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Problem Set 3

3.1. Let $x(t) = \varphi(t, t_0, x_0)$ solve $x' = f(t, x)$ with the initial condition $x(t_0) = x_0$. Find $\frac{\partial}{\partial x_0}\varphi(t, t_0, x_0)$ for given t_0 and x_0 if

(a) $f = 2x + t^2x^2 - x^3$, $x(0) = 0$

(b) $f = \ln(1 - x) - x^2 - t^2x^2$, $x(0) = 0$

(c) $f = t(1 - x^2)$, $x(t_0) = 1$, $t_0 \in \mathbb{R}$

3.2. Let $x(t) = \varphi(t, t_0, x_0)$ solve $x' = f(t, x)$ with the initial condition $x(t_0) = x_0 \in \mathbb{R}$ and $f \in C^2(\mathbb{R}^2)$. Derive the equation for $\frac{\partial^2}{\partial x_0^2}\varphi(t, t_0, x_0)$ and compute $\frac{\partial^2}{\partial x_0^2}\varphi(t, 0, 1/2)$ for the equation presented by me, i.e. $f = e^{2x} - e$. You can consider all information I have deduced as already known. Expand then $\varphi(t, 0, h + 1/2)$ for small h up to the second-order term.

3.3. Let $x(t) = \varphi(t, t_0, x_0, \lambda)$ be the solution to $x' = f(t, x, \lambda)$, $x(t_0) = x_0 \in \mathbb{R}$ with λ being a real parameter. Consider a beautifully smooth $f \in C^1(\mathbb{R}^3)$. Find the equation for $\frac{\partial \varphi}{\partial \lambda}$ and apply your discovery to compute $\frac{\partial \varphi}{\partial \lambda}(t, 1, 0, 1)$ for $x' = \lambda \cos(\lambda\pi)x + t\lambda$.

3.4. Let I be an open interval, $f = f(t, x, x') \in C^1(I \times \mathbb{R}^2)$ and $x(t) = \varphi(t, t_0, x_0^1, x_0^2)$ solve

$$\begin{aligned} x'' &= f(t, x, x'), \\ x(t_0) &= x_0^1, \\ x'(t_0) &= x_0^2. \end{aligned}$$

Derive equations for $u(t) = \frac{\partial}{\partial x_0^1}\varphi(t, t_0, x_0^1, x_0^2)$ and $v(t) = \frac{\partial}{\partial x_0^2}\varphi(t, t_0, x_0^1, x_0^2)$. Finally, apply the result to $f = 4x' + 21x - 3$, $t_0 = 0$ and compute the respective partial derivatives.

3.5. **Food for thought:** Bobek the rabbit is hiding in one of five hats that are lined up in a row. The hats are numbered 1 to 5. Each night Bobek hops into an adjacent hat, exactly one number away. Each morning you can peek into a single hat to test whether Bobek is inside. Can you think up a strategy to find him?

Problem set 3. Solutions

3.1a) Let $u(t) := \frac{d\varphi}{dx_0}(t,0,0)$

Then $u' = f_x u = (2 + 2t^2x - 3x^2)u$

$u(0) = 1$

$x \equiv 0$ is the solution to the given equation

$$\Rightarrow \left.\begin{array}{l} u' = 2u \\ u(0) = 1 \end{array}\right\} u(t) = e^{2t}$$

3.1b) $u' = f_x u = \left(-\frac{1}{1-x} - 2x - 2t^2x\right)u$

$u(0) = 1$

$x \equiv 0$ solves the equation

$$\Rightarrow \left.\begin{array}{l} u' = -u \\ u(0) = 1 \end{array}\right\} u(t) = e^{-t}$$

3.1c) $u(t) := \frac{d\varphi}{dx_0}(t, t_0, 1)$

$\Rightarrow u' = f_x u = -2xtu$

$u(t_0) = 1$

$x \equiv 1$ solves the equation

$$\Rightarrow \left.\begin{array}{l} u' = -2tu \\ u(t_0) = 1 \end{array}\right\} \left.\begin{array}{l} u(t) = c \cdot e^{-t^2} \\ u(t_0) = 1 = c \cdot e^{-t_0^2} \end{array}\right\} \Rightarrow u(t) = e^{t_0^2 - t^2}$$

