Legendreovy polynomy: $P_n(t) = \frac{1}{2^n n!} D^n\left[(t^2-1)^n\right]$

$n = 0, 1, \dots$

- $g_n(t) = (x^2-1)^n$ .. polynom st. $2n$ .. $\Rightarrow$ $P_n(t)$ st. $\underline{n}$
- tvoří OG systém v $L^2(-1,1)$. BÚNO: $n \geq m$

$$\left(P_n(t), P_m(t)\right) = C_{nm} \int_{-1}^{1} D^n g_n(t) \cdot D^m g_m(t) =$$

$$= C_n \cdot C_m \int_{-1}^{1} D^n\left[(x^2-1)^n\right] \cdot D^m\left[(t^2-1)^m\right]$$

$$= C_n \cdot C_m \left[\underbrace{D^{n-1}\left[(t^2-1)^n\right]}_{=0\ x=\pm 1} D^{m}\left[(t^2-1)^m\right]\right]_{-1}^{1}$$

$$- C_m \cdot C_n \int_{-1}^{1} D^{n-1}\left[(t^2-1)^n\right] \cdot D^{m+1}\left[(t^2-1)^m\right] dt$$

$$= \dots (-1)^n \int_{-1}^{1} (x^2-1)^n D^{m+n}\left[(t^2-1)^m\right] dt$$

důkaz: • $n > m$: druhý člen $\equiv 0$;

- $n = m$: $(-1)^n (2n)! \int_{-1}^{1} (x+1)^n (x-1)^n$

$$= (-1)^n (-1)^n (2n)! \frac{n!}{n \cdot (n+1) \dots 2n} \int_{-1}^{1} (x-1)^{2n}$$

$$= \frac{(2n)!\, n!}{n \cdot (n+1) \dots (2n)} \cdot \frac{2^{2n+1}}{2n+1}$$

$$\|P_n\|^2 = \frac{1}{2^n n!} \cdot \frac{1}{2^n n!} \frac{(2n)!\, n!}{n(n+1) \dots (2n)} = \frac{2}{2n+1}$$

$f(t) = \mathrm{sgn}(t)$; $\quad F_f = \, ? = \sum_{k=0}^{\infty} c_k P_n(t)$;

kde $\quad c_k = \dfrac{(\mathrm{sgn}(t), P_n(t))}{(P_n(t), P_n(t))}$.

$=$

$C_{2m} = 0$ .. $P_{2m}(t)$ je sudá funkce. (2m-tá derivace sudé fce.).

$$(\mathrm{sgn}(t), P_{2m+1}(t)) = \int_{-1}^{1} \mathrm{sgn}(t) \cdot P_{2m+1}(t) = 2 \cdot \int_0^1 P_{2m+1}(t) =$$

$$= 2 c_n \int_0^1 D^{2m+1}\left[(t^2-1)^{2m+1}\right] = 2 c_n \cdot \left[D^{2m}\left[(t^2-1)^{2m+1}\right]\right]_0^1$$

$$= 2 \cdot c_n \cdot D^{2m}\left[(x^2-1)^{2m+1}\right]\Big|_{x=0}$$

$$(x^2-1)^{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} \boxed{x^{2k}} (-1)^{2m+1-k}$$

$$2k = 2m$$

$$\underline{\underline{k = m}}.$$

$$(-1)^{2m+1-m} \cdot \binom{2m+1}{m} \cdot (2m)! = (-1)^{m+1} \cdot \frac{(2m+1)!}{m!\,(m+1)!} (2m)! \frac{1}{m!\,2^m}$$

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!\,(m+1)!}$$

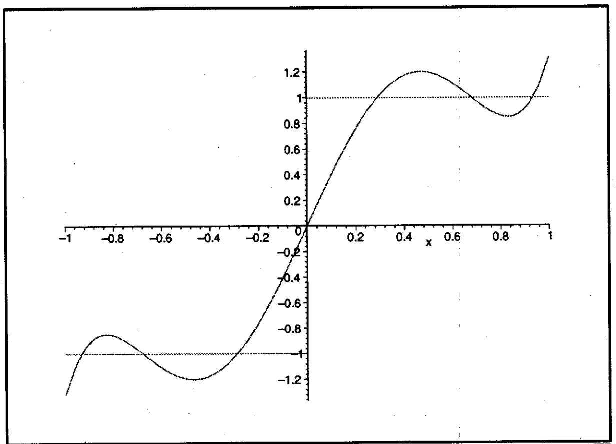
$=$

$$\mathrm{sgn}(t) = \sum_{m=0}^{\infty} \binom{2m+1}{m} \cdot (2m)!\,(-1)^{m+1} \frac{1}{m!\,2^m} \cdot \frac{2m+1}{m} \cdot P_n(t)$$

