14. Funkce více proměnných

$f: \mathbb{R}^N \to \mathbb{R}^M$ $\quad \underset{\sim}{x} \in \mathbb{R}^N$ $\quad \forall i$

$\underset{\sim}{x} = (x_1, \dots x_N)$ $\quad |x_i| \le \|\underset{\sim}{x}\| \le |x_1| + \dots + |x_N|$

$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \sum_{i=1}^{N} x_i y_i$

norma $\|\underset{\sim}{x}\| = \sqrt{\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle} = (x_1^2 + x_2^2 + \dots + x_N^2)^{\frac{1}{2}}$

Cauchy–Schwarz $|\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle| \le \|\underset{\sim}{x}\| \cdot \|\underset{\sim}{y}\|$

$\mathbb{R}^N$ s metrikou: $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \|\underset{\sim}{x} - \underset{\sim}{y}\|$.
(spojitost, limita, okolí atd...)
(Kap 13).

$f: A \to \mathbb{R}^M$ je spojitá $(\forall \underset{\sim}{a} \in A)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall \underset{\sim}{x} \in A)$
$A \subset \mathbb{R}^N$
$[\|\underset{\sim}{x} - \underset{\sim}{x}_0\| < \delta \Rightarrow \|f(\underset{\sim}{x}) - f(\underset{\sim}{x}_0)\| < \varepsilon]$.

Příkl. ① lineární zobrazení $A: \mathbb{R}^N \to \mathbb{R}^M$ je vždy spojité.

$$\underset{\sim}{x} \longmapsto \underline{A}\underset{\sim}{x}^T = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{pmatrix}$$

matice $M \times N$.

$M \times N$ matice

$$\underline{A}\underset{\sim}{x}^T = \begin{pmatrix} \langle \underset{\sim}{\alpha}_1, \underset{\sim}{x} \rangle \\ \vdots \\ \langle \underset{\sim}{\alpha}_M, \underset{\sim}{x} \rangle \end{pmatrix} = y = (y_1, \dots y_M).$$

$y_j = \langle \underset{\sim}{\alpha}_j, \underset{\sim}{x} \rangle$

$|y_j| \le \|\underset{\sim}{\alpha}_j\| \cdot \|\underset{\sim}{x}\|$.

$\|y\| \le M \cdot |y_j| \le K \cdot \|\underset{\sim}{x}\|$

A je lipsch. zobraz. A. $\Rightarrow$ spojité!

② polynom $p(\underline{x}): \mathbb{R}^N \to \mathbb{R}$ je spojitý.

projekce $\pi_i : \underline{x} \longmapsto x_i$ spojité (<u>lineární</u>)

obecný polynom: součet, součin a násobení $\pi_i(\underline{x})$
(spojité dle V. 13.).

$$p(x,y) = x^2 - xy + y^2$$
$$\mathbb{R}^2 \longrightarrow \mathbb{R}.$$

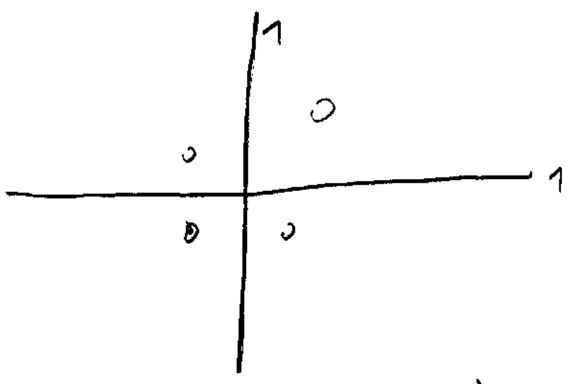
Def.: $f: U(\underline{a}) \to \mathbb{R}$; $\underline{a} \in \mathbb{R}^N$. Parciální derivací f v bodě $\underline{a}$ dle x_i rozumíme
$$\frac{\partial f}{\partial x_i}(a) = \lim_{t\to 0} \frac{1}{t}\left[f(\underline{a}+t\underline{e}_i) - f(\underline{a})\right]$$
$$\underline{e}_i = (0, \dots 1, 0 \dots)$$
↑ i-té pozice.

