Ordinary Differential Equations
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Problem Set 4

4.1. Solve the systems corresponding to the following matrices:

(a) $A = \begin{pmatrix} -2 & -3 \\ 6 & 7 \end{pmatrix}$

(b) $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

4.2. Solve

$$\begin{aligned} x' &= -y - t, \qquad & x(0) &= 1, \\ y' &= x + t, \qquad & y(0) &= 0. \end{aligned}$$

Calculating the integral is completely voluntary.

4.3. Find a 2×2 matrix such that $(x(t), y(t)) = (\sinh(t), e^t)$ is a solution.

4.4. Which of the following functions $(x(t), y(t))$

(a) $(3e^t + e^{-t}, e^{2t})$

(b) $(3e^t + e^{-t}, e^t)$

(c) $(3e^t + e^{-t}, te^t)$

(d) $(3e^t, t^2 e^t)$

(e) $(e^t + 2e^{-t}, e^t + 2e^{-t})$

can be solutions of a first-order autonomous homogeneous system?

4.5. Function $u\colon \mathbb{R} \to \mathbb{R}$ fulfills $u(0) = 0$, $u'(0) = 1$ and $u''(t) \geq -u(t)$ for all $t \in [0, \pi]$. Show that $u(t) \geq \sin(t)$ for all $t \in [0, \pi]$.

Hint: Rewrite the given inequality as $u''(t) + u(t) = f(t)$ with $f \geq 0$ and solve this ODE by means of variation of constants. Recall that you have to work with a first-order ODE in the first place.

4.6. Is there a real matrix A such that

$$\exp(A) = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix}, \quad \alpha, \beta > 0?$$

What if α and β are distinct?

Hint: If desperate, recall $\sigma(\exp(A)) = \exp(\sigma(A))$. If still desperate, try $A = \begin{pmatrix} a & \pi \\ -\pi & a \end{pmatrix}$.

4.7. By means of the matrix exponential show $A^2 = -I \Rightarrow \sigma(A) = \{\pm i\}$ for any complex square matrix A.

Hint: In other words $A^2 = -I \Rightarrow \sigma(\pi A) = \{\pm \pi i\}$.

4.8. **Food for thought:** There are 25 mechanical horses and a single racetrack. Each horse completes the track in a pre-programmed time, and the horses all have different finishing times, unknown to you. You can race 5 horses at a time. After a race is over, you get a list with the order the horses finished, but not the finishing times of the horses. What is the minimum number of races you need to identify the fastest 3 horses?

Problem set 4 solutions

4.1 $x' = e^{At} x_0$, where

a) $e^{At} = C \cdot e^{Dt} \cdot C^{-1}$

?D,C: $\det(A-\lambda I) = \begin{vmatrix} -2-\lambda & -3 \\ 6 & 7-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4$
$= (\lambda-4)(\lambda-1)$

$\Rightarrow \lambda_1 = 1 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = 4 \quad\quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow e^{Dt} = \begin{pmatrix} e^{t} & 0 \\ 0 & e^{4t} \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

b) $e^{At} = C \cdot e^{Dt} \cdot C^{-1}$

?D,C: $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1)$

$\Rightarrow \lambda_1 = -1 \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = 1 \quad\quad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow e^{Dt} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}$

$C = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

4.2: $\begin{pmatrix} x \\ y \end{pmatrix}(t) = e^{tA}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t e^{-sA}\begin{pmatrix} -s \\ s \end{pmatrix} ds\right)$, where

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow A^2 = -I \Rightarrow \forall k \in \mathbb{N}_0 : \begin{array}{l} A^{2k} = (-1)^k I \\ A^{2k+1} = (-1)^k A \end{array}$$

$$\Rightarrow e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{k=0}^{\infty} \frac{t^{2k} A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} A^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} A = \cos t \, I + \sin t \, A$$

$$= \underline{\underline{\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}}}$$

$$\Rightarrow e^{-sA} \begin{pmatrix} -s \\ s \end{pmatrix} = \underline{\underline{\begin{pmatrix} -s \cos s + s \sin s \\ -s \sin s + s \cos s \end{pmatrix}}}$$

<u>4.3</u> $\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} \frac{e^t - e^{-t}}{2} \\ e^t \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}'(t) = \begin{pmatrix} \frac{e^t + e^{-t}}{2} \\ e^t \end{pmatrix} = \begin{pmatrix} -x + y \\ y \end{pmatrix}$

$$= \underline{\underline{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}}$$

