5a $\Phi(y) = \int_1^3 2y - yy' + x(y')^2 \, dx \; ; \quad y(1) = 1, \; y(3) = 4$

$f = 2y - yR + xR^2$

$f_y = 2 - R$

$f_R = -y + 2xR$

(E.L.) $-(-y + 2xy')' + 2 - y' = 0$

$\cancel{y'} - 2y' - 2xy'' + 2 - \cancel{y'} = 0$

$xy'' + y' = 1$

$$\boxed{x^2 y'' + xy' = x}$$

rovnice Eulerova typu $\quad x^2 y'' + xy' = 0$

$y = |x|^\lambda = x^\lambda \quad (x > 0)$

$y' = \lambda x^{\lambda-1}, \; y'' = \lambda(\lambda-1) x^{\lambda-2}$

$[\lambda(\lambda-1) + \lambda] x^\lambda = 0$

$\lambda^2 = 0 \; : \; F.S. \{1, \ln x\}$

partikulární řešení: $\quad y_p = x$

obecné řešení: $\quad y_{ob} = x + C_1 + C_2 \ln x$

$y(1) = 1$, $y(3) = 4$ $\Rightarrow \quad C_1 = 0; \quad C_2 = \frac{1}{\ln 3}$.

5b $\Phi(y) = \int_1^2 (xy' + y)^2 \, dx \; ; \quad y(1) = 1, \; y(2) = \frac{1}{2}$.

$f = (xR + y)^2$

$f_y = 2(xR + y)$

$f_R = 2x(xR + y)$

(E.L.) $-\left(2x(xy' + y)\right)' + 2(xy' + y) = 0$

$-2x\,(xy' + y)' = 0 \quad / 2x \neq 0$ na $(1,2)$.

$(xy)' = xy' + y = C$

$$xy = cx + d \qquad y(1) = 1$$
$$y = c + \frac{d}{x} \qquad y(2) = \frac{1}{2} \quad \Rightarrow \quad y = \frac{1}{x}.$$

(5c) $\phi(y) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(y - \frac{1}{2}(y')^2\right) \sin x \, dx \qquad y\left(\frac{\pi}{4}\right) = -\ln\sqrt{2}$

$y\left(\frac{\pi}{2}\right) = 0.$

$f = \left(y - \frac{1}{2} r^2\right) \sin x$

$f_y = \sin x$

$f_r = -r \sin x$

(E.L.) $-\left(-y' \sin x\right)' + \sin x = 0$

$(y' \sin x)' = -\sin x$

$y' \sin x = c + \cos x$

$y' = \frac{c}{\sin x} + \frac{\cos x}{\sin x}$

$\boxed{y_{ob.} = c \cdot \ln\left(\operatorname{tg} \frac{x}{2}\right) - \ln \sin x + d}$

$f_{rr} = -\sin x < 0$

$\Rightarrow$ není min.

(Legendreova věta).

okr. podm. $\Rightarrow$ $\boxed{y = -\ln \sin x}$

(5d) $\phi(y) = \int_0^{\pi} (y')^2 - \frac{16}{9} y^2 + 2y \sin x\,; \qquad y(0) = 0$

$y(\pi) = -\frac{\sqrt{3}}{2}.$

$f = r^2 - \frac{16}{9} y^2 + 2y \sin x.$

$f_y = -\frac{32}{9} y + 2 \sin x$

$f_r = 2r$

(E.L.) $(-2y')' - \frac{32}{9} y + 2 \sin x$

$y'' + \frac{16}{9} y = \sin x$

F.S. $\left\{\cos \frac{4}{3}x, \sin \frac{4}{3}x\right\}$

$y_p = A \cos x + B \sin x$

$\dots A = \frac{9}{7} \dots \boxed{y = \sin \frac{4x}{3} + \frac{9}{7} \sin x}$

(5e) $\phi(y) = \int_0^{\pi} (y')^2 - \frac{25}{9} y^2 + 68 y e^x$; $y(0) = 9$, $y(\pi) = 9 \cdot e^{\pi}$

$f = r^2 - \frac{25}{9} y^2 + 68 y e^x$

$f_y = -\frac{50}{9} y + 68 e^x$

$f_r = 2r$

(E.L.) $(-2y')' - \frac{50}{9} y + 68 e^x = 0$

$y'' + \frac{25}{9} y = 34 e^x$.

F.S.: $\{\cos \frac{5}{3}x, \sin \frac{5}{3}x\}$.

$y_p = A \cdot e^x$; $A = 9$.