Obecněji, derivací ve směru $\underline{v} \in \mathbb{R}^N$ ($\underline{v} \neq \underline{0}$) rozumíme
$$\frac{\partial f}{\partial \underline{v}}(\underline{a}) = \lim_{t\to 0} \frac{1}{t}\left[f(\underline{a}+t\underline{v}) - f(\underline{a})\right].$$
$$= \varphi'(0);\ \varphi(t) = f(\underline{a}+t\underline{v}).$$

Pozn.: • parc. derivace: derivuj dle x_i, ostatní složky pevné

• $\frac{\partial f}{\partial x_i}(\underline{a}) = \frac{\partial f}{\partial \underline{e}_i}(\underline{a})$.

zcela nedostatečný pojem:
$$f(x,y) = \begin{cases} 1; & xy = 0 \\ 0; & xy \neq 0 \end{cases}$$



$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{1}{t}\left(f(t,0) - f(0,0)\right)$$
$$= \lim_{t\to 0} \frac{0}{t} \equiv 0 \text{ na } P(0,\delta).$$
$$= \lim_{t\to 0} 0 = 0.$$

f nespojitá v bodě $(0,0)$.
Heine:

• $\exists$ fce nespojitá v $(0,0)$; ačkoliv $\frac{\partial f}{\partial \tilde{v}}(0,0)=0 \quad \forall \tilde{v} \neq 0$.

Def.: Gradientem $\tilde{f}: R^N \to R^M$ rozumíme matici $M \times N$.

$$\nabla \tilde{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}, & \dots & \frac{\partial f_1}{\partial x_N} \\ \frac{\partial f_2}{\partial x_1} & \dots & \frac{\partial f_2}{\partial x_N} \\ \frac{\partial f_M}{\partial x_1}, & \dots & \frac{\partial f_M}{\partial x_N} \end{pmatrix} \qquad \tilde{f} = (f_1, \dots f_M), \quad f_j = f_j(x_1, \dots x_N)$$

Def.: Totálním diferenciálem funkce $\tilde{f}: R^N \to R^M$ v bodě $\tilde{a}$ rozumíme lineární zobrazení $L: R^N \to R^M$, splňující

$$\lim_{\tilde{h} \to \tilde{0}} \frac{1}{\|\tilde{h}\|} \left[\tilde{f}(\tilde{a}+\tilde{h}) - \tilde{f}(\tilde{a}) - L(\tilde{h}) \right] = 0.$$

Značíme $L = d\tilde{f}(\tilde{a})$.

Pozn.: • ekvivalentně $f(\tilde{a}+\tilde{h}) = f(\tilde{a}) + L(\tilde{h}) + r(\tilde{h})$;

kde $r(\tilde{h}) = o(\|\tilde{h}\|)$; $\tilde{h} \to \tilde{0}$.

dobrá lineární aproximace — tj. $\frac{r(\tilde{h})}{\|\tilde{h}\|} \to 0$.

• $f: R \to R$: $f'(a) = A \Leftrightarrow \lim_{t \to 0} \frac{f(a+t) - f(t) - At}{t} = 0$

Věta 14.1. Nechť $f: \mathbb{R}^n \to \mathbb{R}^m$ má v bodě $\underset{\sim}{a}$ totální diferenciál.

Potom 1. f je spojité v $\underset{\sim}{a}$

2. $\exists \frac{\partial f}{\partial v}(\underset{\sim}{a})$ pro $\forall$ $\underset{\sim}{v} \in \mathbb{R}^n$ nenulové. a rovná se $[df(\underset{\sim}{a})]\underset{\sim}{v}$.

---

dk.: 1. spojití $\Leftrightarrow \lim_{\underset{\sim}{x} \to \underset{\sim}{a}} f(\underset{\sim}{x}) = f(\underset{\sim}{a})$ Věta

$\Leftrightarrow \lim_{\underset{\sim}{h} \to \underset{\sim}{o}} f(\underset{\sim}{a}+\underset{\sim}{h}) = f(\underset{\sim}{a})$.

