28. Aplikace teorie distribucí

(ODR) $\quad \underbrace{a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y}_{\mathcal{K}[y]} = f(t)$

Věta 28.1. Nechť $y(t)$ je řešení rce $\mathcal{K}[y] = 0$

s počátečními podmínkami

$$y(0) = 0$$
$$y'(0) = 0$$
$$\vdots$$
$$y^{(n-1)}(0) = \frac{1}{a_0} \qquad // \; y^{(n-2)}(0) = 0$$

Potom funkce $\tilde{y}(t) := \begin{cases} 0 \, ; & t \le 0 \\ y(t) \, ; & t > 0 \end{cases}$

je fundamentální řešení rce (ODR); tj.

$\mathcal{K}[\tilde{y}] = \delta_0$ ve smyslu distribucí

dk. $\quad \mathcal{K}[\tilde{y}] = \sum_{j=0}^{n} a_j \left(\frac{d}{dt}\right)^{n-j} \tilde{y} = (*)$

Lemma: $f(t)$ je po částech C^1; $t_1, \dots, t_n$ nespojitosti

$$\Rightarrow \underbrace{\frac{d}{dt}}_{\text{distributivní}} f(t) = \underbrace{f'(t)}_{\text{bodová}} + \sum_j \{f(t_j+) - f(t_j-)\} \delta_{t_j}$$

pozoruji: $\tilde{y} \in C^{n-2} \Rightarrow \left(\frac{d}{dt}\right)^{\ell} \tilde{y} = \tilde{y}^{(\ell)} \quad ; \ell = 0, \dots, n-1$

ale $\quad \left(\frac{d}{dt}\right)^{n} \tilde{y} = \tilde{y}^{(n)} + \frac{1}{a_0} \delta_0$

$$(*) = \underbrace{\sum_{j=0}^{n} a_j \tilde{y}^{(n-j)}}_{\text{splněno pro s.v. } t} + a_0 \cdot \frac{1}{a_0} \delta_0 = \delta_0$$

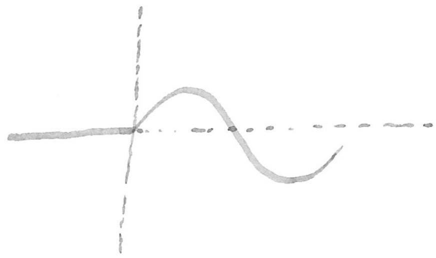
Příkl.: ① $y' + \lambda y = 0$ $\quad y(0) = 1 \quad y(t) = e^{-\lambda t}$

f.ř.: $\tilde{y}(t) = \begin{cases} 0 & ; t \le 0 \\ e^{-\lambda t} & ; t > 0 \end{cases}$

② $y'' + \omega^2 y = 0 \quad y(0) = 0$

$y'(0) = 1 \quad y(t) = \frac{1}{\omega} \sin(\omega t)$

f.ř.: $\tilde{y}(t) = \begin{cases} 0 & ; t \le 0 \\ \frac{1}{\omega} \sin \omega t & ; t > 0 \end{cases}$



(T) $\quad \partial_t u - \Delta u = 0 \; ; \; u = u(x,t)$

$x \in \mathbb{R}^n$

$t > 0$

Věta 28.2. Fundamentální řešení rce (T), Greenova funkce, má tvar $G(x,t) = \frac{h(t)}{(2\sqrt{\pi t})^n} \exp\left(-\frac{|x|^2}{4t}\right)$

// $h(t)$ je Heaviside

$$\partial_t u(x,t) - \Delta u(x,t) = \delta_{(0,0)}(x,t) = \delta_0(x) \otimes \delta_0(t) \quad \Big| \; \mathcal{F}_x \; ; \; t \text{ pevné} \in \mathbb{R}$$

$$\partial_t \hat{u}(\xi,t) + 4\pi^2 |\xi|^2 \hat{u}(\xi,t) = 1(\xi) \otimes \delta_0(t) = \delta_0(t)$$

ξ fixuji: $\xi \in \mathbb{R}^n$ pevné, řeším ODR v prom. t

Příkl. 1: $\quad \hat{u}(\xi,t) = h(t) e^{-4\pi^2 |\xi|^2 t} = \hat{G}(\xi,t)$

aplikací vzorce vpravo s $b = 4\pi^2 t$:

$$G(x,t) = h(t) \left(\frac{\pi}{4\pi^2 t}\right)^{\frac{n}{2}} e^{\frac{-\pi^2 |x|^2}{4\pi^2 t}}$$

$$\boxed{\begin{aligned} & e^{-\pi|x|^2} \xrightarrow{\mathcal{F}} e^{-\pi|\xi|^2} \\ & f(cx) \xrightarrow{\mathcal{F}} \frac{1}{c^n} \hat{f}(\xi/c) \\ & e^{-b|x|^2} \xrightarrow{\mathcal{F}} \left(\frac{\pi}{b}\right)^{\frac{n}{2}} e^{-\frac{\pi^2|\xi|^2}{b}} \\ & e^{-b|\xi|^2} \xrightarrow{\mathcal{F}_{-1}} \left(\frac{\pi}{b}\right)^{\frac{n}{2}} e^{-\frac{\pi^2|x|^2}{b}} \end{aligned}}$$

$\partial_t u - \Delta u = \delta_{(0,0)}$ $\quad x \in \mathbb{R}^n$
$\quad t \in \mathbb{R}$

$$G(x,t) = \left(\frac{1}{4\pi t}\right)^{n/2} e^{-\frac{|x|^2}{4t}} h(t)$$

fundamentální řešení (tepelné jádro)

Motivace 1. $f = f(x,t)$ dáno: $G * f(x,t)$ je řešení

$$\partial_t u - \Delta u = f$$

Motivace 2. $\partial_t u - \Delta u = 0$

$$u(x,0) = u_0(x)$$

formálně F_x: $\partial_t \hat{u}(\xi,t) + 4\pi|\xi|^2 \hat{u}(\xi,t) = 0$

$$\hat{u}(\xi,0) = \hat{u}_0(\xi)$$

$$\hat{u}(\xi,t) = \underbrace{e^{-4\pi|\xi|^2 t}}_{\hat{G}(\xi,t)} \hat{u}_0(\xi)$$

$$G \underset{(x)}{*} u_0(x,t) = u(x,t)$$

Pozn.: $G(x,t)$ má tyto vlastnosti

1. $G(x,t) > 0$; C^∞ v $\mathbb{R}^n \times (0,\infty)$
2. $\underbrace{\int_{\mathbb{R}^n} G(y,t)\,dy}_{\hat{G}(0,t)=1} = 1 \quad \forall t > 0$ pevné
3. $(\partial_t - \Delta) G \equiv 0$ pro $t > 0$ — pilný výpočet (d. cv.)
4. $\lim_{t\to 0+} G(x,t) = \delta_0(x)$

Věta 28.3. Nechť $u_0(x) \in C \cap L^\infty(\mathbb{R}^n)$

Definujme $u(x,t) = [G \underset{x}{*} u_0](x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, u_0(y)\,dy$

Potom platí
1. $u \in C^\infty(\underbrace{\mathbb{R}^n \times (0,\infty)}_{Q})$ omezená; $\sup_Q u \le \sup_{\mathbb{R}^n} u_0$
2. $\partial_t u - \Delta u = 0$ v Q
3. $\lim_{t\to 0+} u(x,t) = u_0(x)$ pro $\forall x \in \mathbb{R}^n$

dů.:

1. $$D^\alpha_{x,t} u = \int_{\mathbb{R}^n} D^\alpha_{x,t} G(x-y,t)\, u_0(y)\, dy$$

majoranta: nezávislá na $x \in \mathbb{R}^n$, $t \in (\delta, \infty)$; $\delta > 0$ pevné

$$|u_0(y)| \leq K$$

$$\left| D^\alpha_{x,t} G(x-y,t) \right| \leq p\left(\tfrac{1}{t}, x\right) e^{\frac{-|x-y|^2}{4t}} \quad // p \text{ jako polynom}$$

$$|u(x,t)| \leq \int_{\mathbb{R}^n} |G(x-y,t)| \underbrace{|u_0(y)|}_{\leq K := \sup_{\mathbb{R}^n} u_0} dy \leq K \underbrace{\int_{\mathbb{R}^n} G(x-y,t)\, dy}_{1} = K$$

$\forall x,t$

2. $$(\partial_t - \Delta)\left[G *_x u_0\right](x,t) = \left\{ \underbrace{(\partial_t - \Delta) G}_{= 0 \text{ pro } t > 0} * u_0 \right\}(x,t) = 0$$