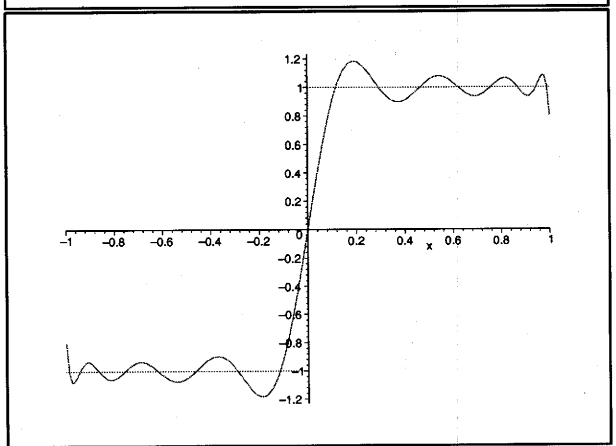
$\{P_n(x)\}_{n=0}$ .. uzavřený systém v $L^2(-1,1) \Rightarrow$ konverguje v $L^2(-1,1)$.

① Rozvoj 'sgn(x)' pomocí Legendre polynomů. $\{P_n(t)\}$ v $L^2(-1,1)$

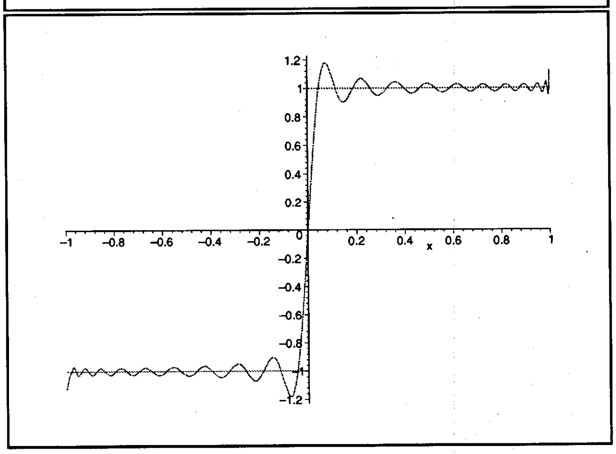
$$s_n(t) = \sum_{k=0}^{n} (-1)^k \frac{(4k+3)\,(2k)!}{2^{2k+1}\, k!\, (k+1)!} \; P(2k+1, x)$$



$n=2$



$n=7$



$n=20$

Hermitovy polynomy: $H_n(x) = (-1)^n e^{x^2} D^n[e^{-x^2}]$

$D[\exp(-x)^2] = -2x\, e^{-x^2}$ zřejmě $H_n(x)$ .. polynom st. n

$D^2[\quad] = e^{-x^2}[-2 + (-2x)\cdot(-2x)]$

$$(H_n(x), H_m(x)) = \int_{\mathbb{R}} H_n(x) H_m(x)\cdot e^{-x^2}\,dx = \int_{\mathbb{R}} \boxed{H_n(x)}\, H_m(x)\, e^{-x^2}\,dx$$

BÚNO: $n \geq m$

$$= \int_{\mathbb{R}} \cancel{e^{x^2}}\, D^n(e^{-x^2}) \cdot e^{x^2} D^m(e^{-x^2})\, \cancel{e^{-x^2}}$$

$$= (-1)^n \cdot \int_{-\infty}^{\infty} D^n[e^{-x^2}]\cdot \underbrace{H_m(x)}_{\text{polynom st. } m}\,dx$$

$$= (-1)^n \left[D^{n-1}[e^{-x^2}]\cdot H_m(x)\right]_{-\infty}^{\infty} - (-1)^n \int_{-\infty}^{\infty} D^{n-1}(e^{-x^2})\cdot DH_m$$

$$(-1)^{2n} \int_{-\infty}^{\infty} e^{-x^2}\cdot \underbrace{D^n H_m(x)}_{(2)}\,dx$$

dostanu: $n > m$: $(2) \equiv 0 \Rightarrow H_n, H_m$ ortog.

$n = m$: $= \int_{-\infty}^{\infty} e^{-x^2}\cdot = \sqrt{\pi}\cdot 2^n\, n!$

$(2) = D^n H_n(x) = D^n[(+2x)^n] = 2^n\cdot n!$

lec. e^x: $e^x \notin \mathcal{L}^2(\mathbb{R})$, ale $e^x \in L^2(\mathbb{R}, e^{-x^2})$.