$f(\underset{\sim}{a}+\underset{\sim}{h}) = f(\underset{\sim}{a}) + L(\underset{\sim}{h}) + R(\underset{\sim}{h})$; $\underset{\sim}{h} \to \underset{\sim}{o}$

$\downarrow$

$L(\underset{\sim}{o}) = 0$

lineární spojité

$R(\underset{\sim}{h}) = \underbrace{\frac{R(\underset{\sim}{h})}{\|\underset{\sim}{h}\|}}_{\to 0} \cdot \underbrace{\|\underset{\sim}{h}\|}_{\to 0} \to 0.$

2. $\frac{1}{t}[f(\underset{\sim}{a}+t\underset{\sim}{v}) - f(\underset{\sim}{a})]$

$= f(\underset{\sim}{a}+t\underset{\sim}{v}) = f(\underset{\sim}{a}) + L(t\underset{\sim}{v}) + R(t\underset{\sim}{v})$;

$\frac{1}{t}[f(\underset{\sim}{a}+t\underset{\sim}{v}) - f(\underset{\sim}{a})] = \frac{1}{t}[L(t\underset{\sim}{v}) + R(t\underset{\sim}{v})]$

$= L(\underset{\sim}{v}) + \underbrace{\frac{R(t\underset{\sim}{v})}{\|t\underset{\sim}{v}\|}} \underbrace{\frac{\|t\underset{\sim}{v}\|}{t}}$

$t \to 0$??

$\to 0$ $\underbrace{\frac{|t|}{t}\|v\| = \pm\|v\|}$

vlastnost omezeného spojitého   omezené

Důsledek: $df(a) = \nabla f(a)$. ve smyslu reprezentace...

$$[df(a)]v = Av^T; \quad \Big/ \; v = e_i = (0, \dots 1, \dots)$$

$$[df(a)]e_i = \frac{\partial f}{\partial x_i}(a)$$

↑ i-tý sloupec matice $\nabla f(a)$

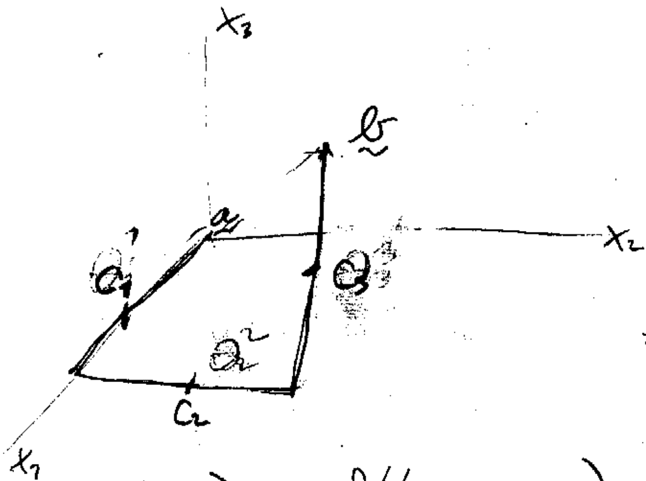
Věta 14.2. $f: \mathbb{R}^N \to \mathbb{R}^M$, $a \in \mathbb{R}^N$

1. Jsou-li $\frac{\partial f}{\partial x_i}$ omezené na $U(a,\delta)$, je f v a spojité.
2. Jsou-li $\frac{\partial f}{\partial x_i}$ spojité v bodě a, má zde f totální dif.

(a platí $df(a) = \nabla f(a)$.)

Dk.: $N=3$; $M=1$..

$b \neq a$



$$\begin{aligned} f(b) - f(a) &= f(b_1, b_2, b_3) - f(a_1, a_2, a_3) \\ &= f(b_1, a_2, a_3) - f(a_1, a_2, a_3) \\ &+ f(b_1, b_2, a_3) - f(b_1, a_2, a_3) \\ &+ f(b_1, b_2, b_3) - f(b_1, b_2, a_3) \\ &= P_1 + P_2 + P_3. \end{aligned}$$

$$\varphi(t) = f(t, a_2, a_3); \; t \in [a_1, b_1]$$

Lagrange: $\varphi(b_1) - \varphi(a_1) = \varphi'(\theta_1)(b_1 - a_1)$; $\theta_1 \in (a_1, b_1)$

$$P_1 = \frac{\partial f}{\partial x_1}\underbrace{(\theta_1, a_2, a_3)}_{\theta^1 \in \mathbb{R}^3} \cdot (b_1 - a_1).$$

analogicky: $P_2 = \frac{\partial f}{\partial x_2}(\theta^2)\,(b_2 - a_2) \qquad \theta^2 = (b_1, c_2, a_3)$

