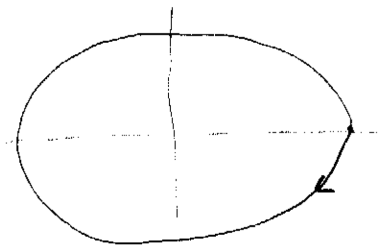
D1 param: $\varphi(t) = (t, t^2)$; $t \in [-1, 1]$

$dx = dt$

$dy = 2t\,dt$ — výsledek: $-\frac{14}{15}$

D2



param: $\varphi(t) = (a\cos t, b\sin t)$; $t \in [0, 2\pi]$

$\circlearrowleft$ – NENÍ VE SHODĚ

$\varphi'(t) = (-a\sin t, b\cos t)$.

$$-\int_0^{2\pi} (a\cos t + b\sin t)(-a\sin t) + (a\cos t - b\sin t)(b\cos t)\,dt$$

$$= \int_0^{2\pi} \underbrace{a^2\cos t\sin t}_{0} + \underbrace{ab\sin^2 t}_{\pi} - \underbrace{ab\cos^2 t}_{\pi} + \underbrace{b^2\sin t\cos t}_{0}\,dt \quad 0$$

---

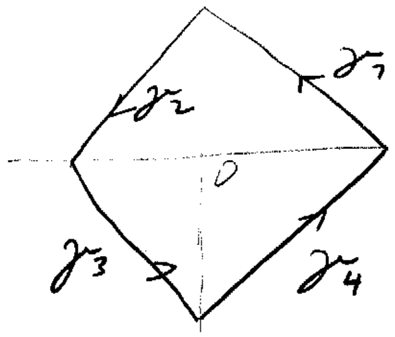
zkouška: Greenova věta $-\int_\gamma \underset{\sim}{F} \cdot \underset{\sim}{ds} = \iint_M \operatorname{rot} \underset{\sim}{F}\, d\lambda_2 = 0.$

M – „vnitřek" γ

$\underset{\sim}{F} = (x+y, x-y)$

$$\operatorname{rot} \underset{\sim}{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

D3 $\displaystyle\int_\gamma \frac{dx+dy}{|x|+|y|}$;



γ_1: $\varphi(t) = (1-t, t)$; $t \in [0,1]$.

$\varphi'(t) = (-1, 1)$

$$\int_{\gamma_1} = \int_0^1 \frac{-1+1}{|1-t|+|t|}\,dt = \int_0^1 0\,dt = 0.$$

podobně: $\displaystyle\int_{\gamma_2} = -2$; $\displaystyle\int_{\gamma_3} = 0$; $\displaystyle\int_{\gamma_4} = 2$.

---

zkouška: $\underset{\sim}{F} = \left(\frac{1}{|x|+|y|}, \frac{1}{|x|+|y|}\right) = (1,1)$ na γ.

leč: $(1,1) = \nabla(x+y)$; $\Rightarrow \displaystyle\int_\gamma = 0$ (uzavřená křivka)

potenciál $\underset{\sim}{F}$

D4 $\displaystyle\int_P \underset{\sim}{F}\cdot d\underset{\sim}{S}$; $\underset{\sim}{F} = (x, y, z)$

$P = \{x^2+y^2+z^2 = 1\}$; orientace <u>ven</u>

(a) snadný výpočet: Gaussova věta →

$$\int_P \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_\Theta \operatorname{div}\underset{\sim}{F}\,d\lambda_3 = 3\lambda_3(\Theta) = 4\pi.$$

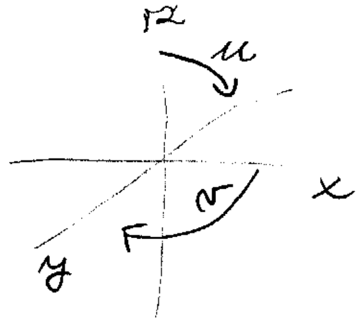
$\Theta = \{x^2+y^2+z^2 < 1\}$; $\operatorname{div}\underset{\sim}{F} = 3$