3. $$u(x,t) - u_0(x) = \int_{\mathbb{R}^n} G(x-y,t)\, u_0(y)\, dy - u_0(x) =$$

$x \in \mathbb{R}^n$ pevné

$$= \int_{\mathbb{R}^n} G(x-y,t) \left\{ u_0(y) - u_0(x) \right\} dy =$$

$$\overset{(*)}{=} \int_{|x-y|<\sigma} \dots + \int_{|x-y|>\sigma} \dots = I_1 + I_2$$

(*) $\varepsilon > 0$ dáno: $\exists\, \sigma > 0$, BÚNO $\sigma < 1$

$$|x-y| < \sigma \Rightarrow |u_0(x) - u_0(y)| < \varepsilon$$

$$|I_1| \leq \int_{|x-y|<\sigma} |G(x-y,t)| \cdot \varepsilon\, dy \leq \int_{\mathbb{R}^n} \varepsilon\, G(x-y,t)\, dy = \varepsilon$$

$$|I_2| \leq \int_{|x-y|>\sigma} |G(x-y,t)| \cdot \left\{ |u_0(x)| + |u_0(y)| \right\} dy$$

$$\leq 2K \int_{|x-y|>\sigma} \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x-y|^2}{4t}}\, dy \leq \quad \Bigg| \; e^{-\frac{|x-y|^2}{4t}} = e^{-\frac{|x-y|^2}{8t}} \cdot e^{-\frac{|x-y|^2}{8t}} \leq$$

$$\leq e^{\frac{-\sigma^2}{4t}} \cdot e^{-\frac{|x-y|^2}{8}}$$

BÚNO $t \in (0,1)$

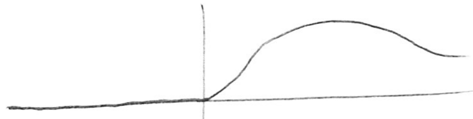
$$\leq 2K \int_{\mathbb{R}^n} \underbrace{\frac{e^{-\frac{\sigma^2}{4t}}}{(4\pi t)^{n/2}}}_{\varphi(t)} \cdot \overbrace{e^{\frac{-|x-y|^2}{8}}}^{L^1 \text{ fce nezávislá na } t} dy \longrightarrow 0 \quad \varphi(t) \to 0,\ t \to 0+$$

celkem: $|I_2| < \varepsilon$ pro t malé

$$|u(x,t) - u_0(x)| = |I_1 + I_2| \le 2\varepsilon$$

Příklad (Tichonov 1935)

$$\varphi(t) = \begin{cases} e^{-\frac{1}{t^2}} & ; \ t > 0 \\ 0 & ; \ t \le 0 \end{cases}$$



$\in C^\infty(\mathbb{R})$; $\varphi^{(n)}(0) = 0 \quad \forall n$

$$u(x,t) := \sum_{\ell=0}^{\infty} \underbrace{\varphi^{(\ell)}(t)}_{\le \frac{\ell!}{t^{2\ell}} e^{-\frac{1}{4t^2}}} \frac{x^{2\ell}}{(2\ell)!}$$

... konvergence
$u \in C^\infty$

$$\boxed{\begin{aligned} \partial_t u - \Delta u &= 0 \\ u(x,0) &= 0 \end{aligned}}$$

netriviální řešení

$$u(x,0) = \sum_{\ell=0}^{\infty} \varphi^{(\ell)}(0) \frac{x^{2\ell}}{(2\ell)!} = 0$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} = \sum_{\ell=1}^{\infty} \varphi^{(\ell)}(0) \frac{2\ell(2\ell-1)\, x^{2(\ell-1)}}{2\ell\cdot(2\ell-1)[2(\ell-1)]!}$$

$$= \sum_{\ell=0}^{\infty} \varphi^{(\ell+1)}(t) \frac{x^{2\ell}}{(2\ell)!} = \partial_t u$$

nejednoznačnost $\Leftarrow$ úloha není korektně zadaná
(chybí okrajová podmínka)

Lze dokázat: $\partial_t u - \Delta u = 0 \quad ; \ t > 0, x \in \mathbb{R}^n$

$u(0,x) = 0$

& navíc: u omezená fce

$\Rightarrow u \equiv 0$ pro $t > 0$

Věta 28.4. Nechť $f(x,t) \in C(\mathbb{R}^n \times [0,\infty))$, omezená

Definuji $u(x,t) = G *_{x,t} f(x,t) = \iint_{\mathbb{R}^n \times (0,t)} G(x-y,t-s) f(y,s)\,dy\,ds$

Potom 1. $u \in C^\infty(Q)$; $\sup_{\mathbb{R}^n \times (0,T)} |u| \leq T \cdot \sup_{\mathbb{R}^n \times (0,T)} f$

2. $\partial_t u - \Delta u = f$ v Q

3. $\lim_{t \to 0+} u(x,t) = 0$

dk. $u(x,t) = \int_0^t \underbrace{\left(\int_{\mathbb{R}^n} G(x-y,t-s) f(y,s)\,dy\right)}_{v_s(x,t)} ds$

klíčové pozorování: $v_s(x,t) = \{G(\cdot, t-s) * f(\cdot, s)\}(x)$

V.28.3: $\forall s$ pevné v_s řeší: $\partial_t v_s - \Delta v_s = 0$; $t > s$

$v_s(x,s) = f(x,s)$

$u \in C^\infty(Q)$ ... nedokazujeme

V28.3.: $\sup_{\substack{x \in \mathbb{R}^n \\ t > s}} |v_s(x,t)| \leq \sup_{x \in \mathbb{R}^n} |f(x,s)|$

$$|u(x,t)| \leq \int_0^t |v_s(x,t)|\,ds \leq t \cdot \sup_{\substack{x \in \mathbb{R}^n \\ s \in (0,t)}} |f(x,s)|$$

$$\leq T \cdot \sup_{\substack{x \in \mathbb{R}^n \\ t \in (0,t)}} |f(x,t)| \quad \Big| \sup_{t \in (0,t)}$$

důsledek: bod 3: $|u(x,t)| \leq \sup_{\substack{x \in \mathbb{R}^n \\ \tau \in [0,t]}} |u(x,\tau) \leq t \cdot K \xrightarrow{t \to 0+} 0$; $K = \sup_{\substack{x \in \mathbb{R}^n \\ \tau \in [0,t]}} |f(x,\tau)| < \infty$

x pevné

2. $\partial_t u = \partial_t \int_0^t v_s(x,t)\,ds = \underbrace{v_t(x,t)}_{f(x,t)} + \int_0^t \underbrace{\partial_t v_s(x,t)}_{\Delta v_s(x,t)}\,ds = f + \Delta \underbrace{\int_0^t v_s(x,t)\,ds}_{u}$

Pozn. Věty 28.3. & 28.4 $\Rightarrow$ $\exists$ řešení pro $\partial_t u - \Delta u = f;\ x \in \mathbb{R}^n,\ t>0$
(+ linearita problému) $\qquad u(0,x) = u_0(x)$

f, u_0 ... spojité, omezené

(RVT-1) $\quad \partial_t u - \Delta u = f(x,t) \quad ; \quad x \in \Omega\ ;\ t > 0$

(RVT-2) $\quad u(x,0) = u_0(x) \quad ; \quad x \in \Omega$

(RVT-3) $\quad u(x,t) = g(x,t) \quad x \in \partial\Omega\ ;\ t > 0$

(RVT-3') $\quad \nabla u(x,t) \cdot n = h(x,t) \quad x \in \partial\Omega\ ;\ t > 0$

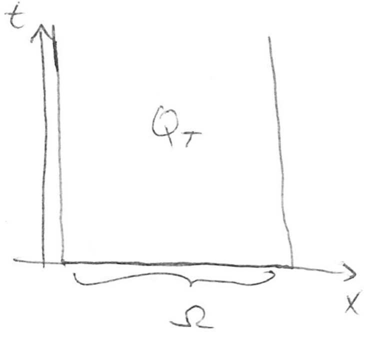
Neumann

Pozn. $\Omega = \mathbb{R}^n$: Cauchyova úloha
$\partial\Omega = \emptyset$ tj. (RVT-3) nemá smysl

Značení: $Q_T = \Omega \times (0,T)$

$\Gamma_T = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0,T]$

parabolická hranice



Def: Nechť $u_0 \in C(\overline{\Omega})$, $g \in C(\partial\Omega \times [0,T])$
Funkce u se nazve klasické řešení (RVT-1,2,3)
jestliže $u \in C(\overline{Q}_T)$; $\partial_t u$, $\Delta u \in C(\overline{Q}_T \setminus \Gamma_T)$
a rce (RVT-1,2,3) platí ve všech bodech příslušného oboru.

? existence, ? jednoznačnost

ad existence řešení

$\Omega = \mathbb{R}^n$: Věta 28.3., 28.4. vzoreček!! (Four. transf.)