$P_3 = \frac{\partial f}{\partial x_3}(\theta^3)\,(b_3 - a_3) \qquad \theta^3 = (b_1, b_2, c_3)$

---

zřejmě: $\theta_i, \theta^i \in \mathcal{P}(\underset{\sim}{a}, \|\underset{\sim}{b} - \underset{\sim}{a}\|)$.

---

ad 1.: $\underset{\sim}{b} \to \underset{\sim}{a}$:

$$|f(\underset{\sim}{b}) - f(\underset{\sim}{a})| \leq \Big|\sum_{i=1}^{3} \underbrace{\frac{\partial f}{\partial x_i}(\theta_i)}_{\leq K \text{ předpoklad}}\,\underbrace{(b_i - a_i)}_{\leq \|\underset{\sim}{b} - \underset{\sim}{a}\|}\Big|$$

$$\leq 3K\|\underset{\sim}{b} - \underset{\sim}{a}\| \to 0;\ \underset{\sim}{b} \to \underset{\sim}{a}.$$

ad 2.: $\underset{\sim}{b} = \underset{\sim}{a} + \underset{\sim}{h}$; $\underset{\sim}{h} \neq 0 \quad h = (h_1, h_2, h_3)$

$$f(\underset{\sim}{a} + \underset{\sim}{h}) - f(\underset{\sim}{a}) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\theta_i)\cdot h_i$$

$\uparrow \pm \frac{\partial f}{\partial x_i}(\underset{\sim}{a})$

$$= \underbrace{\sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\underset{\sim}{a})\, h_i}_{Df(\underset{\sim}{a})\underset{\sim}{h}} + \underbrace{\sum_{i=1}^{3}\Big[\frac{\partial f}{\partial x_i}(\underset{\sim}{\theta_i}) - \frac{\partial f}{\partial x_i}(\underset{\sim}{a})\Big] h_i}_{r_a(\underset{\sim}{h})}.$$

↑ hledaný tvar tot. dif.

? $\frac{r(\underset{\sim}{h})}{\|\underset{\sim}{h}\|} \to 0$; $\underset{\sim}{h} \to \underset{\sim}{o}$.

$$\frac{r(h)}{\|h\|} = \sum_{i=1}^{3} \underbrace{\left[\frac{\partial f}{\partial x_i}(\underset{\sim}{\theta}^i) - \frac{\partial f}{\partial x_i}(\underset{\sim}{a})\right]}_{\to 0;} \underbrace{\frac{h_i}{\|\underset{\sim}{h}\|}}_{\leq 1} \to 0.$$

$\underset{\sim}{\theta}^i \to \underset{\sim}{a}$ pro $\underset{\sim}{h} \to \underset{\sim}{o}$

$\frac{\partial f}{\partial x_i}$ spojité.

Příkl.: $f(x,y) = \frac{x}{y}$; $A = $

$df(1,2) : (h_1, h_2) \mapsto \frac{h_1}{2} - \frac{h_2}{4}$

$\frac{\partial f}{\partial \underset{\sim}{v}}(1,2) = -\frac{9}{4}$; $\underset{\sim}{v} = (-3, 3)$

$\underset{\sim}{f}(\underset{\sim}{x}): \mathbb{R}^N \to \mathbb{R}^M$

totální diferenciál v bodě $\underset{\sim}{a} \in \mathbb{R}^N$

= lineární zobrazení: $L: \mathbb{R}^N \to \mathbb{R}^M$

$$\underset{\sim}{f}(\underset{\sim}{a}+\underset{\sim}{h}) = \underset{\sim}{f}(\underset{\sim}{a}) + L(\underset{\sim}{h}) + \underset{\sim}{\omega}(\underset{\sim}{h});$$

kde $\frac{\underset{\sim}{\omega}(\underset{\sim}{h})}{\|\underset{\sim}{h}\|} \to \underset{\sim}{0}$ pro $\underset{\sim}{h} \to \underset{\sim}{0}$.

