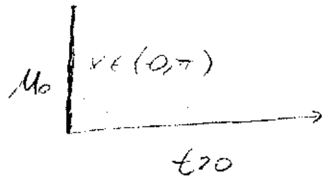
Příklad: (1) $\partial_t u = K\partial_{xx} u$ v Ω

(2) $\partial_x u(0,t)=\partial_x u(\pi,t)=0$

(3) $u(x,0)=u_0(x)$.



$u=u(x,t)$

Pozn: (odvození rce) $I\subset(0,\pi)$ interval libovolný

$$U(I)=\int_I u(x,t)\,dx \leftarrow \text{množství energie}$$

$$\boxed{\frac{d}{dt}U(I)=P(I)} \qquad \text{l.s.: } \int_I \partial_t u\,dx.$$

$$T(t)=-K\partial_x u(x,t)$$

$$P(I)=-K\partial_x u(a,t)+K\partial_x u(b,t)$$

$$=K\left(\partial_x u(a,t)-\partial_x u(b,t)\right)=K\int_a^b \partial_{xx}u(x,t)\,dx.$$

Ansatz: $u(x,t)=\sum_{k=0}^{\infty} c_k(t)\cos kx\ ;+\frac{a_0}{2}$ (A) splňuje (2) $\partial_x[\cos kx]=-k\sin kx=0$, $x=0,\pi$.

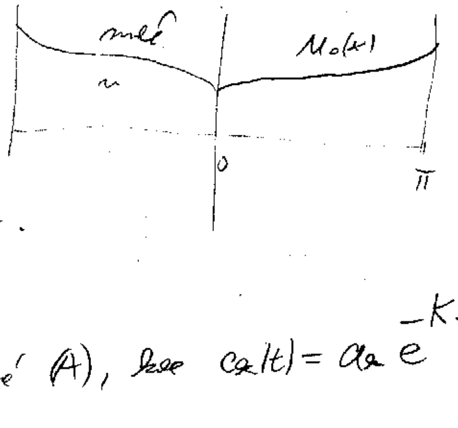
? formálně: $\partial_t u=\sum_{k=1}^{\infty} c_k'(t)\cos kx + \cancel{\frac{a_0}{2}}$

$\partial_{xx}u=\sum_k c_k(t)\left[-k^2\cos kx\right]$

(1): $c_k'(t)=-k^2 c_k(t)$ $\forall k\geq 1$, $\therefore$ $c_k(t)=c_k(0)e^{-Kk^2t}$

(2): $\partial_x u=\sum_k c_k(t)\left[-k\sin kx\right]\Big|_{x=0,\pi}=0$.

(3): $u(x,0)=\sum_{k=0}^{\infty} c_k(0)\cos kx = u_0(x)$



Předpoklad: (P) $u_0(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k\cos kx$;

kde $\sum_{k=1}^{\infty}|a_k|$ konv.

Tvrzení: Nechť platí (P). Potom fce $u(x,t)$, daná (A), kde $c_k(t)=a_k e^{-Kk^2t}$ je spojitá v $\overline{\Omega}$; C^∞ v Ω; řeší (1), (2), (3).

$$u(x,t) = \frac{a_0}{2} + \underbrace{\sum_{k=1}^{\infty} a_k e^{-Kk^2t} \cos kx}_{} \; ;$$

$$|\quad| \le \sum_{k=1}^{\infty} |a_k| e^{-Kk^2t} \le e^{-Kt} \sum_k |a_k| \le C \cdot e^{-Kt}$$

$$\underset{t \to \infty}{u(x,t) \rightrightarrows \frac{a_0}{2}} \; ; \qquad \frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} u_0(x)\,dx \; \dots$$ průměrná hodnota $u_0(x)$ v $(0,\pi)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} u_0(x)\,dx = \frac{2}{\pi}\int_0^{\pi} u_0(x)\,dx$$

?? $u(x,t)$ spojité v $\overline{\Omega} = \{(x,t);\ x \in [0,\pi];\ t \in [0,\infty)\}$.

$$|c_k(t) \cos kx| \le |a_k|; \quad \sum |a_k| \text{ konv.} \; : \; \text{Weierstrass.}$$

Věta: (1) $\sum f_k(y)$ konv. stejn. v Π; $f_k(y)$ spojité v Π $\Rightarrow$ $\sum f_k(y) =: F(y)$ je spojité v Π

(2) $|f_k(y)| \le \alpha_k \; \forall y \in \Pi$; $\sum \alpha_k$ konv. $\Rightarrow$ $\sum f_k(y)$ konv. stejn. v Π.

číslo nezávislé na y

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(3) $f_k(y)$; $\exists f_k'(y)$ vlastní; $\sum f_k'(y)$ konv. stejn.

$\sum f_k(y)$ konv. pro $y_0 \in I$.

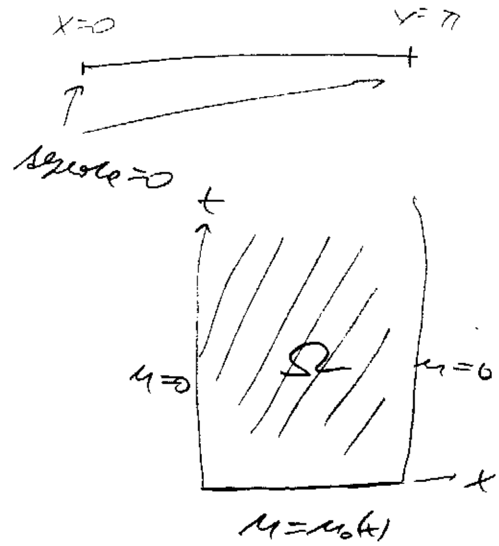
$\Rightarrow F(y) := \sum_k f_k(y)$ má derivaci, a platí

$$F'(y) = \sum_k f_k'(y) \quad \forall y \in I.$$

Příklad: (1) $\partial_t u = \partial_{xx} u \qquad t>0;\ x \in (0,\pi)$

(2) $u(0,t) = u(\pi,t) = 0 \quad \forall t>0$

(3) $u(x,0) = u_0(x)$



Nechť: $u_0(x)$ dáno;

kompatibilita: $u_0(0) = u_0(\pi) = 0$;

$$u_0(x) = \sum_{k=1}^{\infty} b_k \sin kx;$$

Ansatz: $$u(x,t) = \sum_{k=1}^{\infty} c_k(t) \sin kx;$$

$$\left.\begin{aligned} \partial_t u &= \sum_k c_k'(t) \sin kx \\ \partial_{xx} u &= -\sum_k c_k(t)\, k^2 \sin kx \end{aligned}\right\} \Rightarrow$$

formální [illegible]: $c_k'(t) = -k^2 c_k(t)$

$$(3):\quad u_0(x) = \sum_{k=1}^{\infty} \underbrace{c_k(0)}_{b_k} \sin kx.$$

$$c_k(t) = \boxed{b_k e^{-k^2 t}};$$

Nechť $\sum |b_k| < \infty$. Potom $u(x,t) \in C([0,\pi]\times[0,\infty))$
$\in C^\infty(\Omega)$
a splňuje (1), (2), (3).

$$u(x,t) = \sum_{k=1}^{\infty} f_k(x,t);$$ $f_k(x,t)$ spojité;

a dále: Weierstrass: $|f_k(x,t)| \le b_k$

rovnoměrně x, t.