OPRAVA $\quad$ V.28.4. $f \in C(\overline{Q}_T) \not\Rightarrow u \in C^\infty(Q)$

$f \in C^1 \Rightarrow \partial_t \in C(Q)$

V.28.3.: $u_0 \in C(\mathbb{R}^n) \Rightarrow u \in C^\infty(Q)$

$\Omega \subset \mathbb{R}^n$ obecné : Fourierova metoda

$$-\Delta u = \lambda u \; ; \quad x \in \Omega$$

$$u = 0 \; ; \quad x \in \partial\Omega$$

$\Rightarrow \{u_k\} ; \{\lambda_k\} ; \; \lambda_n > 0 \; ; \; \lambda_n \to \infty$

u_n OG v $L^2(\Omega)$

ansatz : $u(x,t) = \sum_k c_k e^{-\lambda_k t} u_k(x)$

... (formálně) řeší (RVT - 1,3) $f \equiv 0$
$g \equiv 0$

(RVT - 2) : $\sum_k c_k u_k(x) = u_0(x)$

---

ad jednoznačnost : princip maxima
energetická nerovnost

Věta 28.5. [Princip maxima pro RVT]
Nechť $\Omega \subset \mathbb{R}^n$ je omezená ; nechť u je klasické řešení (RVT - 1,2,3), nechť $f \leq 0$. Potom

$$\max_{\overline{Q_T}} u = \max_{\Gamma_T} u$$

uzávěr!

dk. : $M := \max_{\overline{Q}_T} u$; $m = \max_{\Gamma_T} u$ (spojitá na kompaktu $\Rightarrow \exists \max$)

sporem ?? určitě $m \leq M$ ($\Gamma_T \subset \overline{Q}_T$)

$m < M$ tj. $\exists (x_0, t_0) \in \overline{Q}_T \setminus \Gamma_T$; $u(x_0, t_0) = M > m = \sup_{\Gamma_T} u$

1. verze : nechť $f < 0$

$x_0 \in \Omega$ je maximum $u(\cdot, t_0)$ vůči Ω

(vnitřní bod) $\Rightarrow$ $\frac{\partial u}{\partial x_i}(x_0, t_0) = 0$

$$\frac{\partial^2 u}{\partial x_i^2}(x_0, t_0) \leq 0$$

$t_0 \in (0, T]$ je maximum $u(x_0, \cdot)$ vůči $(0, T]$

$\Rightarrow \frac{\partial u}{\partial t}(t_0, x_0) \geq 0$.

$$\underbrace{(\partial_t - \Delta) u(x_0, t_0)}_{\geq 0} = \underbrace{f(x_0, t_0)}_{< 0}$$

2. verze $f \leq 0$: stejný předpoklad

$$\tilde{u}(x,t) = u(x,t) + \frac{1}{K}|x - x_0|^2$$

$K > 0$ tak velké, že: $\frac{1}{K}|x - x_0|^2 \leq \frac{M-m}{2} \quad \forall x \in \Gamma_T$

$$\max_{\overline{Q}_T} \tilde{u} \geq \tilde{u}(x_0, t_0) = M$$

$$\max_{\Gamma_T} \tilde{u} \leq \max_{\Gamma_T} u + \max_{\Gamma_T} \frac{1}{K}|x - x_0|^2 \leq m + \frac{M-m}{2}$$

$$= \frac{M+m}{2} < M$$

$\tilde{u}$ má maximum uvnitř, avšak

$$(\partial_t - \Delta)\tilde{u} = (\partial_t - \Delta) u + (\partial_t - \Delta)\frac{1}{K}|x - x_0|^2 = f - \frac{2n}{K}$$

$$\Delta |x - x_0|^2 = \sum_j \frac{\partial^2}{\partial x_j^2}\left(\sum_\ell (x_\ell - x_{0_\ell})^2\right) = 2n$$

spor : 1. verze

Věta 28.5.' [Princip minima]

... $f \geq 0$ ... min ... min ...

Důsledek : Úloha (RVT – 1,2,3) má nejvýše jedno klasické řešení!

dk. u_1, u_2 ... řešení ; $w := u_1 - u_2$

$\Rightarrow$ w je klasické řešení (RVT – 1,2,3) pro $f \equiv 0$

$u_0 \equiv 0$, $g \equiv 0$

V. 28.5. $\Rightarrow$ $\max_{\overline{Q}_T} w = \max_{\Gamma_T} w = 0$; tj: $w \leq 0$

V. 28.5.' $\Rightarrow$ min min 0 ; $w \geq 0$

$w \equiv 0$; tj: $u_1 \equiv u_2$

Lemma 28.1. Nechť u je klasické řešení (RVT-1,2)
s nulovou okrajovou podmínkou,
tj. (RVT-3) pro $g=0$, nebo (RVT-3') pro $h=0$;
nechť $f\equiv 0$. Potom

$$\frac{1}{2}\int_\Omega u^2(x,\tau)\,dx + \int_0^\tau\int_\Omega |\nabla u(x,t)|^2\,dx\,dt = \frac{1}{2}\int_\Omega u^2(x,0)\,dx$$

Dk. $\partial_t u - \Delta u = 0 \quad / \, u\int_\Omega dx$

$$\int_\Omega \underbrace{\partial_t u(x,t)\,u(x,t)}_{\frac{1}{2}\partial_t u^2(x,t)}\,dx = \frac{d}{dt}\frac{1}{2}\int_\Omega u^2(x,t)\,dx$$

pomocný výpočet: $\operatorname{div}(u\nabla u) = |\nabla u|^2 + u\nabla u$

$$\frac{\partial}{\partial x_j}\left(u\frac{\partial u}{\partial x_j}\right) = \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_j} + u\frac{\partial^2 u}{\partial x_j^2}$$

Gaussova věta $\int_{\partial\Omega} F\cdot n\,dS = \int_\Omega \operatorname{div} F\,dx$

$F = u\nabla u$

$$\int_{\partial\Omega} \underbrace{u\nabla u\cdot n}_{\equiv 0}\,dS = \int_\Omega |\nabla u|^2 + u\Delta u\,dx$$

díky okr. podm.

$$\Rightarrow \int_\Omega (-\Delta u)u\,dx = \int_\Omega |\nabla u|^2\,dx$$

čili $\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)\,dx + \int_\Omega |\nabla u(x,t)|^2\,dx = 0 \quad \Big/ \int_0^\tau dt$

$\to$ závěr

Důsledek: jednoznačnost řešení (RVT-1,2,3) resp. (RVT-1,2,3')

$\int_\Omega u^2(x,\tau)\,dx \le \int_\Omega u^2(x,0)\,dx$; tedy $u_0\equiv 0 \Rightarrow u(x,t)\equiv 0$
$\forall t>0$

Dodatečný předpoklad: Ω omezená „rozumná" (platí Gaussova věta)

Pozn.: princip maxima pro $\Omega = \mathbb{R}^n$: NEPLATÍ

Tichonov: $\partial_t u - \Delta u = 0$ v $\mathbb{R}^n \times (0,\infty)$

$u(x,0) = 0$; ale $u \not\equiv 0$ pro $t > 0$

$$\max_{\Gamma} u = \max_{\mathbb{R}^n} u(x,0) = 0$$

Věta 28.5.'' u klasické řešení RVT v $\mathbb{R}^n$; $f \leq 0$

navíc u omezená v Q_T. Pak

$$\sup_{\overline{Q_T}} u \leq \sup_{\Omega} u_0$$

Vlnové rovnice

(VR − 1) $\partial_{tt} u - \Delta u = f(x,t)$ v Q_T

(VR − 2) $u(x,0) = u_0(x)$ v Ω

~~(VR − 3)~~ $\partial_t u(x,0) = u_1(x)$

(VR − 3) $u(x,t) = g(x,t)$ v $\cancel{\Gamma_T}$

nebo (VR − 3') $\frac{\partial u}{\partial n}(x,t) = \nabla u(x,t) \cdot n = h(x,t)$

$\Omega \subset \mathbb{R}^n$ oblast; $Q_T = \Omega \times (0,T)$

$$\Gamma_T = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0,T]$$

Cauchyho úloha: $\Omega = \mathbb{R}^n$

-- fundamentální řešení

$$\partial_{tt} u - \Delta u = \delta_{(0,0)}$$

Věta 28.6. Fundamentálním řešením vlnové rovnice je distribuce $H(x,t)$, kde

$n=1$: $H(x,t) = \frac{1}{2}\chi_{(-t,t)}(x)$

$n=2$: $H(x,t) = \frac{1}{2\pi}\frac{1}{\sqrt{t^2 - x^2}}\chi_{(x<t^2)}(x,t)$

$n=3$: $H(x,t) = \frac{1}{4\pi t}\nu_t$; kde ν_t je Dirac na sféře $\{x \in \mathbb{R}^3;\ |x| = t\}$

$$\langle \nu_t, \varphi \rangle = \int_{|x|=t} \varphi(x)\, dS(x)$$

dk.: $\partial_{tt} H(x,t) - \Delta H(x,t) = \delta_{(0,0)}(x,t) = \delta_0(x) \otimes \delta_0(t)$ $\quad \mathcal{F}_x$

$$\partial_{tt} \hat{H}(\xi,t) + 4\pi^2 |\xi|^2 \hat{H}(\xi,t) = \delta_0(t)$$

$\xi \neq 0$, pevné: V.28.1 : $\hat{H}(\xi,t) = h(t)\, y(t)$

kde $y'' + 4\pi^2|\xi|^2 y = 0$

$y(0) = 0$

$y'(0) = 1$

$$y(t) = \frac{\sin 2\pi|\xi|t}{2\pi|\xi|}$$

$$\Rightarrow \hat{H}(\xi,t) = \frac{\sin 2\pi|\xi|t}{2\pi|\xi|} h(t) \; ; \; |\xi| \neq 0$$

$$H(x,t) = \mathcal{F}^{-1}_{\xi}\Big(\underbrace{\frac{\sin 2\pi|\xi|t}{2\pi|\xi|}}_{u(\xi)}\Big) h(t)$$