$L := d\underset{\sim}{f}(\underset{\sim}{a})$.

<u>V. 14.2.:</u> $\frac{\partial f_i}{\partial x_j}$ spojité v bodě $\underset{\sim}{a}$ $\Rightarrow$ $\exists\, d\underset{\sim}{f}(\underset{\sim}{a})$ a je reprezentován

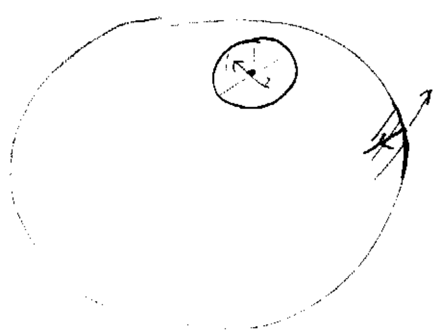
$$D\underset{\sim}{f}(\underset{\sim}{a}) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}.$$

<u>Def.:</u> $\Omega \subset \mathbb{R}^N$ <u>otevř.</u>; $\underset{\sim}{f}: \Omega \to \mathbb{R}^M$

$\underset{\sim}{f} \in C(\Omega)$ ... $\underset{\sim}{f}$ je spojité v Ω ($\Leftrightarrow f_i$ spojité pro $\forall i$)

$\underset{\sim}{f} \in C^1(\Omega)$ ... $f_i, \frac{\partial f_i}{\partial x_j}$ spojité v Ω pro $\forall i, j$.

otevřené?



$\underset{\sim}{a} \in \Omega \Rightarrow \mathcal{U}(\underset{\sim}{a}, \delta) \subset \Omega$ pro jisté $\delta > 0$

Věta 14.3:

1. $f: \mathbb{R}^N \to \mathbb{R}^n$, $g: \mathbb{R}^N \to \mathbb{R}^n$ mají s. d. v bodě $\underset{\sim}{a} \in \mathbb{R}^N$.

Potom $f+g$ má s. d. v bodě $\underset{\sim}{a}$ a platí $d(\underset{\sim}{f}+\underset{\sim}{g})(\underset{\sim}{a}) = d\underset{\sim}{f}(\underset{\sim}{a}) + d\underset{\sim}{g}(\underset{\sim}{a})$.

2. $\underset{\sim}{f}: \mathbb{R}^N \to \mathbb{R}^n$ má s. d. v bodě $\underset{\sim}{a} \in \mathbb{R}^N$,

$\underset{\sim}{g}: \mathbb{R}^n \to \mathbb{R}^k$ má s. d. v bodě $\underset{\sim}{b} = \underset{\sim}{f}(\underset{\sim}{a}) \in \mathbb{R}^n$,

Potom $g \circ f$ má s. d. v bodě $\underset{\sim}{a}$ a platí

$$d(g \circ f)(\underset{\sim}{a}) = d\underset{\sim}{g}(\underset{\sim}{b}) \circ d\underset{\sim}{f}(\underset{\sim}{a}).$$

Dk.: 2. $A = d\underset{\sim}{f}(\underset{\sim}{a})$ $\qquad r(0) = w(0) = 0.$

$B = d\underset{\sim}{f}(\underset{\sim}{b})$

$$f(a+h) = f(a) + Ah + r(h) \; ; \quad \frac{r(h)}{\|h\|} \to 0 \; ; \; h \to 0$$

$$g(b+k) = g(b) + Bk + w(k) \; ; \quad \frac{w(k)}{\|k\|} \to 0 \; ; \; k \to 0 .$$

$$g(f(a+h)) = g(\underbrace{f(a)}_{b} + \underbrace{Ah + r(h)}_{k})$$

$$= g(b) + B(Ah + r(h)) + w(Ah + r(h))$$

$$= g(f(a)) + B(Ah) + Br(h) + w(Ah + r(h)).$$

$$(g \circ f)(a+h) = (g \circ f)(a) + (B \circ A)(h) + \eta_1(h) + \eta_2(h) \; ;$$

$$? \quad \frac{\eta_j(h)}{\|h\|} \to 0 \; ; \; h \to 0.$$

B lineární (spojité). $\frac{B\, r(h)}{\|h\|} = B\left(\frac{r(h)}{\|h\|}\right) \to B(0) = 0$

