Př: $\displaystyle\int_0^{2\pi} \frac{1}{2+\sin x}\,dx$; | obecně $\displaystyle\int_0^{2\pi} R(\cos x, \sin x)\,dx$, $R(\cdot,\cdot)$ je rac. fce.

Tvrzení: $R(\cdot,\cdot)$ je rac fce; $R(\cos x, \sin x)$ nemá singularitu pro $x \in [0, 2\pi]$.

Potom
$$\int_0^{2\pi} R(\cos x, \sin x)\,dx = \int_\psi R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{dz}{iz};$$
$$\psi = e^{it};\ t \in [0, 2\pi].$$

Dk: PS: $z = e^{it}$,
$$\frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(e^{it}+e^{-it}\right) = \cos t$$
$$\frac{1}{2i}\left(z-\frac{1}{z}\right) = \frac{1}{2i}\left(e^{it}-e^{-it}\right) = \sin t;$$
$$\frac{dz}{iz} = \frac{\psi'(t)}{i\psi(t)}dt = \frac{ie^{it}}{ie^{it}}dt = \underline{dt}.$$

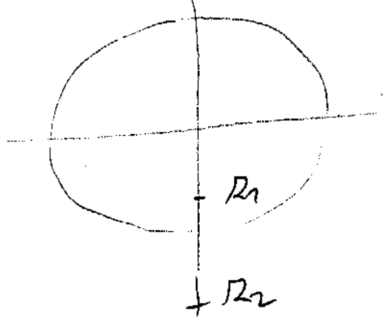
$I = \int_\psi f(z)\,dz$; $\quad f(z) = \dfrac{1}{2+\frac{1}{2i}\left(z-\frac{1}{z}\right)} \cdot \dfrac{1}{iz} = \dfrac{2}{4iz + z\left(z-\frac{1}{z}\right)}$

$$= \frac{2}{\underbrace{z^2+4iz-1}};$$

$$\boxed{I = \frac{2\pi}{\sqrt{3}} \text{ (mod)}}$$

$$z^2+4iz-1 = (z+2i)^2+4-1 = (z+2i)^2+3;$$

$$z_{1,2} = -2i \pm i\sqrt{3} = i\left(-2\pm\sqrt{3}\right)$$



$z_1 = i\left(\sqrt{3}-2\right) \in \operatorname{int}\psi$,

$z_2 = i\left(-\sqrt{3}-2\right) \notin \operatorname{int}\psi$.

$$\operatorname{res}_{z_1} = \frac{2}{z_1-z_2} = \frac{2}{2i\sqrt{3}} = \frac{1}{i\sqrt{3}}$$

C8. $\int_0 \frac{\sin^2 x}{1-2a\cos x+a^2}$ $a \neq 1$ $\boxed{a>1}$. jm. $(a-\cos x)^2+1-\cos^2 x$

$= (a-\cos x)^2+\sin^2 x > 0$

$$g(z)=\frac{\left(\frac{1}{2i}\right)^2\left(z-\frac{1}{z}\right)^2}{1-a\left(z+\frac{1}{z}\right)+a^2}\cdot\frac{1}{iz}\cdot\frac{z^2}{z^2}=\frac{(z^2-1)^2}{z^2\left((1+a^2)z-az^2-a\right)}\left(\frac{i}{4}\right)$$

$$D=(1+a^2)^2-4a^2$$
$$=a^4+2a^2+1-4a^2$$
$$=(a^2-1)^2$$

$$z_{1,2}=\frac{-(1+a^2)\pm(a^2-1)}{-2a}$$

$$=\begin{cases}\frac{-2}{-2a}=\frac{1}{a}\in \operatorname{int} \Psi\\ \frac{-2a^2}{2a}=a\notin \operatorname{int}\Psi.\end{cases}$$

$$g(z)= \qquad \left((1+a^2)z-az^2-a\right)=-a(z-a)\left(z-\frac{1}{a}\right)$$
$$=(a-z)(az-1).$$

$$g(z)=\underbrace{\frac{i}{4}\cdot\frac{(z^2-1)^2}{(a-z)}}_{\mathcal{H}(\operatorname{int}\Psi)}\cdot\frac{1}{z^2(az-1)}$$

$$I=2\pi i\cdot\frac{i}{4}\left(\operatorname{res}_0\tilde{g}(z)+\operatorname{res}_{\frac{1}{a}}\tilde{g}(z)\right);\quad \tilde{g}(z)=\frac{(z^2-1)^2}{z^2(a-z)(az-1)}$$

$$=2\pi i\cdot\frac{i}{4}\cdot\left(-\frac{a^2+1}{a^2}+\frac{a^2-1}{a^2}\right)$$

