Věta 19.4. Buď $\mathbf{F}: \Omega \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ oblast. Potom je ekvivalentní:

(1) $\mathbf{F}$ má v Ω potenciál

(2) integrál z $\mathbf{F}$ nezávisí v Ω na cestě

dk. (1) $\Rightarrow$ (2) ... plyne z Lemmatu 19.2:

buď $U(\mathbf{x}): \Omega \to \mathbb{R}$ potenciál $\mathbf{F}(\mathbf{x})$

$\mathbf{x}_0, \mathbf{x}_1 \in \Omega$ libovolné body

$\gamma \subset \Omega$ ... křivka od $\mathbf{x}_0$ do $\mathbf{x}_1$

$$\Rightarrow \int_\gamma \mathbf{F} \cdot d\mathbf{s} = \underbrace{U(\mathbf{x}_1) - U(\mathbf{x}_0)}_{\text{nezávisí na cestě}}$$

(2) $\Rightarrow$ (1): cíl $\exists\, U: \Omega \to \mathbb{R}$ třídy C^1

t.ž. $DU = \mathbf{F}$ v Ω

fixuji $\mathbf{x}_0 \in \Omega$ pevné, libovolné,
a definuji:

$$U(\mathbf{x}) = \begin{cases} 0, & \mathbf{x} = \mathbf{x}_0 \\ \int_\gamma \mathbf{F} \cdot d\mathbf{s}, & \mathbf{x} \neq \mathbf{x}_0 \end{cases}$$

kde $\gamma \subset \Omega$ je libovolná křivka, jdoucí od $\underset{\sim}{x}_0$ do $\underset{\sim}{x}$.

<u>korektní</u>: γ existuje (Ω oblast) musí být libovolná (díky (2))

slíbený úkol: $$\boxed{\frac{\partial u}{\partial x_j}(\underset{\sim}{x}) = F_j(\underset{\sim}{x}), \quad \forall \underset{\sim}{x} \in \Omega,\ j = 1, \dots n} \quad (*)$$

fixujme $\underset{\sim}{x}, j$:

γ ... (nějaká) křivka od $\underset{\sim}{x}_0$ do $\underset{\sim}{x}$

$h \neq 0$ malé číslo

β_h .. úsečka od $\underset{\sim}{x}$ do $\underset{\sim}{x} + h\underset{\sim}{e}^j$,

kde $\underset{\sim}{e}^j = (0, \dots, 1, 0, \dots)$

j-té pozice

$\underset{\sim}{\varphi}(t) = \underset{\sim}{x} + t\underset{\sim}{e}^j \quad , t \in [0, h]$

.. parametrizace β_h

$\gamma \oplus \beta_h$ .. křivka od $\underset{\sim}{x}$ do $\underset{\sim}{x} + h\underset{\sim}{e}^j$

mohu psát: $\frac{1}{h}\left(U(\underline{x}+h\underline{e}^{j}) - U(\underline{x})\right)$

$$= \frac{1}{h}\left(\int_{\gamma\oplus\beta_h} \underline{F}\cdot d\underline{s} - \int_{\gamma} \underline{F}\cdot d\underline{s}\right)$$

$$= \frac{1}{h}\left(\int_{\gamma} \underline{F}\cdot d\underline{s} + \int_{\beta_h} \underline{F}\cdot d\underline{s} - \int_{\gamma} \underline{F}\cdot d\underline{s}\right)$$

$$= \frac{1}{h}\int_{\beta_h} \underline{F}\cdot d\underline{s} = \frac{1}{h}\int_0^1 \underline{F}(\underline{\psi}(t))\cdot \underline{\psi}'(t)\,dt$$

$$\left(\begin{array}{l} \text{leč } \underline{\psi}'(t) = \underline{e}^{j},\ \forall t \\ \text{tj. } \underline{F}(\underline{\psi}(t))\cdot\underline{\psi}'(t) \\ = \underline{F}(\underline{x}+t\underline{e}^{j})\cdot \underline{e}^{j} = F_j(\underline{x}+t\underline{e}^{j}) \end{array}\right)$$

$$= \frac{1}{h}\int_0^h f_j(t)\,dt, \text{ kde}$$

$$f_j(t) = F_j(\underline{x}+t\underline{e}^{j})$$

zřejmě: $f_j(t)$ je spojité v okolí 0

Věta 9.6. $\Rightarrow \frac{1}{h}\int_0^h f_j(t)\,dt \to f_j(0)$,

tj. $(*)$ platí.

značí: $\underset{\sim}{F} = \nabla U$ spojité $\Rightarrow U \in C^1$.

