Lemma 19.1. [O reparametrizaci.]

Dk. polož $\omega(\tau) = (\underset{\sim}{\varphi})_{-1}(\underset{\sim}{\psi}(\tau))$, $\tau \in (c,d)$

... korektní, neboť $\underset{\sim}{\varphi}, \underset{\sim}{\psi}$ jsou 1-1

KROK 1: $\exists\, \omega'(\tau) \in \mathbb{R}$, pro $\forall \tau \in (c,d)$

volme $\tau_0 \in (c,d)$ pevné, libovolné

$\Rightarrow \alpha_0 = \underset{\sim}{\psi}(\tau_0) \in \gamma$ (ne krajní)

$t_0 = \omega(\tau_0) \in (a,b)$

víme: $\underset{\sim}{\varphi}'(t_0) = (\varphi_1'(t_0), \dots, \varphi_n'(t_0)) \neq \underset{\sim}{0}$

$\Rightarrow \exists j$ t. ž. $\varphi_j'(t_0) \neq 0$,

BÚNO nechť $j = 1$, $\varphi_1'(t_0) > 0$

spojitost φ_1' $\Rightarrow$ $\varphi_1' > 0$ na j. $\mathcal{U}(t_0)$,

a tedy $\varphi_1 : \mathcal{U}(t_0) \to \mathbb{R}$ roste,

tedy je 1-1

$\Rightarrow$ lze psát: $\omega(\tau) = (\varphi_1)_{-1}(\psi_1(\tau))$,

pro $\forall \tau \in \mathcal{V}(\tau_0)$

a tedy konečná $\exists\, \omega'(\tau) \in \mathbb{R}$, $\tau \in U(\tau_0)$
díky Větám 4.3. a 4.4.

<u>KROK 2</u> : $\omega'(\tau) \neq 0$, pro $\forall \tau \in (c,d)$

píšeme $\underset{\sim}{\psi}(\tau) = \underset{\sim}{\varphi}(\omega(\tau))$ $\quad \frac{d}{d\tau}$

$$\underset{\sim}{\psi}'(\tau) = \underset{\sim}{\varphi}'(\omega(\tau))\,\omega'(\tau)$$

?? $\omega'(\tau) = 0 \Rightarrow PS = 0$, tedy $LS = 0$,

neboli $\underset{\sim}{\psi}'(\tau) = \underset{\sim}{0}$, SPOR.

<u>Věta 19.1.</u> [Nezávislost k. i. na par.]

<u>dk</u>: <u>1.</u> je jednoduché (neuzavř.)

buď $\underset{\sim}{\varphi}(t)$, $t \in (a,b)$, $\underset{\sim}{\psi}(\tau)$, $\tau \in (c,d)$

libovolné parametrizace

$\mathcal{L}$. 19.1. $\Rightarrow \exists\, \omega(\tau) : (c,d) \to (a,b)$

(prosté, na, $\omega'(\tau) \in \mathbb{R}, \forall \tau$)

nechť $f : \gamma \to \mathbb{R}$ je dána

$$\int_\gamma f\,ds \overset{\text{pomocí } \underset{\sim}{\varphi}}{=} \int_a^b f(\underset{\sim}{\varphi}(t))\,\|\underset{\sim}{\varphi}'(t)\|\,dt$$

substituce

$t = \omega(\tau)$

$dt = \omega'(\tau)\,d\tau$

$$= \int_c^d f\underbrace{(\underset{\sim}{\varphi}(\omega(\tau)))}_{=\,\underset{\sim}{\psi}(\tau)}\;\underbrace{\|\underset{\sim}{\varphi}'(\omega(\tau))\|\cdot|\omega'(\tau)|}_{\overset{*)}{=}\,\|\underset{\sim}{\psi}'(\tau)\|}\,d\tau$$

$$= \int_c^d f(\underset{\sim}{\psi}(\tau))\,\|\underset{\sim}{\psi}'(\tau)\|\,d\tau \overset{\text{pomocí } \underset{\sim}{\psi}}{=} \int_\gamma f\,ds$$

