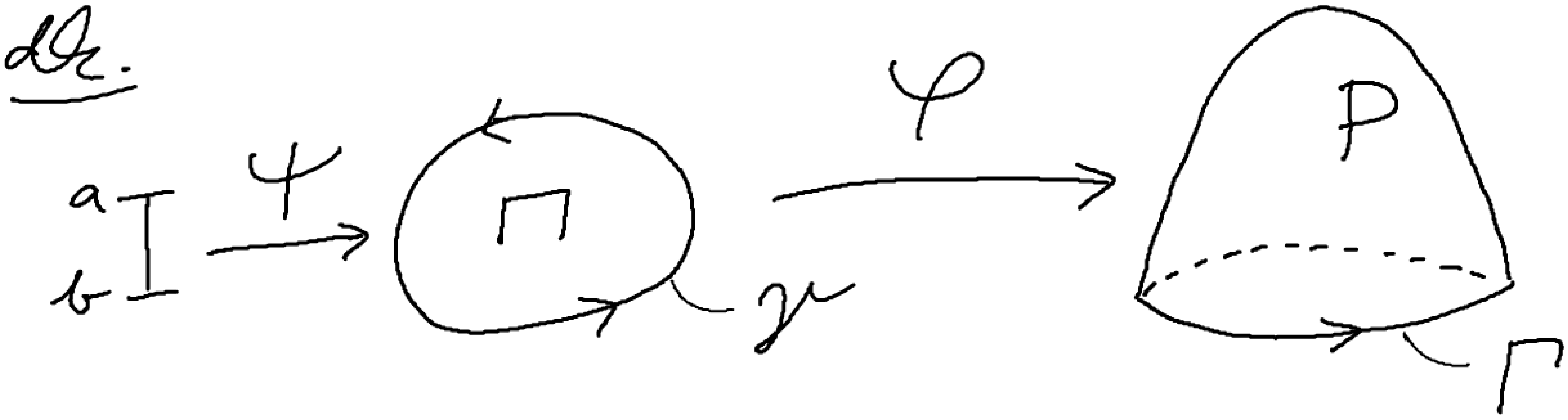
Věta 20.5. [Stokesova v R^3.]

Dk.



$(\psi(t), [a,b])$ ... parametrizace $\gamma = \partial \Pi$

$(\varphi(u), \Pi)$ ... parametrizace P

$(\chi(t), [a,b])$ ... parametrizace Γ,

kde $\chi = \varphi \circ \psi$

TRIK: pomocné fce $G : \Pi \to R^2$, kde

$$\boxed{G_i(u) = F_j(\varphi(u)) \frac{\partial \varphi_j}{\partial u_i}(u) \qquad i = 1,2,\ u \in \Pi}$$

Pozn. sčítáme přes opakované indexy, tj. zde $\sum_{j=1}^{3}$ („sčítací konvence")

Věta 19.7. (Green) $\Rightarrow$ $\boxed{\int_{\gamma} \underset{\sim}{G} \cdot \underset{\sim}{ds} = \int_{\Gamma} \operatorname{rot} \underset{\sim}{G} \, du}$

... ukážeme, že toto je hledaný vzorec...

... dlouhý výpočet; stále užíváme sčítací konvenci...

<u>1. L. S.</u>

$$\int_{\gamma} \underset{\sim}{G} \cdot \underset{\sim}{ds} = \int_a^b G(\psi(t)) \cdot \psi'(t)\, dt$$

$$= \int_a^b \underbrace{G_i(\psi(t))\, \psi_i'(t)}_{=\ F_j(\varphi(\psi(t))) \frac{\partial \varphi_j}{\partial u_i}(\psi(t))\, \psi_i'(t)}\, dt$$

$$F_j(\varphi(\psi(t))) \underbrace{\frac{\partial \varphi_j}{\partial u_i}(\psi(t))\, \psi_i'(t)}_{=\ \frac{d}{dt}\{\varphi_j(\psi(t))\}}$$

$$= F_j(x(t)) \frac{d}{dt}\{\underbrace{\varphi_j(\psi(t))}_{=\ x_j(t)}\}$$

$$= F(x(t)) \cdot x'(t)$$

tj. $L.S. = \int_a^b F(\varphi(t))\cdot \varphi'(t)\,dt = \int_\Gamma \underset{\sim}{F}\cdot d\underset{\sim}{s}$

