pomocné vzorečky:

$$\frac{1}{1-q} = 1+q+q^2+\cdots = \sum_{k=0}^{\infty} q^k \;; \quad |q|<1.$$

$$\int_0^1 x^a \ln x = \underbrace{\left[\frac{x^{a+1}}{a+1}\ln x\right]_0^1}_{=0} - \frac{1}{a+1}\int_0^1 x^a = -\frac{1}{(a+1)^2}$$

$$\forall\, a>-1.$$

$$\int_0^\infty e^{-ax}\cos bx = \frac{a}{a^2+b^2}\;; \quad \int_0^\infty e^{-ax}\sin bx = \frac{b}{a^2+b^2}$$

$$\forall a>0;\; b\in\mathbb{R}$$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{k+1}}{k+1}\;; \;\forall |x|<1.$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\;, \;\forall x\in\mathbb{R}$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}\;; \;\forall x\in\mathbb{R}$$

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

(C1) $\int_0^1 \ln\left(\frac{1+x}{1-x}\right) = \int_0^1 \ln(1+x) - \ln(1-x)\, dx$

$$= \int_0^1 \ln(1+x)\,dx + \int_0^1 (-\ln(1-x))\,dx = I_1 + I_2.$$

$$\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1} \quad ; \quad x \in (0,1):$$

$$I_1 = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{k+1}}{k+1}\,dx \overset{(*)}{=} \sum_{k=0}^{\infty} \int_0^1 (-1)^k \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+2)(k+1)}.$$

ověření $(*)$: „teleskopické sumy": $f_k(t) = (-1)^k g_k(t)$

$g_k(t) \searrow 0 \ \forall x \in (0,1)$ stejně.

$$\Rightarrow \left|\sum_{k=0}^{m} f_k(x)\right| \le |f_0(x)| = t \in L^1(0,1).$$

– Lebesgueova věta; majoranta $g(t) = t$.

$$-\ln(1-x) = -\sum_{k=0}^{\infty} (-1)^k \frac{(-x)^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1};$$

$f_k \ge 0$; měřitelné (spojité)

Levi: $I_2 = \int_0^1 \sum_{k=0}^{\infty} \frac{t^{k+1}}{k+1}\,dx = \sum_{k=0}^{\infty} \int_0^1 \frac{x^{k+1}}{k+1}\,dx = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)}.$

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celkem: $I_1 + I_2 = \sum_{k=0}^{\infty} \frac{(-1)^k + 1}{(k+1)(k+2)}$, $\quad (-1)^k + 1 = \begin{cases} 2; & k \text{ sudé} \\ 0; & k \text{ liché} \end{cases}$

$$= \sum_{l=0}^{\infty} \frac{2}{(2l+1)(2l+2)} = \sum_{l=0}^{\infty} \frac{1}{(2l+1)l}.$$

(C2) $\int_0^\infty e^{-x} \cos x^{\frac{1}{2}} \, dx$.

$$f(t) = e^{-x} \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(x^{1/2})^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \underbrace{(-1)^k \frac{x^k}{(2k)!} \cdot e^{-x}}_{f_k(t)}$$

$f_k(t)$ -- měřitelné (spojité).

$$I = \int_0^\infty \sum_{k=0}^{\infty} f_k(t) \overset{(*)}{=} \sum_{k=0}^{\infty} \int_0^\infty f_k(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \underbrace{\int_0^\infty x^k e^{-x}}_{k!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k)!}.$$

$f_k(t)$ mění znaménko × Levi;

?? majoranta: $\left| \sum_{k=0}^{n} f_k(x) \right| \le \sum_{k=0}^{n} |f_k(x)| \le g(t) := \sum_{k=0}^{\infty} |f_k(t)|$.

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ověříme, že $g \in L(0,\infty) \Leftrightarrow \int_0^\infty g < \infty$.

$$\int_0^\infty g(t)\,dx = \int_0^\infty \sum_{k=0}^{\infty} |f_k| = \sum_{k=0}^{\infty} \int_0^\infty |f_k| = \sum_{k=0}^{\infty} \frac{k!}{(2k)!} =: c_k < \infty$$

Leviho věta

u této řady konvergenci ověříme např. podílovým kritériem:

$$\frac{c_{k+1}}{c_k} = \frac{(k+1)!}{(2k+2)!} \cdot \frac{(2k)!}{k!} = \frac{k+1}{(2k+2)(2k+1)} \to 0$$

(C3) $\int_0^1 \ln x \cdot \ln(1-x)\,dx = \int_0^1 \ln x \cdot \sum_{k=0}^{\infty} (-1)^k \frac{(-x)^{k+1}}{k+1} = \int_0^1 \sum_{k=0}^{\infty} f_k(x)$;

$f_k(x) = -\ln x \cdot \frac{x^{k+1}}{k+1}$ -- měřitelné (spojité)
nezáporné ($-\ln x > 0$ v $(0,1)$)

... Leviho věta :.. $= \sum_{k=0}^{\infty} \int_0^1 (-\ln x) \cdot \frac{x^{k+1}}{k+1} = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)^2}$.