*) výpočet: $\underset{\sim}{\psi}(\tau) = \underset{\sim}{\varphi}(\omega(\tau))$

$\underset{\sim}{\psi}'(\tau) = \underset{\sim}{\varphi}'(\omega(\tau))\cdot\omega'(\tau)$

$\|\underset{\sim}{\psi}(t)\| = \|\underset{\sim}{\varphi}'(\omega(\tau))\|\cdot|\omega(\tau)|$

Věta 19.2. [Nezávislost int. 2. druhu na parametrizaci.]

dk. buď: γ ... jednoduchá, neuzavř., orientovaná křivka

$(\varphi(t), [a,b])$ ... parametrizace ve shodě s orientací

$(\psi(\tau), [c,d])$ ... libovolná parametrizace

$\underset{\sim}{F}(\underset{\sim}{x}) : \gamma \to \mathbb{R}^n$ daná fce

$$\int_\gamma \underset{\sim}{F} \cdot \underset{\sim}{ds}$$

$$= \int_a^b \underset{\sim}{F}(\underset{\sim}{\varphi}(t)) \cdot \underset{\sim}{\varphi}'(t)\, dt \quad \left| \begin{array}{l} t = \omega(\tau) \\ dt = \omega'(\tau) \\ (\ldots \mathcal{L}\, 19.1) \end{array} \right.$$

$$= \int_c^d \underset{\sim}{F}(\underset{\sim}{\varphi}(\omega(\tau))) \cdot \underset{\sim}{\varphi}'(\omega(\tau)) \cdot \underbrace{|\omega'(\tau)|}_{\pm \omega'(\tau)} d\tau$$

$$= \pm \int_c^d \underset{\sim}{F}(\underset{\sim}{\psi}(\tau)) \cdot \underset{\sim}{\psi}'(\tau)\, d\tau$$

pomocí vztahu: (viz důkaz L.19.1)

$$\underset{\sim}{\psi}(\tau) = \underset{\sim}{\varphi}(\omega(\tau))$$

$$\underset{\sim}{\psi}'(\tau) = \underset{\sim}{\varphi}'(\omega(\tau))\,\omega'(\tau)$$

přičemž: $\pm \iff \omega'(\tau) > 0 / < 0$

$\iff \omega(\tau)$ roste/klesá

$\iff \underset{\sim}{\varphi}, \underset{\sim}{\psi}$ vyjadřují orientaci shodnou/opačnou

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Věta 19.3. [Vlastnosti k.i.].

dk. P1. buď φ ... parametrizace γ

$$\int_\gamma (f+g)\,ds = \int_a^b \underbrace{(f+g)(\underset{\sim}{\varphi}(t))}_{(f(\underset{\sim}{\varphi}(t)) + g(\underset{\sim}{\varphi}(t)))} \cdot \|\varphi'(t)\|\,dt$$

$$= \int_a^b f(\underset{\sim}{\varphi}(t)) \cdot \|\underset{\sim}{\varphi}'(t)\|\,dt + \int_a^b g(\underset{\sim}{\varphi}(t)) \cdot \|\underset{\sim}{\varphi}'(t)\|\,dt$$

$$= \int_\gamma f\,ds + \int_\gamma g\,ds$$

<u>P3.</u> $$\left| \int_\gamma f\,ds \right| = \left| \int_a^b f(\underset{\sim}{\varphi}(t)) \cdot \|\varphi'(t)\|\,dt \right|$$

$$\leq \int_a^b |f(\underset{\sim}{\varphi}(t))| \cdot \|\varphi'(t)\|\,dt$$

$$\leq M_1 \int_a^b \|\varphi'(t)\|\,dt = M_1 \cdot \ell(\gamma)$$

neboť: $M_1 = \sup\{ |f(\underset{\sim}{x})| ;\ \underset{\sim}{x} \in \gamma \}$

