at}}_{\substack{\text{scaling} \\ \text{by factor } e^{at}}} \underbrace{\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}}_{\text{rotation by angle } bt} X_0$$

$\Rightarrow$

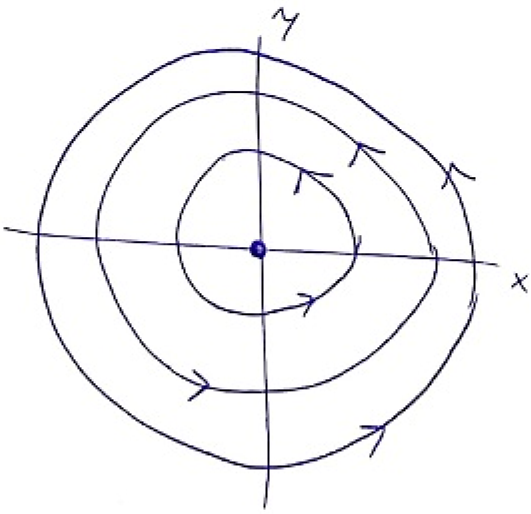


$\Rightarrow$ I. $a < 0$



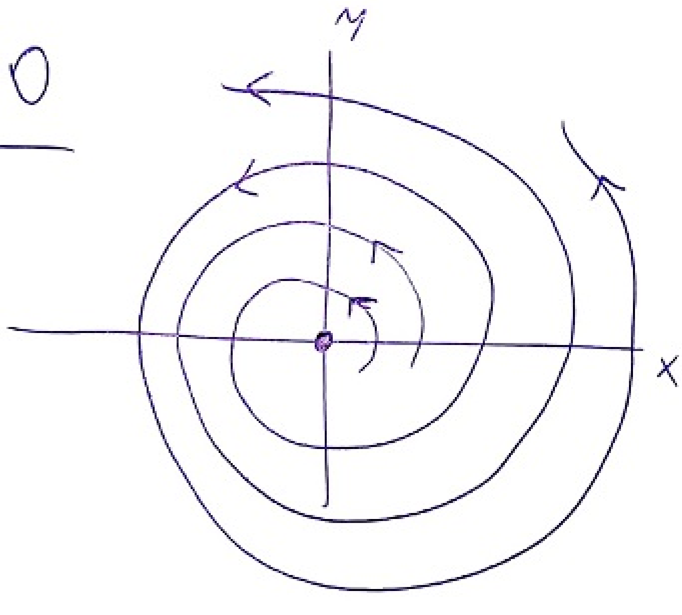
trajectories are logarithmic spirals tending to 0 for $t \to \infty$

II. $a = 0$ ... $|X(t)| = |X_0|$ $\forall t$



trajectories are concentric circles with period $\frac{2\pi}{b}$

III. $a > 0$



trajectories are logarithmic spirals tending to ∞ for $t \to \infty$

5.2 stationary points:

$$0 = xy - 2x - y + 2$$
$$0 = xy + yz + xz \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \Big\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{x_0}, \underbrace{\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}}_{x_1} \Big\}$$
$$0 = 2y(z+1)$$

we want to investigate stability by means of the Hartman-Grobman theorem.

! We need to check the given stat. points are hyperbolic

$$\nabla f = \begin{pmatrix} y-2 & x-1 & 0 \\ y+z & x+z & y+x \\ 0 & 2z+2 & 2y \end{pmatrix}$$

I. x_0

$$\nabla f(x_0) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\det(\nabla f(x_0) - \lambda I) = \lambda(2+\lambda)(1-\lambda) + 2(2+\lambda)$$
$$= -(\lambda+2)(\lambda-2)(\lambda+1)$$

$\sigma(\nabla f(x_0)) \cap \{\operatorname{Re} z = 0\} = \emptyset \Rightarrow x_0$ is hyperbolic

$\Rightarrow$ H-G theorem applicable $\Rightarrow$ stability is resolved by Lyapunov's theorem for $\nabla f(x_0)$:

$\sigma(\nabla f(x_0)) \cap \{\operatorname{Re} z > 0\} \neq \emptyset \Rightarrow x_0$ unstable

$\underline{\text{II. } x_1}$ $\quad \nabla f(x_1) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

$$\det(\nabla f(x_1) - \lambda I) = -(\lambda - 4)\left(\lambda - \frac{1+\sqrt{5}}{2}\right)\left(\lambda - \frac{1-\sqrt{5}}{2}\right)$$

$\overset{\text{like before}}{\Longrightarrow}$ $\underline{x_1 \text{ unstable}}$

$\underline{5.3}$

$$\nabla f_\mu = \begin{pmatrix} \cos(x) & -\sin(y) - \mu \exp(\mu y) \\ \frac{1}{1+y^2} & -2\cos(2y) - \frac{2xy}{(1+y^2)^2} \end{pmatrix}$$

$$\nabla f_\mu(0) = \begin{pmatrix} 1 & -\mu \\ 1 & -2 \end{pmatrix}$$

$$\det(\nabla f_\mu(0) - \lambda I) = (\lambda - 1)(\lambda + 2) + \mu$$
$$= \left(\lambda + \frac{1+\sqrt{9-4\mu}}{2}\right)\left(\lambda + \frac{1-\sqrt{9-4\mu}}{2}\right)$$

$\mu \neq \frac{9}{4} \Rightarrow 0$ is a hyp. stat. point $\overset{\text{H-G th.}}{\Longrightarrow}$ stability dictated by Lyapunov's theorem

I. $\underline{\mu > \frac{9}{4}} \Rightarrow \operatorname{Re} \lambda_{1,2} = -\frac{1}{2} < 0 \Rightarrow 0$ is asymptotically stable (in particular for "large μ")

II. $\lambda_2 := -\frac{1}{2} + \frac{\sqrt{9-4\mu}}{2} > 0 \Leftrightarrow \underline{\underline{\mu < 2}} \Rightarrow 0$ is unstable

III. $\underline{\mu = \frac{9}{4}}$: we don't know!

5.4

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = \lambda^2 - \lambda(a+d) + ad - bc$$
$$= \lambda^2 - \lambda\, tr(A) + \det(A)$$

Vieta's formulas: $\lambda_1 + \lambda_2 = tr(A)$
$\lambda_1 \cdot \lambda_2 = \det(A)$

$\Rightarrow Re\,\sigma(A) < 0 \iff [tr(A) < 0 \;\&\; \det(A) > 0]$

5.5 (Just a mindless application of the Hurwitz theorem – practice makes perfect!)

$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \;,\; a_0 > 0$

$$A = \begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

$\det_1 = a_1 > 0$

$\det_2 = a_1 a_2 - a_0 a_3 > 0$

$\det_3 = a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 > 0$

$\det_4 = a_4 \cdot \det_3 > 0 \iff a_4 > 0$