$\hookrightarrow$ neexistuje univerzální vzorec

pro dimenzi $n=1$

$$n=1: \quad u(\xi) = \frac{\sin 2\pi\xi t}{2\pi\xi}$$

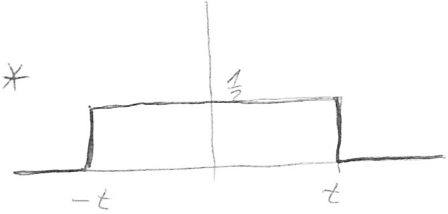
$$2\pi\xi\, u(\xi) = \sin 2\pi\xi t$$

$$\text{(LS)} \quad -i\, 2\pi i \xi\, u(\xi) = (-i)\Big\{\frac{d}{dx} u(x)\Big\}^{\wedge}(\xi)$$

$$(\delta_b)^{\wedge}(\xi) = e^{-2\pi i b\xi} = \cos 2\pi b\xi - i \sin 2\pi b\xi$$

$$\Rightarrow \sin 2\pi\xi t = -\frac{i}{2}(\delta_{-t} - \delta_t)^{\wedge}(\xi) = \text{(PS)}$$

$$\underbrace{\frac{1}{2}(\delta_{-t} - \delta_t)}_{*} = \frac{d}{dx} u(x)$$

$*$ 

$\boxed{n = 2,3}$ ... viz cvičení

Pozn. $f(x,t)$ ... dostatečně regulární fce

$$u := H *_{x,t} f \quad \ldots \quad (\partial_{tt} - \Delta) u = (\partial_{tt} - \Delta u) H *_{x,t} f$$

$$\underbrace{[(\partial_{tt} - \Delta) H]}_{\delta_{(0,0)}} *_{x,t} f = f$$

počáteční úloha: $\partial_{tt} u - \Delta u = 0 \quad ; x \in \mathbb{R}^n ; t > 0$

$$\left.\begin{aligned} u(x,0+) &= 0 \\ \partial_t u(x,0+) &= \delta_0(x) \end{aligned}\right\} \text{ ve smyslu } \lim_{t \to 0+}$$

$\mathcal{F}_x$: $\partial_{tt} \hat{u}(\xi,t) + 4\pi^2 |\xi|^2 = 0 \quad ; t > 0$

$$\hat{u}(\xi, 0+) = 0$$

$$\partial_t \hat{u}(\xi, 0+) = 1$$

$\Rightarrow \quad \hat{u}(\xi,t) = \hat{H}(\xi,t)$; tedy $u(x,t) = H(x,t)$; $t > 0$

$u_1(x)$ ... dáno, dost hladká fce

$u(x,t) := (u_1 *_x H)(x,t)$

$\Rightarrow u$ řeší $\quad \partial_{tt} u - \Delta u = 0$

$$u(x, 0+) = 0$$

$$\partial_t u(x,0+) = u_1(x)$$

- - - - -

$$\partial_{tt} u - \Delta u = 0$$

$$u(x,0+) = \delta_0(x)$$

$$\partial_t u(x,0+) = 0$$

pozorování: $u(x,t) := \partial_t H(x,t)$ je hledané řešení

$$(\partial_{tt} - \Delta) u = (\partial_{tt} - \Delta) \partial_t H(x,t) = \partial_t \{ \underbrace{(\partial_{tt} - \Delta) H}_{0 \text{ pro } t > 0} \}$$

$$u(x,0+) = \partial_t H(x,0+) = \delta_0(x)$$

$$\partial_t u(x,0+) = \partial_{tt} H(x,0+) = \Delta \underbrace{H(x,0+)}_{0} = 0$$

Věta 28.7.* Nechť f, u_0, u_1 jsou dostatečně hladké. Pak funkce $u(x,t) := (f *_{x,t} H)(x,t) + (u_1 *_x H)(x,t) +$

$$+ \partial_t (u_0 *_x H)(x,t)$$ je klasické

řešení (VR-1) v $\mathbb{R}^n$ splňující (VR-2)

Přímé řešení viz web

Pozn.: RVT vs. VR $\quad u_0(x) = \chi_{(-1,1)}(x)$; $n=1$

① RVT : $u(x,0) = u_0(x)$; $x \in \mathbb{R}^n$

$$\Rightarrow u(x,t) = (u_0 *_x G)(x,t) =$$

$$= \int_{\mathbb{R}} u_0(y) \frac{e^{-\frac{(x-y)^2}{4t}}}{(4\pi t)^{1/2}} \, dy = \frac{1}{(4\pi t)^{1/2}} \int_{-1}^{1} e^{-\frac{(x-y)^2}{4t}} \, dy$$

$u \in C^\infty \quad \forall t > 0 \quad x \in \mathbb{R}^n$

$u > 0 \quad$ –"–

② VR : $u(x,0) = u_0(x)$

$\partial_t u(x,0) = 0$

$$\Rightarrow u(x,t) = \partial_t (u_0 *_x H)(x,t) = (u_0 *_x \partial H)(x,t) \overset{(71)}{=\!=}$$

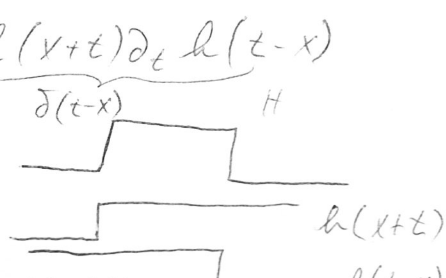
$H(x,t) = \frac{1}{2} \chi_{(-t,t)}(x)$ $\quad$ pak $\partial_t H(x,t) = \frac{1}{2}(\delta_{-t} + \delta_t)$

$$\hat{H} = \frac{\sin 2\pi |\xi| t}{2\pi |\xi|} = \frac{\sin 2\pi \xi t}{2\pi \xi} \quad ; n=1$$

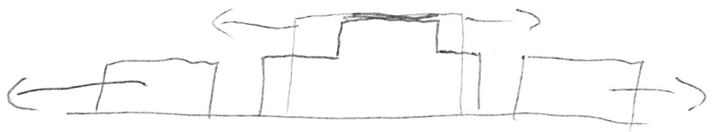
$$\partial_t \hat{H} = \cos 2\pi |\xi| t = \left\{ \tfrac{1}{2} \delta_{-t} + \tfrac{1}{2} \delta_t \right\}^\wedge$$

$$\partial_t H(x,t) = \partial_t \big(h(x+t)\, h(t-x) \big) = \overbrace{\partial_t h(x+t)}^{\delta(x+t)} h(t-x) +$$

$$+ \underbrace{h(x+t) \partial_t h(t-x)}_{\delta(t-x)}$$



(71) $\frac{1}{2}(\delta_{-t} + \delta_t) * u_0 = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right]$



Def.: Funkce $u(x,t)$ se nazývá klasické řešení (VR), pokud $u \in C(\overline{Q_T})$; $\partial_{tt}, \Delta u \in C(\overline{Q_T} \setminus \Gamma_T)$, přičemž (VR − 1,2,3) nebo (VR − 1,2,3') platí ve všech bodech

Lemma 28.2. Nechť u je klasické řešení (VR), nechť Ω je omezená a nechť $f \equiv 0$; $g = 0$ nebo $h = 0$.