$y_{ob.} = 9e^x + c_1 \cos x + c_2 \sin x$

$\boxed{y = 9 \cdot e^x}$

(5f) $\phi(y) = \int_0^1 x^2 y' + 2xy \, dx$; $y(0) = 0$, $y(1) = 1$.

$f = x^2 r + 2xy$

$f_y = 2x$

$f_r = x^2$

(E.L.) $(-x^2)' + 2x = 0$

$0 = 0$

$\Rightarrow$ Každá funkce $y(t)$ je extremála (splňující okr. podm.)

$\ldots D\phi(y; h) = 0 \quad \forall y; \forall h$

$\rightsquigarrow \phi \equiv$ konst. ??

$\phi(y) = \int_0^1 x^2 y' + 2xy \, dx = \int_0^1 (x^2 y)' \, dx = \left[x^2 y(t) \right]_{x=0}^{x=1}$

$= 1 \cdot y(1) - 0 \cdot y(0) = \underline{\underline{1}} \quad \forall y \in \underline{\mathcal{M}}$.

*) **Pozn.:** Je-li $f=f(t,y,z)$ kvadratické vůči y, z;

je výhodné uvažovat $\boxed{\phi(y_0+h)-\phi(y_0)}$

kde: y_0 ... extremála

h ... libovolný element $C_0^1([a,b])$.

$\rightsquigarrow$ víme obvykle mé znaménko;

protože y_0+h je obecný prvek M, plyne z toho, že y_0 je (globální) extrém.

Př.: (5a) $\quad | \; y_0 = x + \frac{\ln x}{\ln 3}$ -- nalezená extremála;

$$\phi(y_0+h)-\phi(y_0) = \int_1^3 2(y_0+h)-(y_0+h)(y_0'+h')+x(y_0'+h')^2 - \left(2y_0 - y_0y_0' + x(y_0')^2\right)dx$$

$$= \int_1^3 2h \underbrace{-y_0h'}_{\text{per-partes}} - y_0'h - hh' + \underbrace{2xy_0'h'}_{\text{per-partes}} + (h')^2\,dx$$

$$-(-y_0)'h \qquad\qquad -(2xy_0')'h$$

$$= \int_1^3 \Big[\underbrace{2-y_0'-(-y_0+2xy_0')'}_{\equiv 0\text{, neboť } y_0 \text{ řeší (E.L.) rovnici.}}\Big]h + \left((h')^2 - hh'\right)dx$$

$$= \underbrace{\int_1^3 (h')^2 dx}_{\geq 0} - \underbrace{\int_1^3 hh'dx}_{\left[\frac{1}{2}h^2\right]_1^3 = 0} \geq 0 \quad \text{-- } y_0 \text{ je globální minimum.}$$

(5e) $y_0 = 9e^x$ .. nalezená extremála

$$\phi(y_0+h)-\phi(y_0) = \int_0^{\pi} (y_0'+h')^2 - (y_0')^2 - \frac{25}{9}\left[(y_0+h)^2 - y_0^2\right] + 68e^x h \, dx$$

$$= \int_0^{\pi} \underbrace{2y_0'h'}_{} + (h')^2 - \frac{50}{9} y_0 h - \frac{25}{9} h^2 + 68e^x h \, dx$$

$(-2y_0')'h$ .. integrací per-partes...

$$= \int_0^{\pi} \underbrace{\left[(-2y_0')' - \frac{50}{9} y_0 + 68e^x\right] h}_{=0 \text{ díky (E.L.) rovnici}} + (h')^2 - \frac{25}{9} h^2 \, dx$$

celkem: $\phi(y_0+h)-\phi(y_0) = \int_0^{\pi} (h')^2 - \frac{25}{9} h^2 \, dx$.

volme $h(x) = \sin x \in C_0^1([0,\pi])$.

$$\int_0^{\pi} \cos^2 x - \frac{25}{9}\sin^2 x \, dx = \frac{\pi}{2}\left(1 - \frac{25}{9}\right) < 0.$$

volme $h(x) = \sin 2x \in C_0^1([0,\pi])$

$$\int_0^{\pi} 4\cos^2 2x - \frac{25}{9}\sin^2 2x \, dx = \frac{\pi}{2}\left(4 - \frac{25}{9}\right) > 0.$$

$\Rightarrow$ NENÍ EXTRÉM

Pozn.: užíváme vzorečk $\int_0^{\pi} \sin^2 kx = \int_0^{\pi} \cos^2 kx = \frac{\pi}{2}$

$\forall k \in \mathbb{N}$.