$\int_{\mathbb{R}} e^{2x}\cdot e^{-x^2}\,dx < \infty$

$\|H_n\|^2 = \sqrt{\pi}\cdot 2^n\, n!$

e^x do $H_n(x)$: $\quad F_{e^x} = \sum_{n=0} c_n H_n(x)$,

$$c_n = \frac{(e^x, H_n(x))}{\|H_n\|^2}; \qquad \|H_n\|^2 = \sqrt{\pi}\cdot 2^n \cdot n! \checkmark$$

$$(e^x, H_n(x)) = \int_{\mathbb{R}} e^x H_n(x)\cdot e^{-x^2}\,dx = (-1)^n \int_{\mathbb{R}} e^x D^n\{e^{-x^2}\}\,dx$$

n-krát p.p.

$$= \cancel{(-1)^{2n}} \cdot \int_{\mathbb{R}} e^x \cdot e^{-x^2}\,dx \qquad \cancel{e^{x+x^2}} \qquad x - x^2 = \tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2$$

$$= \int_{\mathbb{R}} e^{x-x^2}\,dx = \sqrt[4]{e}\cdot \int_{\mathbb{R}} e^{-(x-\frac{1}{2})^2}\,dx = \sqrt[4]{e}\int_{\mathbb{R}} e^{-y^2}\,dy = \sqrt[4]{e}\cdot\sqrt{\pi}$$

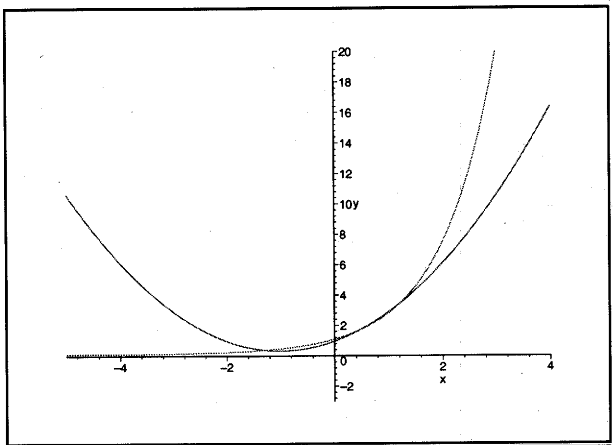
$x - \frac{1}{2} = y$

$$F_{e^x} = \sum_{n=0}^{\infty} \frac{\sqrt[4]{e}}{2^n n!} H_n(x) = e^x \text{ ve sm. } L^2_{e^{-x^2}}(\mathbb{R}, e^{-x^2}),$$

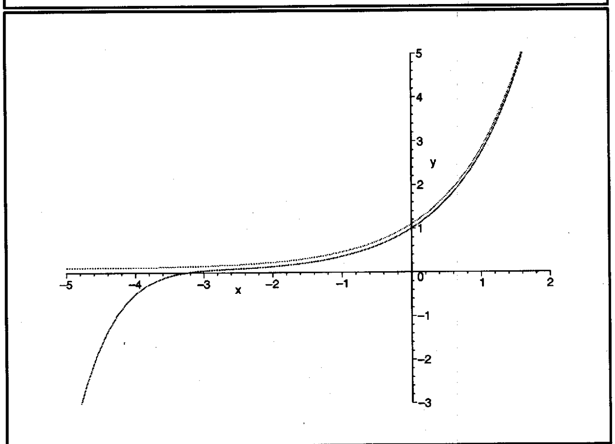
O.K. — maple!

(2) Rozvoj 'e^x' pomocí Hermitových polynomů $H(n,x)$ v $L^2(\mathbb{R}; e^{-x^2})$.

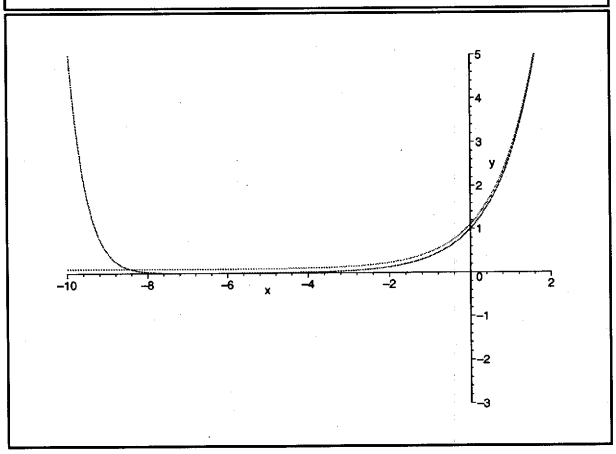
$$s_n(t) = \sum_{k=0}^{n} \frac{e^{1/4}}{2^k k!} H(k,x)$$



$n=2$



$n=7$



$n=20$.