„vnitřek" sféry

D4 b) (z definice)

|sférické souřadnice|

$x = \sin u \cos v$
$y = \sin u \sin v$
$z = \cos u$



$\varphi_u = (\cos u \cos v, \cos u \sin v, -\sin u)$ $\quad \Omega: \begin{array}{l} u \in (0,\pi) \\ v \in (0, 2\pi) \end{array}$

$\varphi_v = (-\sin u \sin v, \sin u \cos v, 0)$

$\varphi_u \times \varphi_v = (\sin^2 u \cos v, +\sin^2 u \sin v, \cos u \sin u)$

orientace? $\begin{array}{l} v = 0 \\ u = \frac{\pi}{2} \end{array}$: bod $(1,0,0)$;

$\varphi_u \times \varphi_v = (1, 0, 0)$

SMĚŘUJE VEN.

$$\int_P \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint_\Omega (\sin u \cos v, \sin u \sin v, \cos u) \cdot (\sin^2 u \cos v, +\sin^2 u \sin v, \cos u \sin u)\, du\, dv$$

$$\int_0^\pi \left(\underbrace{\int_0^{2\pi} \sin^3 u \cos^2 v + \sin^3 u \sin^2 v + \cos^2 u \sin u \, dv}_{} \right) du$$

$\rightarrow \sin^3 u = \sin^2 u \cdot \sin u$

$$= 2\pi \int_0^\pi \sin u \, du = \boxed{4\pi}$$

třetí způsob: $\underset{\sim}{F} = \underset{\sim}{n}$ na P;

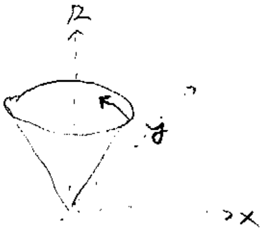
$$\int_P \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_P \underset{\sim}{n} \cdot \underset{\sim}{n}\, dS = \int_P dS = \text{obsah}(P) = 4\pi \cdot 1^2$$

D5 $\displaystyle\int_S (y-z)\,dy\,dz + (z-x)\,dz\,dx + (x-y)\,dx\,dy$;

S ... vnější strana kužele.. $x^2+y^2=z^2$; $0 \le z \le h$.

1) přímo:

$$x = r\cdot\cos t \qquad r\in(0,h);\ t\in(0,2\pi).\ \Omega$$
$$y = r\cdot\sin t$$
$$z = r$$



$$\varphi_r = (\cos t,\ \sin t,\ 1)$$
$$\varphi_t = (-r\sin t,\ r\cos t,\ 0)$$

$$\varphi_r\times\varphi_t = (-r\cos t,\ -r\sin t,\ r)$$
$$t=0, r=1:\quad (-1,\ 0,\ 1) \qquad \text{orientace:}$$

$$I = -\int_\Omega r(\sin t-1)\cdot(-r\cos t) + r(\cos t-1)(-r\sin t) + r(\cos t-\sin t)\,r\ dr\,dt = 0.$$

2) pomocí Stokesovy věty:

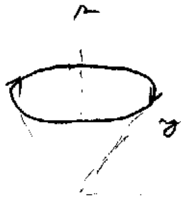
$$\int_S F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy = \int_\varphi \vec{T}\cdot d\varphi$$

kde $\operatorname{rot}\vec{T} = \vec{F}$

a φ .. okraj S

(dobrá parametrizace:)

$$\int_S \vec{F}\cdot dS = \int_\varphi T_1\,dx + T_2\,dy + T_3\,dz;$$



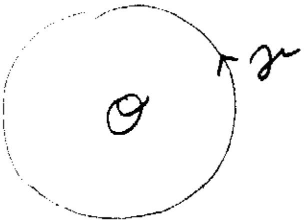
zde: φ.... okraj kužele:

$$\operatorname{rot} T = \left(\frac{\partial T_3}{\partial x_2}-\frac{\partial T_2}{\partial x_3},\ \frac{\partial T_1}{\partial x_3}-\frac{\partial T_3}{\partial x_1},\ \frac{\partial T_2}{\partial x_1}-\frac{\partial T_1}{\partial x_2}\right).$$

