② $\sum \sin\left(\frac{x}{2^k}\right)$

... nekonv. stejn. v $\mathbb{R}$; neboť $f_k(x) \not\rightrightarrows 0$.

$$\sigma_k = \sup_{x\in\mathbb{R}} \left|\sin\left(\frac{x}{2^k}\right)\right| = 1 \quad \left(\text{volme } x = \frac{2^k\pi}{2}\right).$$

.. konv. abs. stejn. v $[-\Pi,\Pi]$; $\Pi>0$ pevné:

Weierstrass: $|f_k(x)| \le \left|\frac{x}{2^k}\right| \le \frac{\Pi}{2^k} \quad \forall k\ \forall x\in[-\Pi,\Pi]$

$\sum \frac{\Pi}{2^k}$ konv. (geom. řada; $q=\frac{1}{2}$)

⑥ $\sum (x^2+k^2)^{-\frac{1}{2}} \cos\left(\frac{2k\pi}{3}\right)$ .. konv. stejn. v $\mathbb{R}$.

<u>Dirichlet</u>: $\frac{1}{(x^2+k^2)^{\frac{1}{2}}} \rightrightarrows 0$; $\forall x$ pevné: klesá v $\underline{k}$.

$$\sigma_k = \sup_{x\in\mathbb{R}} \left|\frac{1}{(x^2+k^2)^{\frac{1}{2}}}\right| = \frac{1}{k}; \text{ tedy } \sigma_k \to 0.$$

$\sum \cos\left(\frac{2k\pi}{3}\right)$ .. omezené částečné součty

(stejnoměrně – nezávisí na $x$.)

⑧ $\sum (-1)^k (1-x)\, x^k$

$f_k(x) = (-x)^k (1-x)$;

konv. stejn. (abs.) v $[-\eta,\eta]$; $\eta\in(0,1)$ pevné:

Weierstrass: $|f_k(x)| \le |x|^k (1+|x|) \le \eta^k\cdot 2 \quad \forall k, \forall |x|\le\eta$

$\sum \eta^k$ konv. (geom. řada; $q=\eta<1$).

$x \in (-1,1]$ : $\quad s_n(x) = \frac{1-x}{1+x} \cdot \left(1-(-x)^{n+1}\right) \to \frac{1-x}{1+x} =: s(x)$.

$\sum f_k(x)$ konv. stejn. $\iff$ $s_n \rightrightarrows s$ v (daném $I$).

---

$I = (-1,1]$ : $\quad \sigma_n = \sup_{x \in I} |s_n(x) - s(x)| = \sup_{x\in I} \left| \frac{1-x}{1+x} (-x)^{n+1} \right|$

NE!!

$$= \sup_{x \in I} \underbrace{\left( \frac{1-x}{1+x} |x|^{n+1} \right)}_{g_n(x)}.$$

$$\sigma_n \geq \lim_{x \to -1+} g_n(x) = \frac{2}{0+} \cdot 1 = +\infty$$

---

$I = [0,1]$ : ANO : $\quad \sigma_n = \sup_{x \in [0,1]} \frac{(1-x)}{1+x} x^{n+1} \leq \sup_{x \in [0,1]} (1-x)x^n$

$\sigma_n \to 0$ (viz A3 – posloupnosti)

---

?? abs. stejn. v $[0,1]$ : NE.

$\sum |f_k(x)| = \sum (1-x)x^k$ ;

$$\sigma_n(x) = 1 - x^{n+1} \to \sigma(x) = \begin{cases} 1; & x \in [0,1) \\ 0; & x = 1 \end{cases}.$$

$\left.\begin{array}{l} |f_k(x)| \text{ spojité} \\ \sigma_n(x) \text{ nespojité} \end{array}\right\} \Rightarrow$ nekonv. stejn.

(9) $\sum \sin^2 x \cos^k x$ ;

$$s_n(t) = \sin^2 x \cdot \sum_{k=0}^{n} (\cos x)^k = \begin{cases} 0; & x = m\pi \\ \sin^2 x \cdot \dfrac{1-(\cos x)^{n+1}}{1-\cos x} & x \neq m\pi \end{cases}$$

leč: $\sin^2 x = 1-\cos^2 x = (1-\cos x)(1+\cos x)$.

---

$$s_n(t) = (1+\cos x)\left(1-(\cos x)^{n+1}\right) \to 1+\cos x \ ; \quad x \neq m\pi$$

$$0 \quad \to 0 \quad ; x \neq m\pi.$$

---

stačí vyšetřit na $[0, 2\pi]$.

$f_k(t)$ .. spojité

$s(t)$ .. nespojité v $0, 2\pi$ (zleva i zprava)

$\Rightarrow$ nekonv. stejn. na $[0,\delta]$ ani $[2\pi-\delta]$.