Potom

$$\int_\Omega |\partial_t u(x,\tau)|^2 + |\nabla_x u(x,\tau)|^2 \, dx = \int_\Omega |u_1(x)|^2 + |\nabla_x u_0(x)|^2 dx$$

dk. $\partial_{tt} u - \Delta u = 0$; $\partial_t u \quad \int_\Omega dx$

$$\int_\Omega \underbrace{\partial_{tt} u(x,t)\, \partial_t u(x,t)}_{\frac{1}{2}\frac{\partial}{\partial t}(\partial_t u(x,t))^2} \, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega (\partial_t u)^2(x,t)\, dx$$

$$\operatorname{div}(\nabla u\, \partial_t u) = \Delta u\, \partial_t u + \underbrace{\nabla u \cdot \nabla \partial_t u}_{\cancel{\frac{1}{2}\partial_t}\ \frac{1}{2}\frac{d}{dt}|\nabla u|^2}$$

$$\nabla u \cdot \nabla \partial_t u = \sum_j \frac{\partial u}{\partial x_j}\partial_t\left(\frac{\partial u}{\partial x_j}\right)$$

Gauss : $\int_{\partial\Omega} \partial_t u \nabla u \cdot n \, dS = \int_\Omega \Delta u\, \partial_t u \, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx$

Dirichlet : $u(x,t) \equiv 0 \quad \forall x \in \partial\Omega$; $t \in (0,\tau)$

$\partial_t u(x,t) \equiv 0$

Neumann : $\nabla u \cdot n \equiv 0$

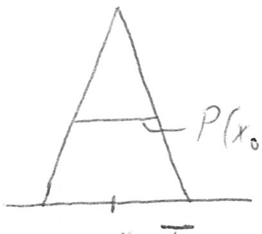
$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} \{|\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2\}\, dx = 0 \quad \Big/ \int_0^{\tau} dt$$

Důsl: jednoznačnost řešení

Lemma 28.3. [Princip šíření vlny]

Nechť $u(x,t)$ je klasické řešení (VR), nechť koule $B(x_0,t) \subset \Omega$, nechť $f \equiv 0$ na $P(x_0,t) := \{(x,t) : |x-x_0| < T-t\}$

Potom
$$\int_{B(x_0,T-t)} |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2\, dx \le \int_{B(x_0,T)} |u_1(x)|^2 + |\nabla u_0(x)|^2\, dx$$



Dk.
$$e(t) := \int_{B(x_0,T-t)} (\partial_t u)^2(x,t) + |\nabla u(x,t)|^2\, dx$$

stačí ukázat $e'(t) \le 0 \implies e(t) \le e(0)$ ... závěr

$$e'(t) = \frac{d}{dt}\int_{B(x_0,T-t)} (\partial_t u)^2(x,t) + |\nabla u(x,t)|^2\, dx =$$

pomocný výpočet:
$$\frac{d}{dt}\int_{\Omega(t)} g(x,t)\,dx = \int_{\Omega(t)} \frac{\partial g}{\partial t}(x,t)\,dx + \int_{\partial\Omega(t)} g(x,t) \underbrace{V_n(x,t)}_{\text{normálová složka pohybu } \partial\Omega(t)} dS$$

$$= \int_{B(x_0,T-t)} 2\partial_t u\, \partial_{tt} u + 2\nabla u \cdot \nabla u_t\, dx - \int_{\partial B(x_0,T-t)} (\partial_t u)^2 + |\nabla u|^2 \cdot dS \quad // \ V_n \text{ je tu } = 1$$

per partes: $\operatorname{div}(u_t \nabla u) = \nabla u_t \cdot \nabla u + u_t \Delta u \quad \Big/ \int_{B(x_0,T-t)}$

$$\int_{\partial B(x_0,T-t)} u_t \nabla u \cdot n \, dS = \int_{B(x_0,T-t)} \nabla u_t \cdot \nabla u + u_t \Delta u \, dx$$

$$e'(t) = \int_{B(x_0,T-t)} 2\partial_t u \, \partial_{tt} u - 2\partial_t u \Delta u \, dx + \int_{\partial B(x_0,T-t)} 2\partial_t u \nabla u \cdot n - |\partial_t u|^2 - |\nabla u|^2 dS$$

$$|2\partial_t u \nabla u \cdot n| \leq 2|\partial_t u||\nabla u| \underset{\text{Young}}{\leq} |\partial_t u|^2 + |\nabla u|^2$$

$|n| = 1$

2. integrand ≤ 0

Důsl: $u, \tilde{u}$ ... řeší (VR-1) pro p.s. $f, \tilde{f}$

s poč. podm. u_0, u_1 resp. $\tilde{u}_0, \tilde{u}_1$

nechť $u_0 = \tilde{u}_0$, $u_1 = \tilde{u}_1$ v $B(x_0,T)$

nechť $f = \tilde{f}$ v $P(x_0,T)$

Pak $u = \tilde{u}$ v $P(x_0,T)$

dk. L.28.3. na $w = u - \tilde{u}$

$\Rightarrow \partial_t w, \nabla_x w \equiv 0$ v $P(x_0,T)$

$\Rightarrow w =$ konst. v $P(x_0,T)$

$w(x,0) = 0 \Rightarrow w \equiv 0$

Důsl.: Jednoznačnost řešení (VR) pro Cauchyho úlohu ($\Omega = \mathbb{R}^n$)

Def. Nechť $\Omega \subset \mathbb{R}^n$ je oblast. Laplaceovou rovnicí rozumíme

$$-\Delta u = 0 \qquad x \in \Omega$$

Poissonovou rovnicí rozumíme

$$-\Delta u = f(x) \; ; \quad x \in \Omega$$

Věta 28.8. Fundamentálním řešením Poissonovy rovnice v $\mathbb{R}^n$ je

$$\Phi(x) = -\frac{1}{2\pi} \ln|x| \quad ; \quad n=2$$

$$\Phi(x) = \frac{1}{(n-2)\beta_n} |x|^{2-n} \quad ; \quad n \geq 3$$

Značení: α_n ... objem jednotkové koule v $\mathbb{R}^n$

β_n ... povrch jednotkové sféry v $\mathbb{R}^n$

platí: $\beta_n = n\,\alpha_n$

sférická Fubiniho věta: $\displaystyle\int_{B(0,R)} f(x)\,dx = \int_0^R \left(\int_{|y|=r} f(y)\,dS(y) \right) dr$

speciálně $\displaystyle\int_{B(0,R)} F(|x|)\,dx = \int_0^R F(r)\,\beta_n\, r^{n-1}\,dr$

Důsl. $|x|^a \in L^1_{loc}(\mathbb{R}^n) \underset{(d.cv)}{\Leftrightarrow} a > -n$

V 28.8.: $\quad -\Delta\Phi = \delta_0 \;$ v $\mathcal{D}'(\mathbb{R}^n)$

náčrt důkazu: $\left.\begin{aligned} \nabla|x|^{\lambda+2} &= (\lambda+2)|x|^{\lambda} x \\ \Delta|x|^{\lambda+2} &= (\lambda+2)(\lambda+n)|x|^{\lambda} \end{aligned}\right\}$ d. cv.

platí v $\mathcal{D}'$ pro $\lambda > -n$ ($\Leftarrow$ lokální integr.)

? $\lambda \to -n$: $\quad x_+^{\lambda} \in \mathcal{D}'(\mathbb{R})$; $\lambda > -1$ regulární

$$(\lambda+1)\, x_+^{\lambda} \to \delta_0 \; ; \; \lambda \to -1$$

analogicky platí: $(\lambda+n)|x|^{\lambda} \to \beta_n \delta_0 \; ; \; \lambda \to -n$

$$\Rightarrow \Delta|x|^{\lambda+2} \to (-n+2)\,\beta_n\delta_0 \; ; \; \lambda \to -n$$

Pozn.: Vlastnosti $\Phi(x) = \begin{cases} -\dfrac{1}{\beta_2} \ln|x| & ; \; n=2 \\ \dfrac{1}{(n-2)\beta_n} |x|^{2-n} & ; \; n \geq 3 \end{cases}$

1. $\Phi \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1_{loc}(\mathbb{R}^n)$
2. $\nabla\Phi(x) = -\dfrac{x}{\beta_n |x|^n}$; $x \neq 0$; $|\nabla\Phi(x)| \leq C|x|^{1-n} \in L^1_{loc}(\mathbb{R}^n)$
3. $\Delta\Phi(x) = 0 \;\; \forall x \neq 0 \quad$ d. cv.