$\frac{r(h)}{\|h\|} \to 0$ (předpoklad)

$$\frac{w(Ah + r(h))}{\|h\|} = \begin{cases} 0; & Ah + r(h) = 0 \\ \text{jinak} & \frac{w(Ah + r(h))}{\|Ah + r(h)\|} \cdot \frac{\|Ah + r(h)\|}{\|h\|}; \end{cases}$$

$\frac{w(Ah + r(h))}{\|Ah + r(h)\|} \to 0$, $\frac{\|Ah + r(h)\|}{\|h\|}$ omezené

$Ah + r(h) \to 0;\ h \to 0.$

~~$\left\|A\frac{h}{\|h\|} + \frac{r(h)}{\|h\|}\right\| \le \|A\|$~~

$\|Ah\| + \|r(h)\| \le K_A \|h\| + \|r(h)\|$

Důsledek:

$$\nabla(\underset{\sim}{g} \circ \underset{\sim}{f})(\underset{\sim}{a}) = \underset{K \times M}{(\nabla \underset{\sim}{g})(\underset{\sim}{f}(\underset{\sim}{a}))} \cdot \underset{M \times N}{\nabla \underset{\sim}{f}(\underset{\sim}{a})}$$

$K \times N$

$$\nabla F = \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j}$$

po složkách

$$\frac{\partial}{\partial x_j}\left(g_\ell(\underset{\sim}{f})\right)(\underset{\sim}{a}) = \sum_{i=1}^{M} \frac{\partial g_\ell}{\partial y_i}(\underset{\sim}{f}(\underset{\sim}{a})) \frac{\partial f_i}{\partial x_j}(\underset{\sim}{a})$$

„řetízkové pravidlo"

M členů:

$$\frac{d}{dx}\left[g(f(x))\right] = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}(x) \qquad M = 1$$

$f, g : \mathbb{R}^1 \to \mathbb{R}^1$ 1 člen

Příklad: záměna proměnných

$u = u(x,y)$; vyjádřete $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$

pomocí polárních souřadnic (r,φ)

$$x = r\cdot\cos\varphi$$
$$y = r\cdot\sin\varphi$$

$$\tilde{u}(r,\varphi) = u(r\cdot\cos\varphi, r\cdot\sin\varphi)$$

$$\frac{\partial \tilde{u}}{\partial r}(r,\varphi) = \frac{\partial u}{\partial x}(r\cos\varphi, r\sin\varphi)\cdot\cos\varphi + \frac{\partial u}{\partial y}(\ ,\ )\cdot\sin\varphi$$

$$\frac{\partial \tilde{u}}{\partial \varphi}(r,\varphi) = \frac{\partial u}{\partial x}(\ ,\ )(-r)\sin\varphi + \frac{\partial u}{\partial y}(\ ,\ )\cdot r\cos\varphi$$

$$\frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\cdot\cos\varphi + \frac{\partial u}{\partial y}\cdot\sin\varphi \qquad \Big/\ \cos\varphi \qquad \Big/\ \sin\varphi$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}(-r)\sin\varphi + \frac{\partial u}{\partial y}\cdot r\cos\varphi \qquad \Big/\ \frac{-1}{r}\sin\varphi \qquad \Big/\ \frac{1}{r}\cos\varphi$$

$$\frac{\partial \tilde{u}}{\partial r}\cos\varphi - \frac{1}{r}\frac{\partial \tilde{u}}{\partial \varphi}\sin\varphi = \frac{\partial u}{\partial x}$$

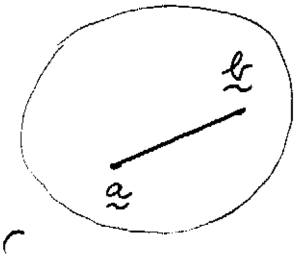
$$\frac{\partial \tilde{u}}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial \tilde{u}}{\partial \varphi}\cos\varphi = \frac{\partial u}{\partial y}$$

Def.: $a, b \in \mathbb{R}^N$:

otevřená úsečka $(a, b) = \{ a + t(b-a);\ t \in (0,1) \}$

uzavřená úsečka $[a, b] = \{ a + t(b-a);\ t \in [0,1] \}$.