---

?? konv. stejn. na $[\delta, 2\pi-\delta]$; $\delta>0$ pevné:

ANO:

$$\sigma_n = \sup_{x\in I} |s_n(t) - s(t)| = \sup_{x\in I} \Big| \underbrace{(1+\cos x)(\cos x)^{n+1}}_{g_n(x)} \Big|$$

$I$ .. om. uzavř.; $g_n \in C^\infty$

$\Rightarrow \exists$ bod. maxima $\begin{cases} \text{krajní bod} \\ g_n'(x) = 0 \end{cases}$

$$g_n'(x) = \left(\cos^{n+1} x + \cos^{n+2} x\right)' = \left((n+1)\cos^n x + (n+2)\cos^{n+1} x\right)(-\sin x)$$
$$= -\cos^n x \cdot \sin x \left((n+1) + (n+2)\cos x\right).$$

$g_n'(x) = 0 \iff \sin x = 0 \;:\; x = \pi \;:\; g_n(\pi) = 0$

$x \in (\delta, 2\pi - \delta)$ $\quad \cos x = 0 \;:\; x = \frac{\pi}{2}, \frac{3\pi}{2} \;:\; g_n\left(\frac{\pi}{2} + k\pi\right) = 0$

$\cos x = -\frac{n+1}{n+2} \;:\; x = \arccos\left(-\frac{n+1}{n+2}\right) =: a_n$

nebo

$x = 2\pi - \arccos\left(-\frac{n+1}{n+2}\right) = b_n$

$$|g_n(a_n)| = |g_n(b_n)| = \Big|\underbrace{\left(1 - \frac{n+1}{n+2}\right)}_{\to 0} \cdot \underbrace{(\cos a_n)^{n+1}}_{\text{omezené.}}\Big| \to 0.$$

celkem: $\sigma_n = \max\limits_{x \in I} |g_n(x)| = \max\{|g_n(\delta)|, |g_n(a_n)|\}$

$\to 0;\ n \to +\infty.$ O.K.

(11) $\sum \exp(-kx)\sin(kx^2)$ ; $x \geq 0$

konv. abs. stejn. pro $x \in [\delta, +\infty)$; $\delta > 0$ pevné:

~ Weierstrass: $|f_k(x)| = |\underbrace{\exp(-kx)}_{\leq \exp(-k\delta)} \cdot \underbrace{\sin kx^2}_{\leq 1}| \leq \exp(-k\delta)$, $\forall k$ $\forall x \in [\delta, +\infty)$.

$\sum \exp(-k\delta)$ konv. (geom. řada $q = e^{-\delta} < 1$).

---

v okolí 0: trik $f_k(x) = \mathrm{Im}\left(\exp[k(ix^2 - x)]\right) = \mathrm{Im}\,[Q(x)]^k$

$$Q(x) = \exp\left(ix^2 - x\right).$$

$x > 0$: $|Q(x)| < 1$ .. $s_n(x) \to s(x) = \mathrm{Im}\left(\frac{1}{1-Q(x)}\right)$

$x = 0$: $s_n(0) = 0$ $\forall n$ ; $s(0) = 0$.

---

ukážeme: $\lim_{x \to 0+} s(x) \neq 0$ ; nespojitá v 0 zprava;

$s_n(x)$ -- spojité: $\nexists$ v $(0, \delta)$.

---

$$s(x) = \mathrm{Im}\left(\frac{1}{1-Q(x)}\right) = \mathrm{Im}\left(\frac{1}{1 - e^{-x}\cos x^2 + i\left(e^{-x}\sin x^2\right)}\right)$$

$$= \frac{e^{-x}\sin x^2}{(1 - e^{-x}\cos x)^2 + (e^{-x}\sin x^2)^2} = \frac{\sin x^2}{e^{x} + e^{-x} - 2\cos x}$$

$2\times$ l'Hosp. (například) $\quad \lim_{x \to 0+} s(x) = \frac{1}{2} \neq 0.$

(12) $\sum \frac{kx}{x^2+k^2} \cdot \operatorname{arctg}\left(\frac{x}{k}\right)$

NEKONV. STEJN. v $\mathbb{R}$: $f_k \not\rightrightarrows 0$ v $\mathbb{R}$

$$\sigma_k = \sup_{x\in\mathbb{R}} |f_k(x)| \geq f_k(k) = \frac{k^2}{k^2+k^2} \operatorname{arctg}(1) = \frac{\operatorname{arctg}(1)}{2}.$$

: konv. abs. stejn. v $[-M,M]$ $\forall M>0$ pevné

Weierstrass: $|f_k(x)| \leq \frac{k|x|}{x^2+k^2} \operatorname{arctg}\left(\frac{|x|}{k}\right)$ $\forall x\in[-M,M]$ $\forall k$.

$$\leq \frac{kM}{k^2} \cdot \frac{M}{k} = \frac{M^2}{k^2}; \quad \sum \frac{M}{k^2} \text{ konv.}$$

užili jsme odhad:

$|\operatorname{arctg} y| \leq \operatorname{arctg}|y| \leq |y|$ $\forall y\in\mathbb{R}$.