Věta 28.9. Nechť $f(x) \in C_c^2(\mathbb{R}^n)$

Definujme $u := \Phi * f$

Potom $u \in C^2(\mathbb{R}^n)$ a $-\Delta u(x) = f(x)$ pro $\forall x \in \mathbb{R}^n$

dk.

$$u(x) = \int_{\mathbb{R}^n} \underbrace{f(x-y)}_{C_c} \underbrace{\Phi(y)}_{L^1_{loc}} dy \qquad \text{má smysl}$$

$$D^\alpha u(x) = \int_{\mathbb{R}^n} D^\alpha f(x-y)\Phi(y)\,dy \quad ; \quad |\alpha| \leq 2$$

ověření rovnice : $x_0 \in \mathbb{R}^n$ pevné

$$\Delta u(x_0) = \int_{\mathbb{R}^n} \Delta f(x_0-y)\Phi(y)\,dy = \int_{\mathbb{R}^n} \Delta f(y)\Phi(x_0-y)\,dy =$$

$$= \int_B \Delta f(y)\Phi(y-x_0)\,dy \quad ; \quad B \subset \mathbb{R}^n \text{ koule taková, že } \operatorname{supp} f \subset B$$

pomocný vzoreček :

$$\left.\begin{aligned} \operatorname{div}(u\nabla v) &= \nabla u \cdot \nabla v + u\Delta v \\ \operatorname{div}(v\nabla u) &= \nabla v \cdot \nabla u + v\Delta u \end{aligned}\right\} \ominus \Big/ \int_\Omega dx \ ; \text{Gauss}$$

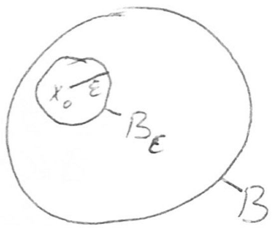
$\ominus \int$:

$$\int_{\partial\Omega} u\nabla v\cdot n - v\nabla u\cdot n\,dS = \int_\Omega u\Delta v - v\Delta u\,dx$$

Greenova identita $\qquad u,v \in C^2$

aplikuji na $f(y)$, $\Phi(y-x_0)$

$\Omega = B_\varepsilon := B \setminus B(x_0,\varepsilon)$; $\varepsilon > 0$ malé



$$\int_{B_\varepsilon} \Delta f(y)\Phi(y-x_0) - f(y)\Delta\Phi(y-x_0)\,dy =$$

$$= \int_{\partial B_\varepsilon} \nabla f(y)\cdot n\,\Phi(y-x_0) - f(y)\nabla\Phi(y-x_0)\cdot n\,dS(y)$$

$$\varepsilon \to 0+:\quad (LS_1) \longrightarrow \int_B \underbrace{\Delta f(y)\,\Phi(y-x_0)}_{L^1(B)}\,dy = \Delta u(x_0)$$

$(LS_2) \equiv 0$ neboť $\Delta \Phi = 0$ mimo počátek (0)

$$(PS) = \int_{\partial B(x_0,\varepsilon)} \nabla f(y)\cdot n\,\Phi(y-x_0) - f(y)\,\nabla\Phi(y-x_0)\cdot n\,dS(y)$$

neboť $f \equiv 0$ na ∂B

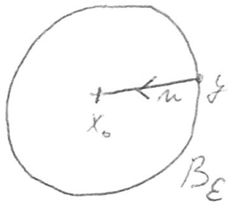
$\nabla f \equiv 0$

$$|(PS_1)| \le \int_{\partial B(x_0,\varepsilon)} C\cdot \underbrace{|y-x_0|}_{\varepsilon}^{2-n}\,dS(y) = C\cdot\varepsilon^{2-n}\cdot\underbrace{\beta_n\varepsilon^{n-1}}_{\text{míra } \partial B(x_0,\varepsilon)} = C\beta_n\varepsilon \longrightarrow$$

pro $n=2$: $|\ln\varepsilon|$

$$\longrightarrow 0\ ;\ \varepsilon \to 0+$$

$$|(PS_2)| = \int_{\partial B(x_0,\varepsilon)} f(y)\,\underbrace{\nabla\Phi(y-x_0)\cdot(-n)}\,dS(y) =$$



$$= -\frac{y-x_0}{|y-x_0|^n}\cdot\frac{1}{\beta_n}$$

$$n = -\frac{y-x_0}{y-x_0}$$

$$\Bigg\}\ \cdot\frac{1}{\beta_n\varepsilon^{n-1}}$$

$$= -\frac{1}{\beta_n\varepsilon^{n-1}}\int_{\partial B(x_0,\varepsilon)} f(y)\,dS(y) = -⨍_{\partial B(x_0,\varepsilon)} f(y)\,dS(y) \longrightarrow -f(x_0)$$

$\varepsilon \to 0+$

↳ průměrový integrál

celkem: $\Delta u(x_0) = -f(x_0)$

Značení: (Průměrový integrál)

$$\fint_M f(x)\,dx = \frac{1}{\lambda_n(M)} \int_M f(x)\,dx$$

$\lambda_n \to$ Leb. míra

$$\fint_{\partial M} f(y)\,dS(y) = \frac{1}{\sigma_{n-1}(\partial M)} \int_{\partial M} f(y)\,dS(y)$$

$\sigma_{n-1} \to$ Leb. míra

dokončení důkazu:

tvrdím $\displaystyle\fint_{\partial B(x_0,\varepsilon)} f(y)\,dS(y) \to f(x_0)$; $\varepsilon \to 0+$

je dáno (víme)

$\eta > 0$ dáno: spojitost $f(y) \Rightarrow \exists \delta > 0 \ldots |f(y) - f(x_0)| < \eta$

pro $|y - x_0| < \delta$

nechť $\varepsilon < \delta$

$$\left| \fint_{\partial B(x_0,\varepsilon)} f(y)\,dS(y) - f(x_0) \right| \leq \fint \underbrace{|f(y) - f(x_0)|}_{\leq \eta} dS(y) \leq \eta$$

celkem $\Delta u(x_0) = - f(x_0)$

Pozn. vedlejší produkt: důkaz V. 28. 8.

$$\int_{\mathbb{R}^n} \Delta f(y)\, \Phi(y - x_0)\,dy = - f(x_0)$$

volme $f = \varphi \in \mathcal{D}(\mathbb{R}^n)$, $x_0 = 0$

$$\int_{\mathbb{R}^n} \Delta \varphi(y)\, \Phi(y)\,dy = - \varphi(0)$$

$\Rightarrow \quad -\Delta \Phi = \delta_0$ ve smyslu $\mathcal{D}'(\mathbb{R}^n)$

Def: Funkce $u(x)$ se nazve harmonická v Ω, pokud
$u(x) \in C^2(\Omega)$ a $-\Delta u(x) = 0 \quad \forall x \in \Omega$

Pozn. $\Omega \subset \mathbb{R}^n$ oblast, $n \geq 2$

$n = 1$: $u(x)$ harmonická $\iff u'' = 0$

$u(x) = Ax + b$ | afinní (lineární)

Značení: $\alpha_n := \lambda_n(B(0,1))$

$\beta_n := \sigma_{n-1}(\partial B(0,1))$

Škálování $\lambda_n(B(x_0, r)) = \alpha_n r^n$

$\sigma_{n-1}(\partial B(x_0, r)) = \beta_n r^{n-1}$

$\beta_n = n \alpha_n$

dk. $\lambda_n(B(0, 1+\varepsilon)) = \alpha_n (1+\varepsilon)^n = \alpha_n(1 + n\varepsilon + o(\varepsilon))$

$\alpha_n + \varepsilon \beta_n + o(\varepsilon)$

Věta 28.10. [O průměru harmonické funkce]

Nechť $u(x)$ je harmonická v Ω,

nechť $B(x_0, R) \subset \Omega$

Potom $u(x_0) = \fint_{B(x_0,R)} u(x)\, dx = \fint_{\partial B(x_0, r)} u(y)\, d\sigma(y) \quad \forall r \in (0, R)$

dk. $\fint_{\partial B(x_0,r)} u(y)\, d\sigma(y) = \Phi(r) = \left| \begin{array}{l} y = x_0 + rx \\ x \in \partial B(0,1) \\ d\sigma(y) = d\sigma(x) \text{ neboť } \fint \end{array} \right.$

$= \fint_{\partial B(0,1)} u(x_0 + rx)\, d\sigma(x) = \frac{1}{\beta_n} \int_{\partial B(0,1)} u(x_0 + rx)\, d\sigma(x)$

$$\Phi'(r) = \frac{1}{\beta_n} \int_{\partial B(0,1)} \nabla u(x_0 + rx) \cdot \underset{\substack{\hookrightarrow = n \\ \text{vnější normála}}}{x} \, d\sigma(x) =$$

$$\overset{\text{Gauss}}{=} \frac{1}{\beta_n} \int_{B(0,1)} \overbrace{\underbrace{\mathrm{div}_x \nabla_x}_{\Delta \text{ v } r^2} u(x_0 + rx)}^{\equiv 0} \, dx = 0$$

$$\Rightarrow \quad \Phi(r) = C \quad ; \quad r \in (0,R)$$

$$\underbrace{\lim_{r \to 0+} \Phi(r)}_{u(x_0) \ldots \text{ viz důkaz V. 28.9.}} = C$$

$$\Rightarrow \Phi(r) = u(x_0) \quad \forall r \in (0,R) \qquad \text{2. rovnost dokázána}$$

1. rovnost:
$$\fint_{B(x_0,R)} u(x)\,dx = \frac{1}{\alpha_n R^n} \int_0^R \Big(\underbrace{\int_{\partial B(x_0,r)} u(y)\,d\sigma(y)}_{\substack{\Phi(r) \cdot \sigma_{n-1}(\partial B(x_0,r)) \\ u(x_0) \cdot \beta_n r^{n-1}}} \Big) dr$$

sfér. Fubiniho integrál

$$= \frac{1}{\alpha_n R^n} \int_0^R u(x_0) \cdot \beta_n r^{n-1} \, dr = u(x_0) \frac{1}{\alpha_n R^n} \cdot \beta_n \frac{R^n}{n}$$

prim. f. $\frac{r^n}{n}$ $\qquad n\alpha_n = \beta_n$

Věta 28.11. Nechť $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$, nechť $\Omega \subset \mathbb{R}^n$ je omezená (oblast)
označme $M := \max_{\partial\Omega} u$

1) Je-li $-\Delta u < 0$ v Ω, pak $\max_{\overline{\Omega}} u = M$

2) Je-li $-\Delta u = 0$ v Ω a existuje-li $x_0 \in \Omega$ tak, že $u(x_0) = M$, pak $u \equiv M$ v $\overline{\Omega}$

dk.