<u>4.4</u> a) $\begin{pmatrix} 3e^t + e^{-t} \\ e^{2t} \end{pmatrix}' = \begin{pmatrix} 3e^t - e^{-t} \\ 2e^{2t} \end{pmatrix} \ldots \; 3e^t - e^{-t} \notin \operatorname{span}\{3e^t + e^{-t}, e^{2t}\}$ ✗

b) $\begin{pmatrix} 3e^t + e^{-t} \\ e^t \end{pmatrix}' = \begin{pmatrix} 3e^t - e^{-t} \\ e^t \end{pmatrix} = \begin{pmatrix} -1 & 6 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3e^t + e^{-t} \\ e^t \end{pmatrix}$ ✓

c) $\begin{pmatrix} 3e^t + e^{-t} \\ te^t \end{pmatrix}' = \begin{pmatrix} 3e^t - e^{-t} \\ e^t + te^t \end{pmatrix} \ldots \; e^t + te^t \notin \operatorname{span}\{3e^t - e^{-t}, e^t + te^t\}$ ✗

d) $\begin{pmatrix} 3e^t \\ t^2 e^t \end{pmatrix}' = \begin{pmatrix} 3e^t \\ 2te^t + t^2 e^t \end{pmatrix} \dots \quad 2te^t + t^2 e^t \notin \operatorname{span}\{3e^t, t^2 e^t\}$

X

e) $\begin{pmatrix} e^t + 2e^{-t} \\ e^t + 2e^{-t} \end{pmatrix}' = \begin{pmatrix} e^t - 2e^{-t} \\ e^t - 2e^{-t} \end{pmatrix} \dots \quad e^t - 2e^{-t} \notin \operatorname{span}\{e^t + 2e^{-t}\}$

X

4.5

The conditions say in other words

$$u''(t) + u(t) = f(t) \geq 0 \qquad \text{for some } f(t)$$

Let $u'(t) =: v(t) \Rightarrow v(0) = 1$

$$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}(t) = e^{Bt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t e^{B(t-s)} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds$$

$$B = -A \text{ from } 4.2 \Rightarrow e^{Bt} = e^{-At} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\Rightarrow u(t) = \sin t + \int_0^t \underbrace{\sin(t-s) f(s)}_{\geq 0 \text{ for } t \in [0,\pi] \text{ and } s \in [0,t]} ds$$

$$\underline{\underline{\geq \sin t}}$$

4.6 let $A := \begin{pmatrix} a & \pi \\ -\pi & a \end{pmatrix}$ for some $a \in \mathbb{R}$

$$\Rightarrow A = aI + \pi \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{B \text{ from } 4.5} \quad \ldots \text{ commuting matrices!}$$

$$\Rightarrow e^{A} = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{a} \end{pmatrix} \cdot e^{B\pi} = \begin{pmatrix} e^{a} & 0 \\ 0 & e^{a} \end{pmatrix} \cdot \begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix}$$

$$= \begin{pmatrix} -e^{a} & 0 \\ 0 & -e^{a} \end{pmatrix}$$

if α, β distinct:

a) $\sigma(A) \subset \mathbb{R} \Rightarrow \forall x \in \underbrace{\sigma(\exp(A))}_{=\{-\alpha,-\beta\} < 0} = \exp(\sigma(A)) > 0$ ↯

b) $\sigma(A) \not\subset \mathbb{R} \Rightarrow \exists x \in \mathbb{C} \setminus \mathbb{R} : \sigma(A) = \{x, \bar{x}\}$

$\Rightarrow \exp(\sigma(A)) = \{e^{\operatorname{Re} x} \cdot e^{\pm i \operatorname{Im} x}\}$

but $|e^{\operatorname{Re} x} \cdot e^{\pm i \operatorname{Im} x}| = e^{\operatorname{Re} x}$

and $|-\alpha| \neq |-\beta|$

and $\exp(\sigma(A)) = \sigma(\exp(A)) = \{-\alpha, -\beta\}$ ↯

4.7: Since the complex logarithm is a multivalued function, the proof is in the end not so elegant as I previously thought and actually leads to the easy version anyway:

$$-I\underset{\substack{\downarrow \\ \text{eigenvector} \neq 0}}{v} = A \cdot Av = A\underset{\substack{\downarrow \\ \text{eigenvalue}}}{\lambda} v = \lambda^2 v \Rightarrow \lambda^2 = -1 \quad \& \left[x \in \sigma \Rightarrow \bar{x} \in \sigma\right]$$

$$\Rightarrow \sigma = \{\pm i\}$$