(výpočet rezidui na další straně..)

$$=-\frac{\pi}{2}\cdot\left(-\frac{2}{a^2}\right)=\frac{\pi}{a^2}.$$

[hmm]

$$\tilde{g}(z) = \frac{(z^2-1)^2}{z^2(a-z)(az-1)} .$$

$$\tilde{g}(z) = \frac{1}{z^2} \cdot h(z) ; \quad h(z) = \frac{(z^2-1)^2}{(a-z)(az-1)} \in \mathcal{H}(U(0)) .$$

$$\rightarrow \operatorname{res}_0 \tilde{g}(z) = h'(0) = -\frac{1+a^2}{a^2}$$

$$h'(z) = \frac{1}{(a-z)^2(az-1)^2} \cdot \left[2z \cdot (\cdots) - (z^2-1)^2 \cdot \left(-(az-1) + a(a-z) \right) \right]$$

$$h'(0) = \frac{1}{a^2} \cdot \left[0 - (-1)^2 \cdot \left(-(-1) + a^2 \right) \right]$$

$$= -\frac{1+a^2}{a^2}$$

$$\tilde{g}(z) = \frac{1}{(z-\frac{1}{a})} \cdot \ell(z) ; \quad \ell(z) = \frac{(z^2-1)^2}{z^2(a-z)a} \in \mathcal{H}(U(0))$$

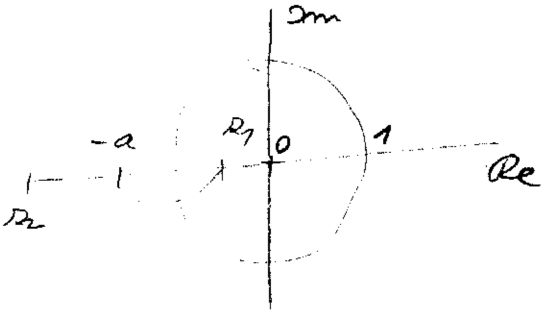
$$\operatorname{res}_{\frac{1}{a}} \tilde{g}(z) = \ell\left(\frac{1}{a}\right) = \frac{(1/a^2-1)^2}{\frac{1}{a^2}(a-\frac{1}{a})a} = \frac{(\frac{1}{a}-a)^2}{\cancel{(a-\frac{1}{a})}a}$$

$$= \frac{1-a^2}{a^2} .$$

Pr. $I = \int_0^{2\pi} \frac{dx}{a+\cos x}$; $a > 1$. $(a \in \mathbb{R})$.

$$f(z) = \frac{1}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{1}{iz} \cdot \frac{2}{2} = \frac{-2i}{z^2 + 2az + 1} = \frac{-2i}{(z+a)^2 + \underbrace{1-a^2}_{<0}} ;$$

$$z_{1,2} = -a \pm \sqrt{a^2-1}$$



$$z_1 = \sqrt{a^2-1} - a \in (-1, 0).$$

$$\sqrt{a^2-1} - a > -1$$

$$\sqrt{a^2-1} > a - 1 \quad \Big|^2$$

$$\cancel{a^2} - 1 > \cancel{a^2} - 2a + 1$$

$$2a > 2$$

$$\underline{\underline{a > 1}}.$$

$$f(z) = \frac{-2i}{(z-z_1)(z-z_2)} = \frac{1}{z-z_1} \cdot \underbrace{\left(\frac{-2i}{z-z_2}\right)}_{\in \mathcal{H}(U(z_1))}$$

$$\operatorname{res}_{z_1} f(z) = g(z_1) = \frac{-2i}{z_1 - z_2} = \frac{-2i}{2\sqrt{a^2-1}}$$

$$I = 2\pi i \cdot \frac{-2i}{2\sqrt{a^2-1}} = \boxed{\frac{2\pi}{\sqrt{a^2-1}}}. \quad \text{O.K. (max.)}$$

Příklad 3: $I=\int_0^{2\pi}\frac{dx}{2+\sin x+\cos x}$; obecně: $\int_0^{2\pi} R(\sin x,\cos x)\,dx$

R – rac. funkce nemá singul. v $(0,2\pi)$.

položme $f(z)=\frac{1}{2+\frac{1}{2i}\left(z-\frac{1}{z}\right)+\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{1}{iz}$

tvrdím: $\int_\varphi f(z)\,dz = I$: $\varphi = e^{it};\ t\in(0,2\pi)$.