1. „slabý" princip maxima" ... dk. viz V.28.5. (RVT)

2. „silný" princip maxima"

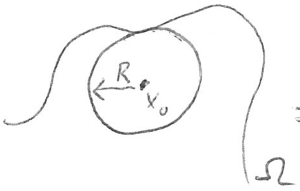
je vidět, že max. je na hranici,
není-li na hranici, pak je to konst. fce.

nechť $u(x_0) = M = \max\limits_{\partial\Omega} \overset{(1.)}{=} \max\limits_{\overline{\Omega}} u$

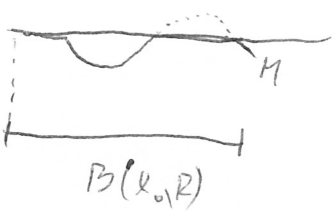
$x_0 \in \Omega$

volme maximální $R > 0 \dots B(x_0,R) \subset \Omega$

V 28.10.: $\underset{M}{\overset{\shortparallel}{u(x_0)}} = \displaystyle\fint_{B(x_0,R)} \overbrace{u(x)}^{\leq M} dx \leq M$ ↳ nutně $=$



$\Rightarrow u(x) \equiv M$ na $B(x_0,R)$ (na kouli)



sporem (?) $u(y) < M$ pro $y \in B(x_0,R) \dots$
... spojitost: $\exists\, \mathcal{U}$ .. okolí y ...
... $u(x) \leq M - \varepsilon$ na $\mathcal{U}$

$$\int_{B(x_0,R)} u(x)\,dx = \int_{\mathcal{U}} u + \int_{B\setminus \mathcal{U}} u \leq \lambda(\mathcal{U})\cdot(M-\varepsilon) + M\lambda(B\setminus\mathcal{U})$$

$$\lambda(\mathcal{U}) + \lambda(B-\mathcal{U}) = \lambda(B)$$

$$< M\cdot\lambda(B_{\text{koule}})$$

Def: [Klasická Dirichletova úloha]

$\Omega \subset \mathbb{R}^n$ oblast, $g(x): \partial\Omega \to \mathbb{R}$ spojité jsou dány
Najděte $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$; $\Delta u = 0$ v Ω a $u(x) = g(x)$

$$\boxed{\begin{array}{l} -\Delta u = 0 \text{ v } \Omega \\ u = g \text{ na } \partial\Omega \end{array}}$$

Teorie potenciálu

řešení úlohy zdvihu na $\partial\Omega$:  nelze : moc rychle se zmenšuje

 lze

Pozn. nyní se omezíme na situaci $\Omega = \mathbb{R}^2 \cong \mathbb{C}$
↳ ztotožníme s

Def.: $f : \Omega \to G$ se nazve konformní, jestliže

1. f je vzájemně jednoznačné
2. f je holomorfní $(\exists f'(z))$
3. $f'(z) \neq 0$

$z = x + iy$

$z \mapsto f_1 + i f_2$

Pozn. f je holomorfní $\Rightarrow$ Cauchy-Riemannovy podmínky

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y} \quad ; \quad -\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}$$

$$\Rightarrow f \in C^\infty \; ; \quad \Delta f_1 = \Delta f_2 = 0$$

$$\Delta f_1 = \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial f_2}{\partial x}\right) = 0$$

záměnnost parc. derivací

holomorfní a C^∞

$$Jf = \det \nabla f = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \underset{\text{C.R.-pod.}}{=} \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 = |f'(z)|^2$$

z 3.: $f'(z) \neq 0$

$$f'(z) = \frac{\partial f_1}{\partial x} + i \frac{\partial f_1}{\partial y}$$

Def.: $f : \Omega \to G$ je konformní:

(i) vzájemně jednoznačné

(ii) $z \to f(z)$ je holomorfní $(\exists f'(z))$

(iii) $f'(z) \neq 0 \quad \forall z \in \Omega$

Pozn. $f : \Omega \to G$ konformní

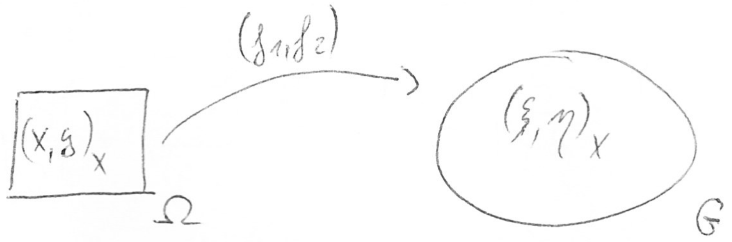
① C.R. podmínky $\dfrac{\partial f_1}{\partial x} = \dfrac{\partial f_2}{\partial y} \; ; \; \dfrac{\partial f_1}{\partial y} = -\dfrac{\partial f_2}{\partial x}$

② $f \in C^\infty(\Omega)$, f_{-1} je též konformní $\Big] \; f'(z) = \dfrac{\partial f_1}{\partial x} + i \dfrac{\partial f_2}{\partial y}$

③ $\Delta f_1 = 0 \; , \; \Delta f_1 = 0$ v Ω

④ $\mathcal{J}f = \det \nabla f = |f'(z)|^2 = \left(\dfrac{\partial f_1}{\partial x}\right)^2 + \left(\dfrac{\partial f_2}{\partial x}\right)^2$

⑤ f zachovává úhly



Pozn. Laplace – Poissonova rovnice

$$\boxed{\begin{aligned} -\Delta u &= f \quad \text{v } \Omega \\ u &= g \quad \text{na } \partial\Omega \end{aligned}}$$

Věta 28.12.

Nechť $f: \Omega \to G$ je konformní, $\Omega, G \subset \mathbb{R}^2$ oblast

nechť $u: \Omega \to \mathbb{R}$, $v: G \to \mathbb{R}$ splňují $u = v \circ f$

Potom

1. $v \in C^2(G) \Rightarrow u \in C^2(\Omega)$ a platí $\Delta u = (\Delta v) \circ f \cdot |f'(z)|^2$

2. $v \in L^1_{loc}(G)$ a $-\Delta v = \delta_0$ v $\mathcal{D}'(G) \Rightarrow u \in L^1_{loc}(\Omega)$

   a $\quad -\Delta u = \delta_{f^{-1}(0)}$ v $\mathcal{D}'(\Omega)$

3. $v \in C^1(G) \Rightarrow u \in C^1(\Omega)$ a platí $E = (\tilde{E} \circ f) \cdot \overline{f'(z)}$, kde

   $E = -(\partial_x u + i \partial_y u)$; $\tilde{E} = -(\partial_\xi v + i\, d_\eta v)$