$$T = \frac{1}{2}\left(z^2+y^2,\ z^2+x^2,\ y^2+x^2\right)$$

$$\varphi = (h\cos t,\ h\sin t,\ h);\quad t\in(0,2\pi)$$
$$\varphi' = (-h\sin t,\ h\cos t,\ 0);$$
$$I = -\frac{1}{2}\int_0^{2\pi} (h^2+h^2\sin^2 t)\cdot(-h\sin t) + (h^2+(h^2\cos^2 t))\cdot h\cos t + 0\ dt = 0$$

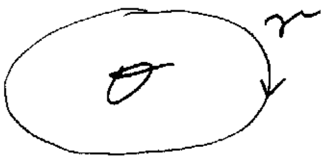
(E1) $\underset{\sim}{F} = (-x^2y, xy^2)$; $\int_\gamma \underset{\sim}{F}\cdot d\underset{\sim}{s} = \iint_\Omega \mathrm{rot}\,\underset{\sim}{F}\, d\lambda_2 = 0$ symetrie Ω dle os



$\mathrm{rot}\,\underset{\sim}{F} = 2xy + 2xy = \underline{4xy}$

(E2) Green: $\int_\gamma \underset{\sim}{F}\cdot d\underset{\sim}{s} = -\iint_\Omega \mathrm{rot}\,\underset{\sim}{F}\, d\lambda_2 = 2\lambda_2(\Omega)$

$= 2\pi ab.$

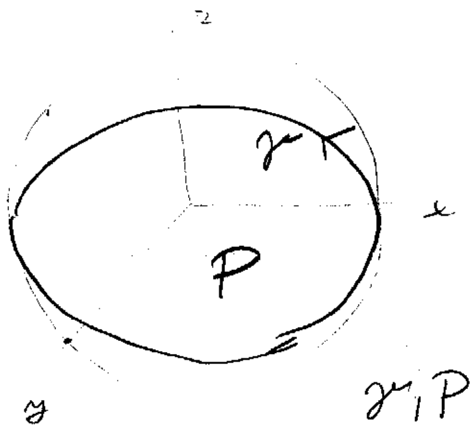


$\mathrm{rot}\,\underset{\sim}{F} = -1-1 = -2$

$\underset{\sim}{F} = (x+y, -x+y)$

(E3) — opravit zadání — Kopáček.

(F1)



$\underset{\sim}{F} = (y, z, x)$;

$\operatorname{rot} \underset{\sim}{F} = (-1, -1, -1)$

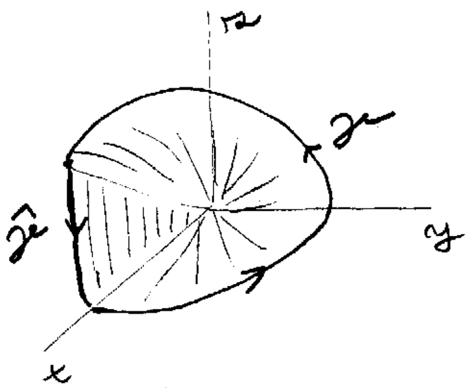
γ, P ... kružnice resp. kruh $(r=1)$

v rovině kolmé na $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

– normála

$$\int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = \int_P \operatorname{rot} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_P \underbrace{\operatorname{rot} \underset{\sim}{F} \cdot \underset{\sim}{n}}_{-\sqrt{3}} \, dS = -\sqrt{3} \cdot \underbrace{\pi}_{\text{plocha } P}$$

(F2)



$\Gamma = \gamma \oplus \hat{\gamma}$ ... okraj zobecněné plochy P.

$$\int_\Gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = \int_\gamma + \int_{\hat{\gamma}} = \int_P \operatorname{rot} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

leč: $\underset{\sim}{F} = (x^2 - yz, y^2 - xz, z^2 - xy)$

$= \nabla u$; $u = \frac{1}{3}(x^3 + y^3 + z^3) - xyz$

odsud (a přímo): $\operatorname{rot} \underset{\sim}{F} = (0, 0, 0)$.