<u>Věta 19.5.</u> Nechť $\underset{\sim}{F} : \gamma \to R^n$, γ je orientovaná křivka, $\underset{\sim}{T} : \gamma \to R^n$ tečný vektor ve shodě s orientací.

Potom: $\int_\gamma \underset{\sim}{F} \cdot \underset{\sim}{ds} = \int_\gamma (\underset{\sim}{F} \cdot \underset{\sim}{T})\,ds$.

<u>dk</u> BÚNO. γ .. jednoduché křivka

$(\varphi(t), [a,b])$ … parametrizace ve shodě s orientací

označ. $f = (\underset{\sim}{F} \cdot \underset{\sim}{T}) : \gamma \to \mathbb{R}$

$$P.S. = \int_\gamma f\,ds = \int_a^b f(\underset{\sim}{\varphi}(t))\,\|\underset{\sim}{\varphi}'(t)\|\,dt$$

avšak: $f(\underset{\sim}{\varphi}(t)) = \underset{\sim}{F}(\underset{\sim}{\varphi}(t)) \cdot \underbrace{\underset{\sim}{T}(\underset{\sim}{\varphi}(t))}$

vše je ve shodě — $(+)\dfrac{\varphi'(t)}{\|\underset{\sim}{\varphi}'(t)\|}$

$$= \int_a^b \underset{\sim}{F}(\underset{\sim}{\varphi}(t)) \cdot \underset{\sim}{\varphi}'(t)\,dt = \int_{\varphi} \underset{\sim}{F} \cdot \underset{\sim}{ds} = L.S.$$

Věta 19.5. [Gauss.]

dk. $g(\Omega, \underset{\sim}{F})$ … značí výrok: věta platí pro pole $\underset{\sim}{F}$ v oblasti Ω

1. KROK: invariance vůči posunutí (jasné) a otočení, tj. $g(\Omega, \underset{\sim}{F}) \Leftrightarrow g(\hat{\Omega}, \hat{\underset{\sim}{F}})$,

kde $\hat{\Omega} = \{ R\underset{\sim}{x},\ \underset{\sim}{x} \in \Omega \}$

$$\hat{\underset{\sim}{F}}(\underset{\sim}{x}) = R\underset{\sim}{F}(R^{-1}\underset{\sim}{x}),\ \underset{\sim}{x} \in \hat{\Omega}$$

přičemž R je matice (lin. zob.) otočení v $\mathbb{R}^2$

P.S. snadné, neboť $\boxed{\hat{F}\cdot\hat{n} = F\cdot n}$

(otočení zachová skal. součin)

L.S. nutno uvěřit, že $\boxed{\operatorname{div}\hat{F} = \operatorname{div} F}$

dk: $\operatorname{div}\hat{F} = \operatorname{tr} D\hat{F}$

$$= \operatorname{tr}\left(R\, DF\, R^{-1}\right)$$

$$= \operatorname{tr} DF = \operatorname{div} F$$

(podobnost matice zachová stopu)

---

2. KROK: princip skládanky

$$\mathcal{G}(\Omega_1, \underset{\sim}{F}) \pm \mathcal{G}(\Omega_2, \underset{\sim}{F})$$