   (o přenášení Δ)

dk. 1. $u(x,y) = v(\underbrace{f_1(x,y)}_{\xi}, \underbrace{f_2(x,y)}_{\eta})$

$$\partial_x u = \partial_\xi v \cdot \partial_x f_1 + \partial_\eta v \cdot \partial_x f_2 \quad / \partial_x$$

$$\partial_{xx} u = (\partial_{\xi\xi} v \cdot \partial_x f_1 + \partial_{\xi\eta} v\, \partial_x f_2) \cdot \partial_x f_1 +$$

$$+ (\partial_{\eta\xi} v\, dx f_1 + \partial_{\eta\eta} v\, \partial_x f_2)\, \partial_x f_2 +$$

$$+ \partial_\xi v\, \cancel{\partial_{xx}} f_1 + \partial_\eta v\, \cancel{\partial_{xx}} f_2$$

$$\Delta f_1 = \Delta f_2 = 0$$

$$\partial_{yy} u = (\partial_{\xi\xi} v\, \partial_y f_1 + \partial_{\xi\eta} v\, \partial_y f_2)\, \partial_x f_1 +$$

$$+ (\partial_{\eta\xi} v\, \partial_y f_1 + \partial_{\eta\eta} v\, \partial_y f_2)\, \partial_y f_2 +$$

$$+ \partial_\xi \cancel{v\, \partial_{yy}} f_1 + \partial_\eta \cancel{\partial_{yy}} f_2$$

$$\Delta u = \partial_{xx} u + \partial_{yy} u = \partial_{\xi\xi} v\left((\partial_x f_1)^2 + \overset{(\partial_x f_2)^2}{(\partial_y f_1)^2}\right) +$$

$$+ \partial_{\eta\eta} v\left((\partial_x f_2)^2 + \overset{(\partial_x f_1)^2}{(\partial_y f_2)^2}\right) +$$

$$+ \partial_{\xi\eta} v \left\{ \underbrace{\partial_x f_2\, \partial_x f_1 + \partial_y f_2\, \partial_y f_1}_{\text{z C.R.} = 0} + \partial_x f_1\, \partial_x f_2 + \partial_y f_1\, \partial_y f_2 \right\}$$

zapojím C.R. $\quad \partial_x f_1 = \partial_y f_2$

$\partial_x f_2 = -\partial_y f_1$

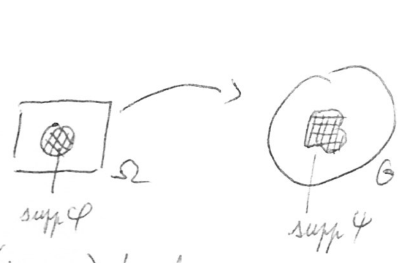
$$\Delta u = \Delta v\, \underbrace{|Jf|}_{|f'(z)|^2}$$

3. podobně

2. $\langle -\Delta u, \varphi \rangle = \langle u, -\Delta\varphi \rangle = -\int_\Omega u \Delta\varphi \, dx dy =$

$\varphi \in \mathcal{D}(\Omega) \quad \varphi = \psi \circ f \ ; \ \psi \in \mathcal{D}(G)$

$\psi = \varphi \circ f_{-1}$



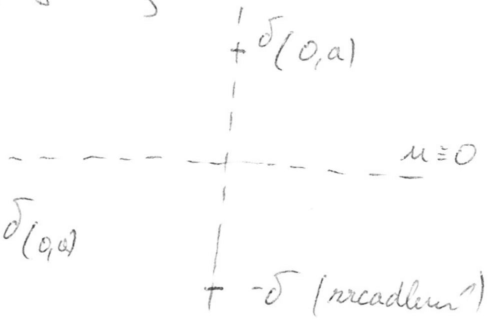
$$= -\int_\Omega (v \circ f)\, \Delta\varphi \, dx dy = -\int_\Omega (v \circ f)\, \underbrace{\Delta(\psi \circ f)}_{\Delta\psi \circ f \cdot |Jf| = \Delta\varphi} dx dy =$$

$$\overset{\text{věta o substituci}}{=} -\int_G v \Delta\psi \, d\xi d\eta = \langle v, -\Delta\psi \rangle = \langle \underbrace{-\Delta v}_{\delta_a}, \psi \rangle = \psi(a) = \varphi(f_{-1}(a)) = \langle \delta_{f_{-1}(a)}, \varphi \rangle$$

Př.:

(1) $-\Delta u = \delta_{(0,a)}$ $\Omega = \{(x,y);\ y>0\}$

$u/_{\partial\Omega} = 0$



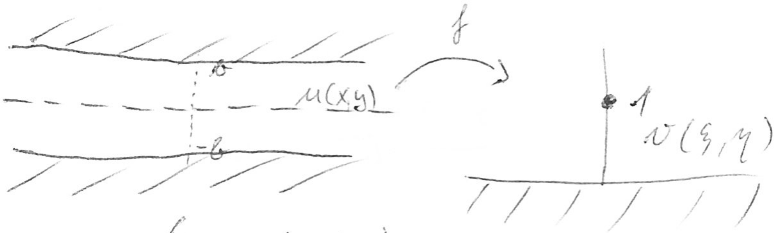
$$u_+ := \frac{-1}{2\pi}\ln\left(x^2+(y-a)^2\right);\quad u-\Delta u_* = \delta_{(0,a)}$$

$$u_- := \frac{-1}{2\pi}\ln\left(x^2+(y+a)^2\right);\quad -\Delta u_- = \delta(0,-a)$$

$$\Rightarrow u = u_+ - u_- = -\frac{1}{4\pi}\left\{\ln\frac{x^2+(y-a)^2}{x^2+(y+a)^2}\right\}$$

(2) $-\Delta u = \delta_0$ $\Omega = \{(x,y);\ |y|<b\}$

$u/_{\partial\Omega} = 0$



$z \mapsto e^z$

$$f(z) := i e^{\frac{\pi z}{2b}} = i e^{\frac{\pi}{2b}(x+iy)} = \qquad \exp\left((z+ib)\cdot\frac{\pi}{2b}\right) =$$

$z = x+iy$

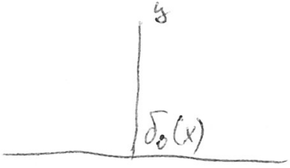
$$= i e^{\frac{\pi x}{2b}}\left(\cos\frac{\pi y}{2b} + i\sin\frac{\pi y}{2b}\right) \qquad = \exp\left(\frac{\pi z}{2b}\right)\cdot e^{\frac{i\pi}{2}}$$

$$\text{př. 1.} \Rightarrow v(\xi,\eta) = -\frac{1}{4\pi}\ln\frac{\xi^2+(\eta-1)^2}{\xi^2+(\eta+1)^2};\qquad \xi = -e^{\frac{\pi x}{2b}}\sin\frac{\pi y}{2b},\quad \eta = e^{\frac{\pi x}{2b}}\cos\frac{\pi y}{2b}$$

(3) $-\Delta u = 0$ v $\Omega = \{(x,y);\ y>0\}$

$u = g$ na $\partial\Omega$

Fundamentální řešení:



x řeší obecný problém

$$\left.\begin{aligned} &-\Delta U = 0 \quad y>0 \\ &\lim_{y\to 0+} U(x,y) = \delta_0(x) \end{aligned}\right\}\ g(x) \text{ rozumná} \Rightarrow u := U * g$$

tj. $u(x,y) = \int_{\mathbb{R}} U(x-s,y)\, g(s)\, ds$

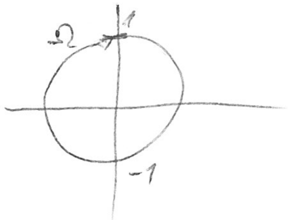
Výpočet U: viz Čihák str. 58 ... $U(x,y) = \frac{1}{\pi}\,\frac{y}{x^2+y^2}$

tj. pro obecnou úlohu $u(x,y) = \frac{1}{\pi}\int_{\mathbb{R}} g(s)\,\frac{y}{(x-s)^2+y^2}\,ds$

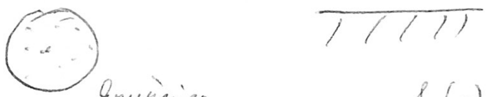
(4) $-\Delta u = 0$ v $\Omega = \{(x,y)\,;\ x^2+y^2>1\}$

$u = h$ na $\partial\Omega$

$h = \begin{cases} 1\,; & y>0 \\ -1\,; & y<0 \end{cases}$



?? $\Omega \xrightarrow{\ f\ } G$



kružnice

$f(0) = -i$

$f(z) = \frac{z-1}{z+1}\cdot i$

$z \mapsto \frac{az+b}{cz+d}$; $ad-bc \neq 0$ derivace $\frac{ad-bc}{(cz+d)^2} \neq 0$

$S \xrightarrow{1-1} S$; $S = \mathbb{C}\cup\{\infty\}$ Riemannova sféra

kružnice/přímka $\longleftrightarrow$ přímka/kružnice

?? přenesení okrajové podmínky $f(1) = 0$

$f(-1) = \infty$

$f(i) = \frac{i-1}{i+1}\cdot i = \frac{(i-1)(1-i)}{(i+1)(1-i)}\cdot i = \frac{1}{2}(1+2i-1)\,i = -1$

$-\Delta v = 0$; $\xi\in\mathbb{R},\ \eta>0$; $v(\xi,0) = -\operatorname{sgn}(\xi)$

3. příklad $\Rightarrow v(\xi,\eta) = -\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{sgn}(s)\,\frac{\eta}{(\xi-s)^2+\eta^2}\,ds = \dots = -\frac{2}{\pi}\operatorname{arctg}\left(\frac{\xi}{\eta}\right)$

$f(z) = \xi + i\eta = \frac{x+iy-1}{x+iy+1}\cdot i = \underbrace{\frac{-2y}{(x+1)^2+y^2}}_{\xi} + i\,\underbrace{\frac{x^2+y^2-1}{(x+1)^2+y^2}}_{\eta}$