a tedy: $\int_\gamma = -\int_{\hat{\gamma}}$ ... param. $\hat{\gamma}$: $\varphi(t) = (1, 0, t)$, $t \in (0,1]$

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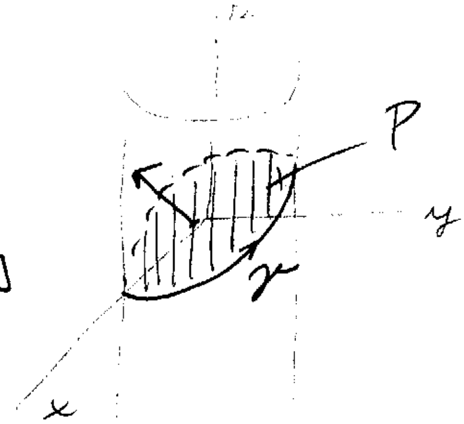
$$\int_{\hat{\gamma}} \underset{\sim}{F} \cdot d\underset{\sim}{s} = \int_0^1 (\cdot, \cdot, t^2) \cdot (0, 0, 1) \, dt = \frac{1}{3}.$$

(F3) $\underset{\sim}{F} = (y-z,\ z-x,\ x-z)$

$\operatorname{rot}\underset{\sim}{F} = (-1, -2, -2)$



(a) přímo: $t \in [0, 2\pi]$

$\psi(t) = (\cos t, \sin t, 1-\cos t)$

$\psi'(t) = (-\sin t, \cos t, \sin t)$

$$\int_\gamma \underset{\sim}{F}\cdot \underset{\sim}{ds} = \int_0^{2\pi} (\sin t + 1 - \cos t,\ 1-2\cos t,\ 2\cos t - 1)\cdot(-\sin t, \cos t, \sin t)\,dt$$

$$= \int_0^{2\pi} -\sin^2 t - 2\cos^2 t\ dt \quad ; \text{ neboť } \int_0^{2\pi} \sin t;\ \cos t;\ \sin\cos t = 0$$

$$= \boxed{-3\pi}.$$

$z = 1-x$

(b) pomocí Stokese: $P: \quad \varphi(u,v) = (u, v, 1-u)$

$u, v \in \Omega = \{u^2+v^2 < 1\}.$

$\partial_u \varphi = (1, 0, -1)$

$\partial_v \varphi = (0, 1, 0)$

$\operatorname{rot}\underset{\sim}{F} \circ \underset{\sim}{\varphi} = (-1, -2, -2)$

$\partial_u\varphi \times \partial_v \varphi = (-1, 0, 1)$ -- $\boxed{\text{NENÍ}}$ ve shodě s kladným obíháním γ

$$\int_P \operatorname{rot}\underset{\sim}{F}\cdot \underset{\sim}{dS} = \int_\Omega \underbrace{(-1,-2,-2)\cdot(-1,0,1)}_{3}\,du\,dv$$

$$= 3\cdot\lambda_2(\Omega) = \boxed{3\pi}$$

změnit znaménko!

(G1) $\int_P \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_\Theta \operatorname{div}\underset{\sim}{F}\, d\lambda_3$; $\quad \underset{\sim}{F} = (x^3, y^3, z^3)$

$\operatorname{div}\underset{\sim}{F} = 3(x^2+y^2+z^2)$

$\Theta = \{x^2+y^2+z^2 < 1\}$.