$\frac{1}{2i}\left(z-\frac{1}{z}\right)=\frac{1}{2i}\left(e^{it}-e^{-it}\right)=\sin t$

$\frac{1}{2}\left(z+\frac{1}{z}\right)= \cdot = \cos t$

$dz = ie^{it}dt;\ \Rightarrow\ \frac{dz}{iz}=dt.$

$\int_\varphi f(z) = 2\pi i\sum_{\xi\in \operatorname{int}\varphi\cap K}\operatorname{res}_\xi f(z).$; K – sing. $f(z)$.

$f(z)=\frac{2}{4iz+z^2-1+i(z^2+1)}=\frac{2}{z^2(i+1)+4iz+i-1}=\frac{1-i}{z^2+2(i+1)z+i}$

$z^2+2(i+1)z+i=0$

$(z+(i+1))^2=i=\left(\frac{i+1}{\sqrt2}\right)^2$

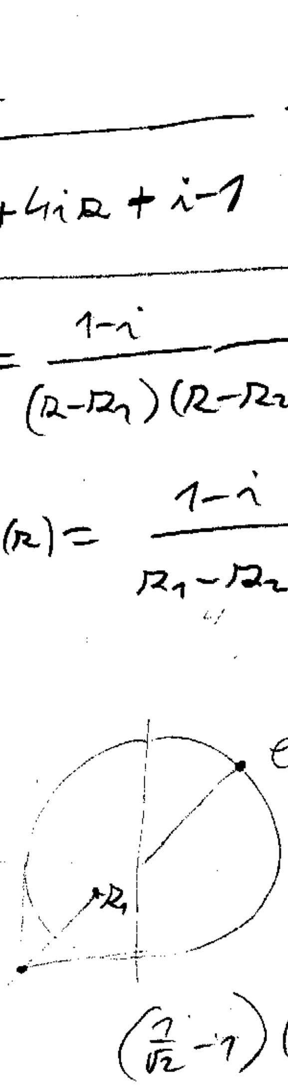
$z_{1,2}=-(i+1)\pm\frac{i+1}{\sqrt2}$

$=(i+1)\left(\pm\frac{1}{\sqrt2}-1\right)$

$f(z)=\frac{1-i}{(z-z_1)(z-z_2)}$

$\operatorname{res}_{z_1} f(z)=\frac{1-i}{z_1-z_2}=\frac{1-i}{(i+1)(\sqrt2)}$

$=$



$e^{\frac{i\pi}{4}}=\frac{1}{\sqrt2}(i+1)$

$\left(\frac{1}{\sqrt2}-1\right)(i+1)\cdot\frac{\sqrt2}{\sqrt2}$

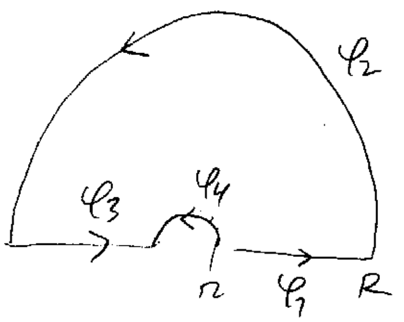
protože $z_1\in\operatorname{int}\varphi$

$I=2\pi i\cdot\operatorname{res}_{z_1} f(z)$

$\boxed{=\sqrt2\cdot\pi}$

Př. $\int_0^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{1}{2}\int_{\mathbb{R}} \frac{1-\cos x}{x^2}\,dx = \frac{1}{2}\int_{\mathbb{R}} \mathrm{Re}\left\{\frac{1-e^{ix}}{x^2}\right\}dx$

$f(z) = \frac{1-e^{iz}}{z^2}$; $\varphi = \varphi_1 \dot{+} \varphi_2 \dot{+} \varphi_3 \dot{+} \varphi_4$



$$0 = \int_{\varphi} f(z)\,dz = \int_{\varphi_1} + \int_{\varphi_2} + \int_{\varphi_3} + \int_{\varphi_4}$$

$$\int_{\varphi_1} + \int_{\varphi_3} = \int_{(-R,-r)\cup(r,R)} \frac{1-e^{it}}{t^2}\,dt = \int_{I_{r,R}} \left(\frac{1-\cos t}{t^2} - i\,\cancel{\frac{\sin t}{t^2}}\right)dt \to 2I\,; \quad \begin{matrix} R \to \infty \\ r \to 0+. \end{matrix}$$