$\qquad = \sup\{ |f(\underset{\sim}{\varphi}(t)) ;\ t \in [a,b] \}$

<u>D3.</u> podobně jako <u>P3</u>, s pomocí:

$$|\underset{\sim}{F}(\underset{\sim}{\varphi}(t)) \bullet \underset{\sim}{\varphi}'(t)| \leq \underbrace{\|\underset{\sim}{F}(\underset{\sim}{\varphi}(t))\|}_{\leq M_2} \cdot \|\varphi'(t)\|$$

neboť: $|\underset{\sim}{u} \bullet \underset{\sim}{v}| \leq \|\underset{\sim}{u}\| \cdot \|\underset{\sim}{v}\| ;\ \forall \underset{\sim}{u}, \underset{\sim}{v} \in R^n.$

__Lemma 19.2__ nechť $\underset{\sim}{F}: \Omega \to \mathbb{R}^m$, γ křivka v Ω od $\underset{\sim}{x}_0$ do $\underset{\sim}{x}_1$, U potenciál $\underset{\sim}{F}$.

Potom: $\displaystyle\int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = U(\underset{\sim}{x}_1) - U(\underset{\sim}{x}_0)$.

__dk.__ __1. KROK__: γ ... jednoduché,

$\underset{\sim}{x}_0, \underset{\sim}{x}_1$ .. p.b. / k.b. γ

$\underset{\sim}{\varphi}(t)$, $t \in [a, b]$ ... ve shodě,

tj. $\underset{\sim}{\varphi}(a) = \underset{\sim}{x}_0$, $\underset{\sim}{\varphi}(b) = \underset{\sim}{x}_1$.

$$L.S. = \int_a^b \underbrace{\underset{\sim}{F}(\underset{\sim}{\varphi}(t)) \cdot \underset{\sim}{\varphi}'(t)}\, dt$$

$$\| $$

$$\sum_{j=1}^m F_j(\underset{\sim}{\varphi}(t))\, \varphi_j'(t)$$

$$= \sum_{j=1}^m \frac{\partial U}{\partial x_j}(\underset{\sim}{\varphi}(t))\, \varphi_j'(t) = \frac{d}{dt}\Big(U(\underset{\sim}{\varphi}(t))\Big)$$

řetízkové pravidlo,
(viz Věta 14.3)

$$= \int_a^b \frac{d}{dt}\left(U(\underset{\sim}{\varphi}(t))\right) dt = \left[U(\underset{\sim}{\varphi}(t))\right]_{t=a}^{t=b}$$

$$= U(\underbrace{\underset{\sim}{\varphi}(b)}_{\underset{\sim}{x}_1}) - U(\underbrace{\underset{\sim}{\varphi}(a)}_{\underset{\sim}{x}_0}) = \text{P.S.}$$

<u>2. KROK</u>: $\gamma = \bigcup_{j=1}^{m} \gamma_j$, kde

$$\text{z.b.}\,\gamma_1 = x_0 , \quad \text{k.b.}\,\gamma_m = x_1$$

$$\text{k.b.}\,\gamma_j = \text{z.b.}\,\gamma_{j+1} , \quad j = 1, \dots, m-1$$

$$\int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = \sum_{j=1}^{m} \int_{\gamma_j} \underset{\sim}{F} \cdot d\underset{\sim}{s} \quad \Big| \; 1.\,\text{KROK}$$

$$= \sum_{j=1}^{m} \Big(U(\text{k.b.}\,\gamma_j) - U(\text{z.b.}\,\gamma_j) \Big)$$

$$= \cancel{U(\text{k.b.}\,\gamma_1)} - U(\underset{\sim}{x}_0) + \cancel{U(\text{k.b.}\,\gamma_2)} - \cancel{U(\text{z.b.}\,\gamma_2)}$$

$$\dots + U(\underset{\sim}{x}_1) - \cancel{U(\text{z.b.}\,\gamma_m)} = U(\underset{\sim}{x}_1) - U(\underset{\sim}{x}_0)$$

(teleskopické suma)