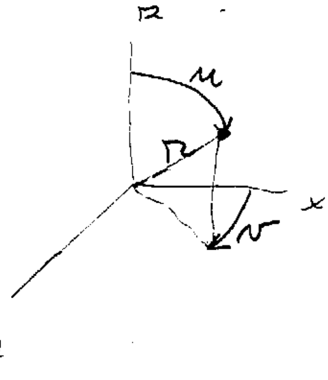
sférické souřadnice: $\quad x = r\cdot \sin u \cos v$

$\varphi: \quad y = r \sin u \sin v$

$z = r \cos u$

$\Omega:$ $r \in (0,1)$, $u \in (0,\pi)$, $v \in (0,2\pi)$

$\boxed{J = r^2 \sin u}$



výpočet: $\nabla\varphi = \begin{pmatrix} \sin u \cos v, & r\cos u \cos v, & -r \sin u \sin v \\ \sin u \sin v, & r \cos u \sin v, & r \sin u \cos v \\ \cos u, & -r \sin u, & 0 \end{pmatrix}$

$J\varphi = \cos u \cdot \begin{vmatrix} r\cos u \cos v, & -r \sin u \sin v \\ r \cos u \sin v, & r \sin u \cos v \end{vmatrix}$

$+ r \sin u \begin{vmatrix} \sin u \cos v, & -r \sin u \sin v \\ \sin u \sin v, & r \sin u \cos v \end{vmatrix}$

$= \cos u \cdot r^2 \cdot (\cos u \cdot \underline{\sin u}) + r \cdot \sin u \cdot (r \cdot \sin^2 u)$

$= r^2 \sin u$;

$r \in (0,1)$

$\int_\Theta 3(x^2+y^2+z^2)\, dx\,dy\,dz = \int_\Omega 3r^2 \cdot r^2 \sin u \; du\,dv\,dr$

$v \in (0,2\pi)$

$= 3\int_0^1 \left(\int_0^\pi \left(\int_0^{2\pi} r^4 \sin u \, dv \right) du \right) dr = 3\cdot \int_0^1 r^4\, dr \cdot \int_0^\pi \sin u \, du \cdot 2\pi$

$= 3 \cdot \frac{1}{5} \cdot 2 \cdot 2\pi = \frac{12\pi}{5}$

(62) $\int_P \underset{\sim}{F}\cdot d\underset{\sim}{S} = \int_\Theta \operatorname{div}\underset{\sim}{F}\, d\lambda_3 = 3\lambda_3(\Theta)$

$\underset{\sim}{F} = (x-y+z,\ y-z+x,\ z-x+y)$

$\operatorname{div}\underset{\sim}{F} = 3.$

$\lambda_3(\Theta) = ?\ ;\quad \Theta = \{\,|x-y+z| + |y-z+x| + |z-x+y| < 1\,\}$

lineární zobrazení $L : \mathbb{R}^3 \to \mathbb{R}^3$

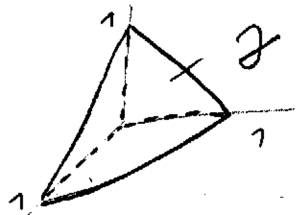
$(x,y,z) \longmapsto (x-y+z,\ y-z+x,\ z-x+y)$

matice: $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$

zjevně $L\Theta = \Omega$; $\quad \Omega \subset \mathbb{R}^3$

$\Omega = \{\,|u| + |v| + |w| < 1\,\}$

sestává z osmi jehlanů



$\lambda_3(\Omega) = 8\cdot\lambda_3(J)$

$= 8\cdot\frac{1}{3}\cdot\frac{1}{2}\cdot 1$ ← výška

↑ obsah podstavy

$= \frac{4}{3}.$

---

věta o substituci: $L\Theta = \Omega \longrightarrow \underbrace{\lambda_3(\Omega)}_{4/3} = \underbrace{|\det A|}_{4}\cdot\lambda_3(\Theta)$

---

celkem tedy: $\int_P \underset{\sim}{F}\cdot d\underset{\sim}{S} = \cdots = 3\cdot\lambda_3(\Theta) = \cancel{3}\cdot\frac{4}{\cancel{3}}\cdot\frac{1}{\cancel{4}} = 1.$

(63) $\int_P \underset{\sim}{F} \cdot dS = \int_K \operatorname{div} \underset{\sim}{F} \, d\lambda_3 ; \quad = 2 \cdot 3 \int_K x \, d\lambda_3 . = \underline{\underline{3}} .$