$\int_{\varphi_2} \to 0$; $R \to +\infty$. $\quad |e^{iz}| = e^{(\mathrm{Re}(iz))} = e^{(-\mathrm{Im}\, z)} \le 1$

$\mathrm{Im}\, z > 0$

$$|f(z)| \le \frac{1+|e^{iz}|}{|z|^2} \le \frac{2}{|z|^2}; \quad \text{(Lemma o velké kružnici).}$$

L. (o malé kruž.). $\quad f(z)\cdot(z-z_0) \to A$; $z \to z_0$

$$\int_{\varphi} f(z)\,dz \to iA(\beta-\alpha)\;;\; \text{pro } r \to 0+$$

$$\varphi = z_0 + re^{it};\; t \in [\alpha,\beta].$$

$z_0 = 0$; $(\alpha,\beta) = (0,\pi]$.

$$f(z)\cdot z = \frac{1-e^{iz}}{z} = \frac{1-(1+iz+o(z))}{z} \to -i;\; z \to 0.$$

$$\int_{\varphi_4} \to -i\cdot i\cdot \pi = \pi\,.$$

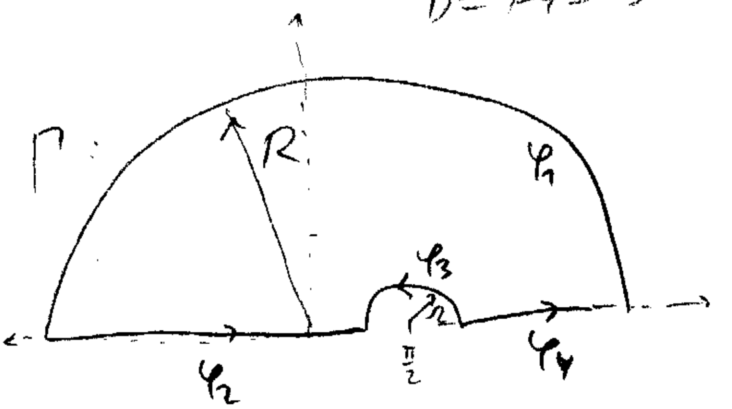
$2I = \pi$; tj. $I = \frac{\pi}{2}$.

Příklad 2. $I = \int_{-\infty}^{\infty} \frac{\cos x}{(2x-\pi)\cdot\left(x^2+\underset{5}{\cancel{x+1}}\right)}\,dt$; $f(z) = \frac{e^{iz}}{(2z-\pi)(z^2+z+1)}$

singularity $f(z) =$ $z = \frac{\pi}{2}$;

$z^2+z+1=0$: $z_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$D = 1-4 = -3$



$\varphi_1: Re^{it};\ t \in [0,\pi]$

$\varphi_2: t;\ t \in (-R, \frac{\pi}{2}-r]$

$\varphi_3: \frac{\pi}{2}+re^{it};\ t \in [0,\pi)$

$\varphi_4: t;\ t \in [\frac{\pi}{2}+r, R]$.

Búno: r malé, R velké:

$$\Gamma = \varphi_1 \oplus \varphi_2 \ominus \varphi_3 \oplus \varphi_4;$$

$$\int_\Gamma f(z)\,dz = 2\pi i \cdot \operatorname{res}_{-\frac{1}{2}+i\frac{\sqrt{3}}{2}} f(z).$$

$e^{it} = \cos t + i \sin t$

$$\int_{\varphi_2} + \int_{\varphi_4} = \left(\int_{-R}^{\frac{\pi}{2}-r} + \int_{\frac{\pi}{2}+r}^{R}\right) \frac{e^{it}}{(2t-\pi)(t^2+t+1)}\,dt$$

$$\operatorname{Re}\left[\int_{\varphi_2} + \int_{\varphi_4}\right] \to I \text{ ; pro } r \to 0+,\ R \to +\infty.$$

$f(z) = e^{iz}h(z)$

$\int_{\varphi_1} \frac{e^{iz}}{(2z-\pi)(z^2+z+1)}\,dz \to 0$ lemma o velké půlkružnici $|h(z)| \le \frac{K}{|z|^3}$; $|z| > R_0$