(C4) $\underbrace{\int_0^1 \frac{x^p \ln x}{1+x^2}\,dx}_{f(x)}$ -- ověřme konvergenci integrálu:

$f(x)$ měřitelné (spojité)

$$\frac{x^p|\ln x|}{2} \le |f(x)| \le x^p|\ln x| = x^p(-\ln x).$$

$$\int_0^1 x^p(-\ln x) = \int_0^\delta + \int_\delta^1 = I_1 + I_2$$

I_2 konv. $\forall p$: $x^p(-\ln x)$ spojité na $[\delta,1]$ (omezený interval)
$\Rightarrow$ omezené

I_1: $p \le -1$: $x^p(-\ln x) \ge x^p(-\ln\delta) = c\cdot x^p \notin \mathcal{L}(0,\delta)$

$p > -1$: volme $\varepsilon > 0$; $p-\varepsilon > -1$

$$x^p(-\ln x) = x^{p-\varepsilon} \cdot \underbrace{x^\varepsilon(-\ln x)}_{\substack{\text{omezené na } (0,\delta) \\ \text{neboť} \to 0 \\ \text{pro } x\to 0+}} \le c\cdot x^{p-\varepsilon} \in \mathcal{L}(0,\delta)$$

neboť $p-\varepsilon > -1$.

Celkem I_{C4} konv.
$\Leftrightarrow p > -1$.

Nadále předpokládáme: $p > -1$.

(C4-pokr.) $\quad \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{k=0}^{\infty} (-1)^k x^{2k}; \quad \forall x \in (0,1)$

$$I_{C4} = \int_0^1 \underbrace{\sum_{k=0}^{\infty} (-1)^k x^{2k+p} \ln x}_{f_k(t)} \overset{(*)}{=} \sum_{k=0}^{\infty} \int_0^1 (-1)^k x^{2k+p} \ln x = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+p+1)^2}.$$

ověření (*): $\quad f_k(t)$ měřitelné (spojité)

„teleskopické sumy": $(-1)^k g_k(t)$; $\quad |g_k(t)| \geq |g_{k+1}(t)|$

$$g_k(t) < 0 \qquad x^{2k+p} |\ln x| \geq x^2 x^{2k+p} |\ln x|$$

$$\left| \sum_{k=0}^{n} f_k(t) \right| \leq |f_0(t)| = x^p |\ln x|. \qquad \in (0,1).$$

víme: $x^p |\ln x| \in \mathcal{L}(0,\infty)$ .... $\int \sum = \sum \int$ dle Lebesg. věty.

C5 $\quad \int_0^{\infty} \underbrace{\frac{\sin x}{1+e^x}}_{f(t)} dx$; konverguje: $f(t)$ měřitelné (spojité)

$$|f(t)| \leq \frac{1}{1+e^t} \leq e^{-x} \in \mathcal{L}(0,\infty).$$

rozvoj do řady: $\quad \frac{1}{1+e^x} = \frac{e^{-x}}{e^{-x}+1} = e^{-x} \cdot \left(\frac{1}{1-(-e^{-x})} \right) = \sum_{k=0}^{\infty} (-1)^k e^{-(k+1)x}.$

$$f(t) = \sum_{k=0}^{\infty} (-1)^k e^{-(k+1)x} \sin x ;= \sum_{k=0}^{\infty} f_k(t)$$

$$I_{C5} = \int_0^{\infty} \sum_{k=0}^{\infty} f_k(t) \overset{*}{=} \sum_{k=0}^{\infty} \int_0^{\infty} f_k(t) = \sum_{k=0}^{\infty} \int_0^{\infty} (-1)^k e^{-(k+1)x} \sin x$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2+1}.$$

ověření (*): Lebesgueovy věty:

$$\left| \sum_{k=0}^{n} f_k(t) \right| \leq \sum_{k=0}^{n} |\sin x| \cdot e^{-(k+1)x} \leq \sum_{k=0}^{\infty} |\sin x| \cdot e^{-(k+1)x} = \underbrace{e^{-x} \cdot \frac{|\sin x|}{1-e^{-x}}}_{=: g(t)}.$$

(C5 – pokr.) ?? integrovatelnost g.

$$\int_0^\infty g = \int_0^K + \int_K^\infty = I_1 + I_2 .$$

$$\lim_{x\to 0+} g(x) = e^{-0} \cdot \lim_{x\to 0} \frac{\sin x}{1-e^{-x}} = \underline{\underline{1}} \quad (\text{l'Hosp.}).$$

$\Rightarrow$ g omezená na $(0,\delta)$; spojitá na $[\delta, K]$ $\Big\}$ $\Rightarrow$ omezená na $(0,K)$ -- I_1 konv.

I_2: $\quad g(x) \le e^{-x} \cdot \dfrac{1}{1-e^{-K}} = C_0 \cdot e^{-x} \in \mathcal{L}(K,\infty)$.

C6 $\displaystyle\int_0^\infty \frac{x}{e^x-1} = \int_0^\infty x \cdot e^{-x} \cdot \frac{1}{1-e^{-x}} = \int_0^\infty \sum_{k=0}^\infty \underbrace{x e^{-(k+1)x}}$

$f_k(x) \ge 0$, měřitelné (spojité)

$$\underset{\text{Leviho věta}}{=} \sum_{k=0}^\infty \int_0^\infty x \cdot e^{-(k+1)x} = \sum_{k=0}^\infty \frac{1}{(k+1)^2} .$$