$\underset{\sim}{F} = (x^2, y^2, z^2)$ ↗ symetrie

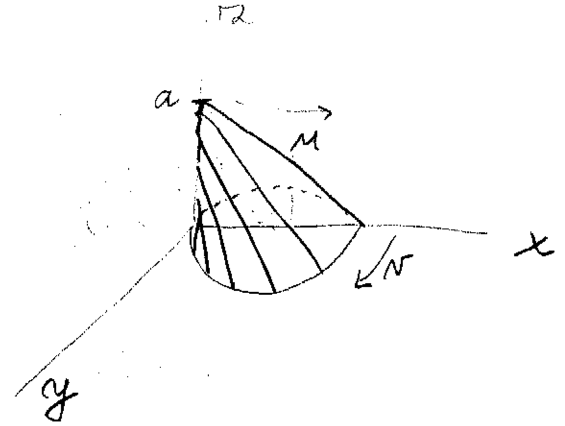
$\operatorname{div} \underset{\sim}{F} = 2(x+y+z)$ $\quad K;$

Fubini: $\int_K x \, dx dy dz = \int_0^1 \left(\int_0^1 \left(\underbrace{\int_0^1 x \, dx}_{\frac{1}{2}} \right) dy \right) dz = \frac{1}{2} .$

(65) $x = u \cdot \cos v$

$y = u \sin v$

$z = a \cdot \cos v - u$



$$\lambda_3(O) = \int_O \operatorname{div} \underset{\sim}{F}\, d\lambda_3 = \int_{\partial O} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_{P_1} + \int_{P_2}$$

podstava plášť

$\underset{\sim}{F} = (x, 0, 0)$

$\operatorname{div} \underset{\sim}{F} = 1$

(a) podstava: $\int = 0$; neboť $\underset{\sim}{F} \perp \underset{\sim}{n}$,

(b) plášť: $\varphi(u,v) = (u\cos v,\ u \sin v,\ a\cos v - u)$

$$v \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$
$$u \in (0, a\cdot\cos v) \quad : \Omega$$

$$\partial_u \varphi = (\cos v, \sin v, -1)$$

$$\partial_v \varphi = (-u \sin v, u\cos v, -a \sin v)$$

$$\partial_u\varphi \times \partial_v \varphi = (-a\sin^2 v + u\cos v, \ldots)$$

$$\int_{P_2} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \int_\Omega u\cos v \cdot (-a \sin^2 v + u \cos v)\, du\, dv$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{a\cos v} -a \cdot u \cos v \sin^2 v + u^2 \cos^2 v \, du \right) dv \qquad \|$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{a^3}{2} \cos^3 v \sin^2 v + \frac{a^3}{3} \cos^5 v \, dv =$$

subst. $t = \sin v$
$dt = \cos v\, dv$
$t \in (-1, 1)$, ..

H1 $\underset{\sim}{F} = (y, x) = \nabla U$ $\qquad \int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = U(2,3) - U(-1,2)$

$U = xy$ $\qquad = 6 - (-2) = \underline{8}$.

H2 $\underset{\sim}{F} = \left(\frac{y}{x^2}, -\frac{1}{x}\right) = \nabla\left(-\frac{y}{x}\right)$

přímo $U(1,2) - U(2,1) = -\frac{2}{1} - \left(-\frac{1}{2}\right) = \underline{-\frac{3}{2}}$

H3 $\underset{\sim}{F} = \left(\sin\frac{y}{x} - \frac{y}{x}\cos\frac{y}{x}, \cos\frac{y}{x}\right)$

$= \nabla U$; kde $U = x \cdot \sin\frac{y}{x}$

přímo: $U(2,\pi) - U(1,\pi) = 2 \cdot \sin\frac{\pi}{2} - 1 \cdot \sin\pi = \underline{2}$

---

pomocí vzorečku: $\int_\gamma \underset{\sim}{F} \cdot d\underset{\sim}{s} = U(k.b.\gamma) - U(z.b.\gamma)$;

pokud $\nabla U = \underset{\sim}{F}$ na okolí γ.