$\int_{\varphi_3} f(z)\,dz$ $\quad f(z)\cdot(z-\frac{\pi}{2}) \to A;\ z \to \frac{\pi}{2}$

$$f(z)\cdot(z-\tfrac{\pi}{2}) = \frac{e^{iz}\cdot\cancel{(z-\frac{\pi}{2})}}{(z^2+z+1)\cdot 2\cdot\cancel{(z-\frac{\pi}{2})}} \to \frac{e^{i\frac{\pi}{2}}}{2\left(\frac{\pi^2}{4}+\frac{\pi}{2}+1\right)} =: A$$

lemma o malé půlkr.: $\int_{\varphi_3} f(z) \to \pi i \cdot A = \pi i \cdot \frac{i}{\frac{\pi^2}{4}+\frac{\pi}{2}+1} = \underbrace{\frac{-\pi}{\frac{\pi^2}{4}+\frac{\pi}{2}+1}}_{K_1}$.

---

reziduová věta: $f(z) \in \mathcal{H}\left(\mathbb{C} \setminus \left\{ \frac{\pi}{2} ; -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right\}\right)$

$$\int_\Gamma f(z)\,dz = 2\pi i \cdot \operatorname{res}_{z_1} f(z) ; \quad z_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$f(z) = \frac{e^{iz}}{(2z-\pi)(z-z_1)(z-z_2)} \Rightarrow \operatorname{res}_{z_1} f = \frac{e^{iz_1}}{(2z_1-\pi)(z_1-z_2)}$$

$$= e^{-\frac{\sqrt{3}}{2}} \cdot \left(\cos\frac{1}{2} - i\sin\frac{1}{2}\right) \cdot \frac{1}{(-1-\pi+i\sqrt{3}) \cdot i\sqrt{3}} =: K_2$$

$$K_2 := 2\pi i \cdot \operatorname{res}_{z_1} f(z) = \frac{2\pi \cdot e^{-\frac{\sqrt{3}}{2}}}{\sqrt{3}} \cdot \frac{\cos\frac{1}{2} - i\sin\frac{1}{2}}{-1-\pi+i\sqrt{3}}.$$

---

celkem: $K_2 = \int_\Gamma f(z)\,dz = \int_{\varphi_2} + \int_{\varphi_4} + \int_{\varphi_1} - \int_{\varphi_3}$; Re

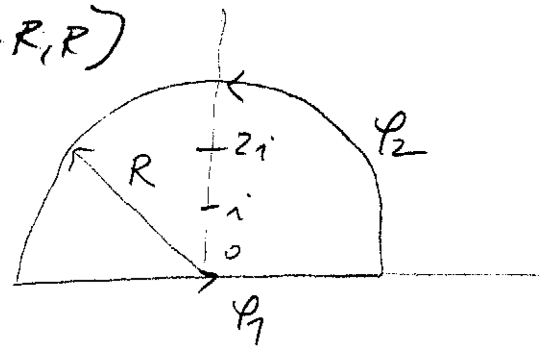
$$\operatorname{Re} K_2 = I - K_1 ;$$

Př: $\int_{\mathbb{R}} \frac{dx}{(x^2+1)(x^2+4)^2}$; obecně: $\int_{\mathbb{R}} R(x)\,dx$;

$$f(z) = \frac{1}{(z^2+1)(z^2+4)^2}; \quad f(z)\in\mathcal{H}(\mathbb{C}\setminus K); \quad K=\{\pm i, \pm 2i\}.$$

$\varphi = \varphi_1+\varphi_2$; $\quad \varphi_1(t)=Re^{it}$; $t\in[0,\pi]$

$\varphi_2(t)=t$; $t\in[-R,R]$



BÚNO: $R>2$:

$$\int_\varphi f(z) = 2\pi i\left(\operatorname{res}_i f(z)+\operatorname{res}_{2i} f(z)\right);$$

$z_0=i$: $$f(z)=\frac{1}{z-i}\cdot\underbrace{\frac{1}{z+i}\cdot\frac{1}{(z^2+4)^2}}_{g(z)\in\mathcal{H}(\mathcal{U}(i))};$$

$$\operatorname{res}_i f(z) = g(i) = \frac{1}{2i}\cdot\frac{1}{(-1+4)^2} = -\frac{i}{18}$$