3.2

Let $u(t) := \frac{\partial}{\partial x_0} \varphi(t, 0, \frac{1}{2})$

$v(t) := \frac{\partial^2}{\partial x_0^2} \varphi(t, 0, \frac{1}{2})$

We know

$$u' = f_x u$$
$$u(0) = 1$$

$$\Rightarrow \frac{\partial^3}{\partial x_0^2 \partial t}\varphi = \frac{\partial}{\partial x_0}\left(f_x \frac{\partial}{\partial x_0}\varphi(t, t_0, x_0)\right)$$
$$= f_{xx}\left(\frac{\partial}{\partial x_0}\varphi\right)^2 + f_x \frac{\partial^2}{\partial x_0^2}\varphi$$

$$\Rightarrow v' = f_{xx} u^2 + f_x v$$
$$= 4e^{2x} u^2 + 2e^{2x} v$$

we know $x = \frac{1}{2}$, $u = e^{2et}$ $\Rightarrow v' = 4e\, e^{4et} + 2e v$

$v(0) = \frac{\partial^2}{\partial x_0^2}\varphi(0, t_0, x_0) = \frac{\partial}{\partial x_0} u(0) = \frac{\partial}{\partial x_0} 1 = 0$

$$\Rightarrow v = 2e^{2et}(e^{2et} - 1)$$

3.8

$$\frac{\partial}{\partial t}\varphi(t,t_0,x_0,\lambda) = f(t,\varphi(t,t_0,x_0,\lambda),\lambda)$$
$$\varphi(t_0,t_0,x_0,\lambda) = x_0$$

$$\Rightarrow \frac{\partial^2}{\partial t\partial\lambda}\varphi = f_x \cdot \frac{\partial\varphi}{\partial\lambda} + f_\lambda$$
$$\frac{\partial\varphi}{\partial\lambda}(t_0,t_0,x_0,\lambda) = 0$$

Let $u := \frac{\partial\varphi}{\partial\lambda}(t,1,0,1)$

$$\Rightarrow u' = f_x u + f_\lambda = \lambda\cos(\lambda\pi)u + t + x(\cos(\lambda\pi) - \pi\lambda\sin(\lambda\pi))$$
$$u(1) = 0$$

$$\lambda = 1 \Rightarrow u' = -u + t - x \qquad , \qquad x' = -x + t$$
$$u(1) = 0 \qquad\qquad x(1) = 0 \Big\} \Rightarrow x = t - 1$$

$$u' = -u + 1 \qquad \Leftarrow$$
$$\Rightarrow \underline{\underline{u = 1 - e^{1-t}}}$$

$\underline{3.4}$

$$\frac{\partial^2}{\partial t^2} \varphi(t, t_0, x_0^1, x_0^2) = f\left(t, \varphi(t, t_0, x_0^1, x_0^2), \frac{\partial \varphi}{\partial t}(t, t_0, x_0^1, x_0^2)\right)$$

$$\varphi(t_0, t_0, x_0^1, x_0^2) = x_0^1$$

$$\frac{\partial}{\partial t} \varphi(t_0, t_0, x_0^1, x_0^2) = x_0^2$$

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let $u(t) := \frac{\partial}{\partial x_0^1} \varphi(t, t_0, x_0^1, x_0^2)$

$v(t) := \frac{\partial}{\partial x_0^2} \varphi(t, t_0, x_0^1, x_0^2)$

$$\Rightarrow \quad u'' = f_x u + f_{x'} u'$$

$$u(t_0) = 1$$

$$u'(t_0) = 0$$

$$v'' = f_x v + f_{x'} v'$$

$$v(t_0) = 0$$

$$v'(t_0) = 1$$

Now let $f := 4x' + 21x - 3, \quad t_0 = 0$

$$\Rightarrow \left.\begin{aligned} u'' &= 21u + 4u' \\ u(0) &= 1 \\ u'(0) &= 0 \end{aligned}\right\} \Rightarrow u = c_1 e^{7t} + c_2 e^{-3t} \Rightarrow \underline{\underline{u = \frac{3}{10} e^{7t} + \frac{7}{10} e^{-3t}}}$$

$$\Rightarrow \left.\begin{aligned} v'' &= 21v + 21v' \\ v(0) &= 0 \\ v(0) &= 1 \end{aligned}\right\} \Rightarrow v = c_1 e^{7t} + c_2 e^{-3t} \Rightarrow \underline{\underline{v = \frac{1}{10} e^{7t} - \frac{1}{10} e^{-3t}}}$$