$$\Rightarrow \mathcal{G}(\Omega_1 \cup \Omega_2, \underset{\sim}{F})$$

$$\mathcal{G}(\Omega_1 \setminus \Omega_2, \underset{\sim}{F})$$

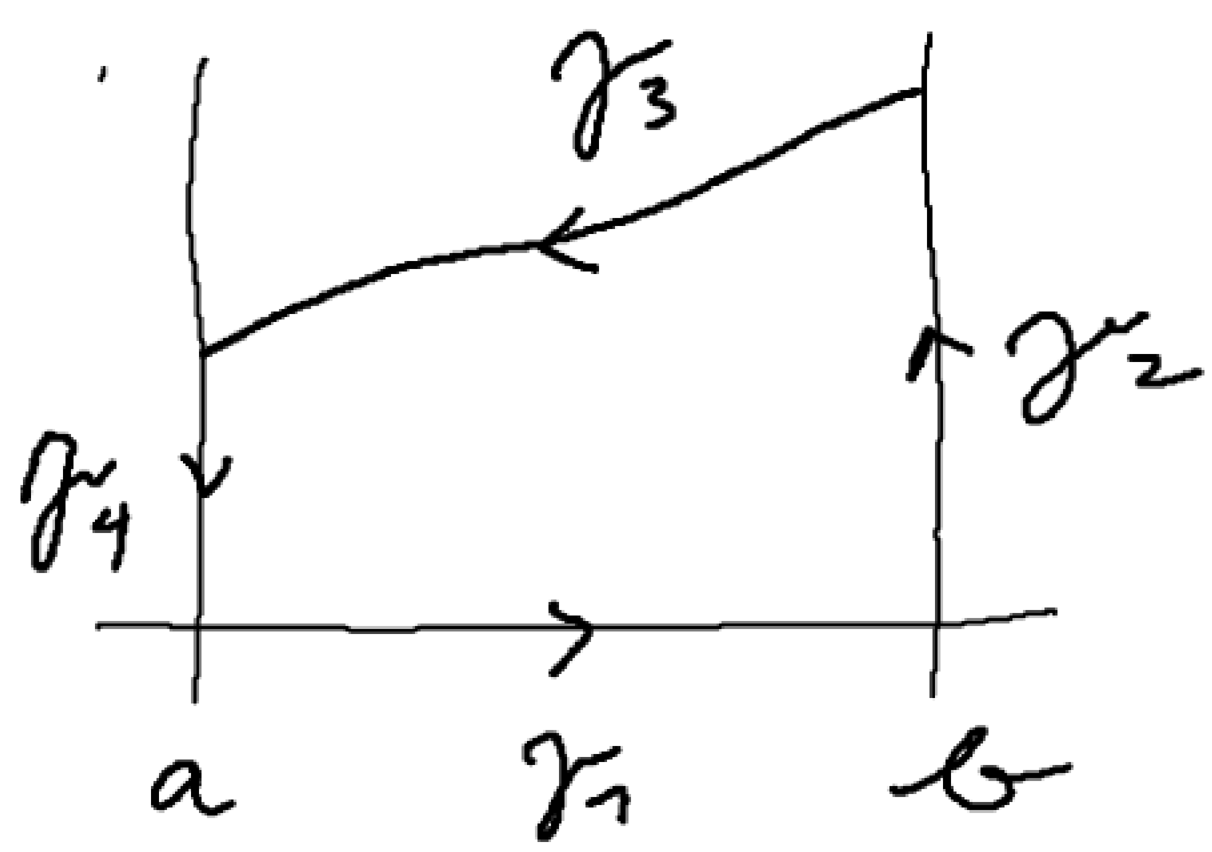
3. KROK speciální případ:

$$\Omega = \{x \in (a,b);\ 0 < y < f(x)\},$$

kde $f : [a,b] \to \mathbb{R}$ je C^1 fce možno BÚNO $F_1 = 0$, tj. $\underset{\sim}{F} = (0, F_2)$,

tj. cílem je
$$\boxed{\int_\Omega \frac{\partial F_2}{\partial y}\, dx\,dy = \int_{\partial\Omega} F_2 n_2 \, ds}$$

kde $\partial\Omega = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4$



ad P.S. $\int_{\gamma_{2,4}} = 0$, neboť $\underset{\sim}{n} = (\pm 1, 0)$

tj. $F_2 n_2 = F_2 \cdot 0 = 0$

γ_1 ... $\underset{\sim}{n} = (0, -1)$, tj. $n_2 = -1$

$\underset{\sim}{\varphi}(x) = (t, 0)$, $t \in [a, b]$

$\underset{\sim}{\varphi}'(t) = (1, 0)$, a tedy $\|\underset{\sim}{\varphi}'(t)\| = 1$

pro $\forall t \in (a, b)$

$$\int_{\gamma_2} F_2 n_2 \, ds = \int_a^b F_2(\underset{\sim}{\varphi}(t)) \cdot \|\varphi'(t)\| \, dt$$

$$= \int_a^b F_2(t, 0) \, dt$$

$\underline{\gamma_3 /}$ .. $\underset{\sim}{\varphi}(t) = (t, f(t)), \ t \in (a, b)$

$$\underset{\sim}{\varphi}'(t) = (1, f'(t))$$

$$\underset{\sim}{T}(\underset{\sim}{\varphi}'(t)) = \frac{\varphi'(t)}{\|\underset{\sim}{\varphi}'(t)\|} = \left(\frac{1}{\|\underset{\sim}{\varphi}'(t)\|}, \frac{f'(t)}{\|\underset{\sim}{\varphi}'(t)\|} \right)$$

$$\underset{\sim}{n}(\underset{\sim}{\varphi}(t)) = \left(\frac{-f'(t)}{\|\underset{\sim}{\varphi}'(t)\|}, \underbrace{\frac{1}{\|\underset{\sim}{\varphi}'(t)\|}}_{n_2(\underset{\sim}{\varphi}(t))} \right)$$

a teda:

$$\int_{\gamma_3} F_2 n_2 \, ds = \int_a^b F_2(\underset{\sim}{\varphi}(t)) \cdot \frac{1}{\|\varphi'(t)\|} \cdot \|\varphi'(t)\| \, dt$$