$z_0=2i$: $$f(z)=\frac{1}{(z-2i)^2}\cdot\underbrace{\frac{1}{z^2+1}\cdot\frac{1}{(z+2i)^2}}_{g(z)\in\mathcal{H}(\mathcal{U}(2i))};$$

$$\operatorname{res}_{2i} f(z) = g'(2i);$$

$$g'(z) = \frac{-1}{(z^2+1)^2(z+2i)^4}\cdot\left\{(z^2+1)(z+2i)^2\right\}'$$

$$= \frac{-1}{(z^2+1)^2(z+2i)^4}\left\{2z(z+2i)^2+(z^2+1)\cdot 2(z+2i)\right\};$$

$$g'(2i) = \frac{-1}{(1-4)^2 (4i)^4} \cdot \left(4i\,(4i)^2 + (1-4)\cdot 2 \cdot 4i \right)$$

$$= \frac{-1}{9 \cdot 256} \cdot \left(-64i - 24i \right) = \frac{88}{9 \cdot 256}\, i = \underline{\frac{11}{288}\, i}.$$

$$g'(z) = \frac{-2}{(z^2+1)^2 (z+2i)^3} \cdot \underbrace{\left\{ z(z+2i) + (z^2+1)\cdot i \right)}_{2z^2 + 2zi + 1}$$

$$z = 2i: \quad \frac{-2}{(1-4)^2 (4i)^3} \left\{ 2i \cdot 4i \right.$$

$$R > 2: \; I = \int_{\varphi} f(z) = 2\pi i \left(-\frac{i}{18} + \frac{11}{288} i \right) = 2\pi i \left(\frac{-16+11}{288} \right) i$$

$$= \frac{10\pi}{288} = \frac{5\pi}{144}.$$

$$I = I_1 + I_2;$$

$$I_1 = \int_{\varphi_1} f(z)\,dz = \int_{-R}^{R} f(t)\,dt \to \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$$

$$I_2 = \int_{\varphi_2} f(z)\,dz \to 0, \text{ pro } R \to +\infty.$$

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závěr: $I = \frac{5\pi}{144}$. [max.]

A9. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+6x^2+25}$;

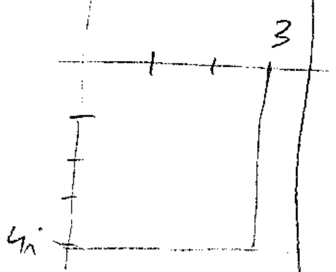
$f(z) = \dfrac{z^2}{z^4+6z^2+25}$; $\qquad a^2-b^2=(a+b)(a-b)$

$$(z^2+3)^2+16 = (z^2+3)^2 - (4i)^2$$
$$= (z^2+\underbrace{3+4i}) (z^2+3-4i)$$

$(1+2i)^2 = 1-4+4i = 4i-3$;

$(1-2i)^2 = 1-4+4i = -3-4i$

$z^2 = 4i-3$;



$\operatorname{arctg} \frac{3}{4}$

$\operatorname{res}_{z_1} = \dfrac{g(z_1)}{h'(z_1)}$

?? rozklad: $\quad z^4+6z^2+25 = (z^2+5)^2 - 10z^2+6z^2$

$$= (z^2+5)^2 - (2z)^2 = (z^2+2z+5)(z^2-2z+5)$$

$z^2+2z+5 = (z+1)^2+4$; $\quad z_{1,2} = -1 \pm 2i$

$z^2-2z+5 = (z+1)^2+4$; $\quad z_{3,4} = 1 \pm 2i$.

horní polorovina: $\quad z_1 = -1+2i$

$z_3 = 1+2i$

$$f(z) = \frac{z^2}{z^4+6z^2+25}; \qquad \frac{g(z)}{h(z)};$$

$$\operatorname{res}_{z_1} f(z) = \frac{g(z_1)}{h'(z_1)}; \qquad h'(z) = 4z^3+12z;$$

$$\frac{(2i-1)^2}{4(2i-1)^3+12(2i-1)} = \frac{2i-1}{4(2i-1)^2+12} = \frac{2i-1}{4(-3-4i)+12} = \frac{2i-1}{-16i}$$

$$= -\frac{1}{8} - \frac{i}{16};$$

$$(2i-1)^2 = -4-4i+1 = -3-4i;$$

$$(2i-1)^3 =$$

$$(1+2i)^2 = 1+4i-4 = -3+4i$$

$$\operatorname{res}_{z_3} f(z) = \frac{2i+1}{4(-3+4i)+12} = \frac{2i+1}{16i} = \frac{1}{8} - \frac{i}{16};$$

$$I = 2\pi i\left(-\frac{i}{8}\right) = \frac{\pi}{4}$$

✓ (maxima)