<u>2. P.S.</u> zkrácené značení:

$$\partial_i F_j = \frac{\partial F_j}{\partial x_i}, \quad \partial_i \varphi_j = \frac{\partial \varphi_j}{\partial u_i}$$

tj. v tomto zápisu: $\operatorname{rot} \underset{\sim}{F} =$

$$= \left(\partial_2 F_3 - \partial_3 F_2,\ \partial_3 F_1 - \partial_1 F_3,\ \partial_1 F_2 - \partial_2 F_1\right)$$

$$\operatorname{rot} G = \frac{\partial G_2}{\partial u_1} - \frac{\partial G_1}{\partial u_2}$$

$$= \frac{\partial}{\partial u_1}\left(F_j(\varphi)\frac{\partial \varphi_j}{\partial u_2}\right) - \frac{\partial}{\partial u_2}\left(F_j(\varphi)\frac{\partial \varphi_j}{\partial u_1}\right)$$

$$= \frac{\partial F_j}{\partial x_k}(\varphi)\frac{\partial \varphi_k}{\partial u_1}\frac{\partial \varphi_j}{\partial u_2} + F_j(\varphi)\frac{\partial^2 \varphi_j}{\partial u_1 \partial u_2}$$

$$- \frac{\partial F_j}{\partial x_k}(\varphi)\frac{\partial \varphi_k}{\partial u_2}\frac{\partial \varphi_j}{\partial u_1} - F_j(\varphi)\frac{\partial^2 \varphi_j}{\partial u_2 \partial u_1}$$

$\shortparallel$

0

$$= \partial_k F_j(\varphi) \cdot \left(\partial_1 \varphi_k \partial_2 \varphi_j - \partial_2 \varphi_k \partial_1 \varphi_j\right)$$

zbývá ukázat, že tento výraz je roven $(\mathrm{rot}\, F) \cdot (\partial_1 \varphi \times \partial_2 \varphi)$

pomocné výpočty:

$$\partial_1 \varphi = (\partial_1 \varphi_1, \partial_1 \varphi_2, \partial_1 \varphi_3)$$

$$\partial_2 \varphi = (\partial_2 \varphi_1, \partial_2 \varphi_2, \partial_2 \varphi_3)$$

$$\Rightarrow \partial_1 \varphi \times \partial_2 \varphi = \left(\partial_1 \varphi_2 \partial_2 \varphi_3 - \partial_2 \varphi_2 \partial_1 \varphi_3,\right.$$

$$\left.\partial_2 \varphi_1 \partial_1 \varphi_3 - \partial_1 \varphi_1 \partial_2 \varphi_3, \partial_1 \varphi_1 \partial_2 \varphi_2 - \partial_2 \varphi_1 \partial_1 \varphi_2\right)$$

tedy dostáváme:

$$(\mathrm{rot}\, F)_1 (\partial_1 \varphi \times \partial_2 \varphi)_1$$

$$= (\partial_2 F_3 - \partial_3 F_2)(\partial_1 \varphi_2 \partial_2 \varphi_3 - \partial_2 \varphi_2 \partial_1 \varphi_3)$$

a snadno ověříme, že toto se rovná:

$$\sum_{j,k=2}^{3} \partial_k F_j \cdot \left(\partial_1 \varphi_k \partial_2 \varphi_j - \partial_2 \varphi_k \partial_1 \varphi_j\right)$$

obecněji pak platí:

$$\boxed{\begin{aligned}&(\operatorname{rot} F)_l\,(\partial_1\varphi\times\partial_2\varphi)_l\\&=\sum_{j,k\neq l}\partial_k F_j\left(\partial_1\varphi_k\,\partial_2\varphi_j-\partial_2\varphi_k\,\partial_1\varphi_j\right)\end{aligned}}$$

... čímž je důkaz hotov...