$$= \int_a^b F_2(t, f(t))\, dt.$$

CELKEM TEDY: $P.S. = \quad (t \leftarrow x)$

$$= \int_a^b F_2(x, f(x)) - F_2(x, 0)\, dx.$$

$L.S. = \int_\Omega \frac{\partial F_2}{\partial y}\, dx dy$ (Fubiniho věta)

$$= \int_a^b \underbrace{\left(\int_0^{f(x)} \frac{\partial F_2}{\partial y}(x, y)\, dy \right)}_{\parallel} dx$$

$$\left[F_2(x, y) \right]_{y=0}^{y=f(x)}$$

$= P.S.$

<u>Věta 19.7.</u> [Green.] Nechť $\Omega \subset \mathbb{R}^2$ je „rozumné" oblast, $G : \overline{\Omega} \to \mathbb{R}^2$ je C^1. Nechť $\partial\Omega$ obíhá kolem Ω v kladném smyslu. Pak

$$\int_{\Omega} \operatorname{rot} \underset{\sim}{G} \, dx\,dy = \int_{\partial\Omega} \underset{\sim}{G} \cdot \underset{\sim}{ds}$$

Dk. <u>TRIK</u> polož $\boxed{\underset{\sim}{F} := (G_2, -G_1)}$

tj: $F_1 = G_2$, $F_2 = -G_1$

... aplikuj Větu 19.6. (Gauss)

L.S. $\operatorname{div} \underset{\sim}{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}$

$= \operatorname{rot} \underset{\sim}{G}$

P.S. $\underset{\sim}{F} \cdot \underset{\sim}{n} = F_1 n_1 + F_2 n_2$

$= G_2 T_2 + (-G_1)(-T_1) = \underset{\sim}{G} \cdot \underset{\sim}{T} = \underset{\sim}{G} \cdot \underset{\sim}{ds}$

neboť $\underset{\sim}{n} = (n_1, n_2) = (T_2, -T_1)$

Věta 19.8. Nechť $\Omega \subset \mathbb{R}^2$ je jednoduše souvislá oblast, $\underset{\sim}{F} : \Omega \to \mathbb{R}^2$ třídy C^1. Nechť $\operatorname{rot} \underset{\sim}{F} = 0$ v Ω. Pak $\underset{\sim}{F}$ má v Ω potenciál.

dk: díky Větě 19.4. a předchozímu pozorování stačí ukázat:

$$\int_\omega \underset{\sim}{F} \cdot \underset{\sim}{ds} = 0, \quad \text{kde } \omega \subset \Omega \text{ je daná uzavřená křivka}$$

označ $\mathcal{O}$ „vnitřek" ω, tj. omezenou oblast t.ž. $\partial \mathcal{O} = \omega$

Ω jednoduše souvislé $\Rightarrow \mathcal{O} \subset \Omega$, a tedy $\underset{\sim}{F}$ je C^1 na okolí $\overline{\mathcal{O}} = \mathcal{O} \cup \omega$

Věta 19.7. (Green)

$$\Rightarrow \quad \underbrace{\int_\omega \underset{\sim}{F} \cdot \underset{\sim}{ds} = \int_{\mathcal{O}} \underbrace{(\operatorname{rot} \underset{\sim}{F})}_{\substack{\parallel \\ 0}} \, dx\, dy}_{\substack{\parallel \\ 0}}$$

<u>Příkl.</u> $\underset{\sim}{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$

$$\Omega = \mathbb{R}^2 \setminus \underset{\sim}{0}$$

<u>vidíme:</u> $\underset{\sim}{F} \in C^1(\Omega)$,

$$\operatorname{rot} \underline{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{y^2-x^2}{x^2+y^2} + \frac{x^2-y^2}{x^2+y^2} = 0$$

<u>leč uvaž:</u> $\gamma \subset \Omega$ kružnice $\circlearrowleft$

$$\underset{\sim}{\varphi}(t) = (\cos t, \sin t), \; t \in [0, 2\pi]$$

$$\int_\gamma \underset{\sim}{F} \cdot \underline{ds} = \int_0^{2\pi} \underset{\sim}{F}(\underline{\varphi}(t)) \cdot \underline{\varphi}'(t)\, dt$$

$$= \int_0^{2\pi} \underbrace{(-\sin t, \cos t) \cdot (-\sin t, \cos t)}_{\equiv 1} dt = 2\pi$$

$\Rightarrow$ $\underset{\sim}{F}$ <u>nemá</u> v Ω potenciál

(chybí jednoduchá souvislost!!)