Laurent -2

(2) $f = \dfrac{1}{\sin z}$; $z_0 = 0$.

$f \sim \dfrac{1}{z}$, $z \to 0$ $\Rightarrow$ jednoduchý pól

$$\frac{1}{\sin z} = \frac{A}{z} + B + Cz + Dz^2 + Ez^3 + \dots$$

$$1 = \left(z - \frac{z^3}{6} + \frac{z^5}{120} + \dots\right)\left(\frac{A}{z} + Cz + Ez^3 + \dots\right)$$

Víme: f liché, $a_{2k} = 0$; $\forall k$.

$$1 = A + z \cdot 0 + z^2\left(C - \frac{A}{6}\right) + z^3 \cdot 0 + z^4\left(E - \frac{1}{6}C + \frac{A}{120}\right)$$

$$\Longrightarrow \quad A = 0$$

$$C - \frac{A}{6} = 0 \Rightarrow C = \frac{1}{6}$$

$$E - \frac{1}{6}C + \frac{A}{120} = 0 \Rightarrow E = \frac{7}{360}$$

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7}{360}z^3 + \dots$$

Pozn.: Tedy také $\operatorname{res}_0 \dfrac{1}{\sin z} = 1$,

$$\frac{1}{\sin z^2} = \frac{1}{z^2} + \frac{z^2}{6} + \frac{7}{360}z^6 + \dots$$

$$\operatorname{res}_0 \frac{1}{\sin z^2} = 0.$$

(ii) $f = \dfrac{1}{\sinh z}$; $z_0 = 0$ ... liché fce; jednoduchý pól:

$$\frac{1}{\sinh z} = \frac{A}{z} + Bz + Cz^3 + \cdots$$

$$1 = \left(z + \frac{z^3}{6} + \frac{z^5}{120} + \cdots\right)\left(\frac{A}{z} + Bz + Cz^3 + \cdots\right)$$

$$1 = A + z^2\left(B + A/6\right) + z^4\left(C + \frac{B}{6} + \frac{A}{120}\right) + \cdots$$

$A = 1$

$0 = B + \dfrac{A}{6} \Rightarrow B = -\dfrac{1}{6}$

$0 = C + \dfrac{B}{6} + \dfrac{A}{120} \Rightarrow C = \dfrac{7}{360}$

$$\boxed{\frac{1}{\sinh z} = \frac{1}{z} - \frac{z}{6} + \frac{7}{360} z^3 + \cdots}$$

(iii) $\dfrac{1}{z^6(z-2)}$; $z_0 = 0$.

pomocný výpočet: $\dfrac{1}{z-2} = -\dfrac{1}{2} \cdot \dfrac{1}{1-\frac{z}{2}} = -\dfrac{1}{2} \displaystyle\sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^k$

$$f(z) = -\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot z^{k-6} = -\frac{1}{2z^6} - \frac{1}{4z^5} - \frac{1}{8z^4} - \cdots \underbrace{- \frac{1}{64z}}_{\text{residuum.}}$$

(iv) $f = \dfrac{1}{(e^z-1)^2}$; $f \sim \dfrac{1}{z^2}$ ... 2-násobný pól v $z_0=0$.

$$(e^z-1)^2 = \left(z + \frac{z^2}{2} + o(z^2)\right)\cdot\left(z + \frac{z^2}{2} + o(z^2)\right)$$
$$= z^2 + z^3 + o(z^3)$$

$$\frac{1}{(e^z-1)^2} = \frac{A}{z^2} + \frac{B}{z} + \cdots$$
$$1 = \left(z^2 + z^3 + o(z^3)\right)\cdot\left(\frac{A}{z^2} + \frac{B}{z} + \cdots\right)$$
$$1 = A + z\cdot(B+A) + \cdots$$
$$A = 1;\ B = -1: \quad \frac{1}{(e^z-1)^2} = \frac{1}{z^2} - \frac{1}{z} + \cdots$$

$$\text{res: } \operatorname{res}_0 = -1.$$

(v) $f = \operatorname{cotg} z = \dfrac{\cos z}{\sin z} \sim \dfrac{1}{z}$; $z \to 0$: jednoduchý pól.

liché: $a_{2k} = 0$

$$\cos z = \sin z \cdot \left(\frac{A}{z} + Bz + Cz^3 + \cdots\right)$$
$$\left(1 - \frac{z^2}{2} + \frac{z^4}{24} + o(z^5)\right) = \left(z - \frac{z^3}{6} + \frac{z^5}{120} + o(z^6)\right)\cdot\left(\frac{A}{z} + Bz + Cz^3 + \cdots\right)$$
$$= A + z^2\left(B - \frac{A}{6}\right) + z^4\left(C - \frac{B}{6} + \frac{A}{120}\right) + \cdots$$

$1 = A$

$-\frac{1}{2} = B - \frac{A}{6}$ $\qquad\Rightarrow\ A = 1$

$\frac{1}{24} = C - \frac{B}{6} + \frac{A}{120}$ $\qquad\Rightarrow\ B = -\frac{1}{3}$, $\ C = -\frac{1}{45}$

$$\operatorname{cotg} z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} + \cdots$$