<u>Tvrzení</u> nechť $\Omega\subset\mathbb{R}^3$ je otevřené, $h:\Omega\to\mathbb{R}$ spojité, $\underset{\sim}{F}:\Omega\to\mathbb{R}^3$ třídy C^1. Potom je ekvivalentní:

(1) $h(x)=\operatorname{div}\underset{\sim}{F}(x)$, pro $\forall x\in\Omega$

(2) $\int_V h\,dx=\int_{\partial V}\underset{\sim}{F}\cdot\underset{\sim}{dS}$, pro $\forall V\subset\Omega$ rozumnou

(3) $h(x)=\lim_{r\to 0+}\frac{1}{|B_r(x)|}\int_{\partial B_r(x)}\underset{\sim}{F}\cdot\underset{\sim}{dS}$

pro $\forall x\in\Omega$, kde $B_r(x)$ je koule o středu x, poloměru r, $|B_r(x)|$ její objem dk. díky Gaussově větě platí:

$$\boxed{\int_{\partial V} \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_V \operatorname{div} F(x)\,dx}$$

a tedy

$$(2) \iff \boxed{(2')\quad \int_V \big(f(x) - \operatorname{div} F(x)\big)\,dx = 0 \quad \text{pro } \forall\, V\subset\Omega \text{ rozumné}}$$

Implikace $(1) \Rightarrow (2')$ je ale zřejmé, opačné implikace viz Lemma D.1, bod (i) výše

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Ohledně bodu (3) stačí ukázat, že:

$$\boxed{\lim_{r\to 0+} \frac{1}{|B_r(x)|} \underbrace{\int_{\partial B_r(x)} \underset{\sim}{F}\cdot d\underset{\sim}{S}}_{\textstyle = \int_{B_r(x)} \operatorname{div} F(u)\,du} = \operatorname{div} \underset{\sim}{F}(x)}$$

dk. $\int_{B_r(x)} \operatorname{div} F(u)\,du$, a užij Lemma D.1, (ii)

Lemma D.1. Nechť $f:\Omega\to\mathbb{R}$ je spojitá.

(i) nechť $\int_V f(x)\,dx = 0$, $\forall V\subset\Omega$.

Pak $f(x)=0$, pro $\forall x\in\Omega$

(ii) $$\frac{1}{|B_r(x)|}\int_{B_r(x)} f(u)\,du \to f(x), \quad r\to 0+.$$

Dk. (i) nepřímo: nechť $\exists x_0\in\Omega$ t.ž. $f(x_0)\neq 0$, BÚNO: $f(x_0)=\Delta>0$

spojitost $f \Rightarrow \exists r>0$ t.ž.

$$f(x)>\frac{\Delta}{2} \quad \text{pro } \forall x\in V,$$

kde $V=B_r(x_0)$

BÚNO: $r>0$ malé $\Rightarrow V\subset\Omega$

odtud je
$$\int_V \underbrace{f(x)}_{>\frac{\Delta}{2}}\,dx \geq |V|\cdot\frac{\Delta}{2}>0.$$

(ii) cíl: $Z(r) \to 0$, pro $r \to 0+$

kde $Z(r) = f(x) - \frac{1}{|B_r(x)|} \int_{B_r(x)} f(u)\,du$

TRIK: $Z(r) = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(x) - f(u)\,du$

$$|Z(r)| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x) - f(u)|\,du$$

$\varepsilon > 0$ dáno: $\exists\, \delta > 0$ t.ž. $\forall u \in B_r(x)$

platí $|f(x) - f(u)| < \varepsilon$

(díky spojitosti f)

nyní, pro $r \in (0, \delta)$ libovolné máme

$$|Z(r)| \leq \frac{1}{|B_r(x)|} \int_{B_r(x)} \underbrace{|f(x) - f(u)|}_{< \varepsilon}\,du < \varepsilon$$

Tím je důkaz hotov.