Množina $\Omega \subset \mathbb{R}^n$ se nazve <u>konvexní</u>, jestliže $a, b \in \Omega$ implikuje $[a, b] \subset \Omega$.



Věta 14.4 Nechť $f \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^N$ je otevřená, konvexní.

Potom pro libovolné $a, b \in \Omega$ existuje $\theta \in (a, b)$ tak, že

$$f(b) - f(a) = \langle \nabla f(\theta), b - a \rangle = [\nabla f(\theta)](b-a)^T$$

Dk.: $\varphi(t) := a + t(b-a)$; $t \in [0,1]$

$g : [0,1] \to \mathbb{R}$

$g(t) = f(\varphi(t))$.

φ ~~spojité~~; C^1

$\nabla \varphi(t) = (b-a)^T$

f je C^1

$\Big\} \Rightarrow g \in C^1(0,1)$

$$g'(t) = \nabla f(\varphi(t))[\varphi'(t)]^T = \sum_{j=1}^{N} \frac{\partial f}{\partial x_j}(\varphi(t)) \frac{\partial \varphi_j}{\partial t}(t) = \langle \nabla f(\varphi(t)), b - a \rangle.$$

Lagrange: $g(1) - g(0) = g'(t_0) = \langle \nabla f(\underbrace{\varphi(t_0)}_{\theta}), b - a \rangle$

$\exists\, t_0 \in (0,1)$ $\qquad\qquad \theta$.

φ spojité v $[0,1]$; $\frac{\partial \varphi_j}{\partial t} = b_j - a_j$ spojité v $(0,1)$

$f \in C^1$ v Ω: $\quad g(t)$ spojité v $[0,1]$

$\qquad\qquad\quad g'(t)$ spojité v $(0,1)$

Def: [derivace vyšších řádů] $f(\underset{\sim}{x}) : \mathcal{U}(\underset{\sim}{a}) \to \mathbb{R}$; $\underset{\sim}{a} \in \mathbb{R}^N$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\ ) := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

obecně: $\dfrac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}(\ )$ ... definuji postupně $\dfrac{\partial}{\partial x_{i_k}}, \dfrac{\partial}{\partial x_{i_{k-1}}}, \dots \dfrac{\partial}{\partial x_{i_1}}$

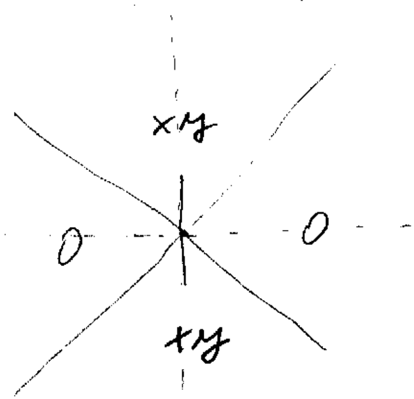
$i_1, i_2, \dots i_k \in \{1, \dots N\}$.

Příkl.: $f(x,y) = \dfrac{x}{y}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y^2}\right) = -\frac{1}{y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\right) = -\frac{1}{y^2}.$$

Protipříkl.:



$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{1}{x}\left[\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)\right]$$

$$= 0.$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{1}{y}\left[\underbrace{\frac{\partial f}{\partial x}(0,y)}_{y} - \underbrace{\frac{\partial f}{\partial x}(0,0)}_{0}\right] = 1.$$

Def: $f \in C^k(\Omega)$ ... $\dfrac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}$ spojité v Ω $\forall\, l = 0, 1, \dots k$, $i_1 \dots i_l$

---

spojité: $\lim\limits_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0.$

$|f(x,y)| \leq |xy|.$

Věta 14.5 [Záměnnost parc. derivací]

$f: \mathbb{R}^2 \to \mathbb{R}$; $f \in C^2(U)$; $U = \mathcal{U}(\underline{a}, \delta)$; $\underline{a} \in \mathbb{R}^2$.

Potom $\frac{\partial^2 f}{\partial x \partial y}(\underline{a}) = \frac{\partial^2 f}{\partial y \partial x}(\underline{a})$.

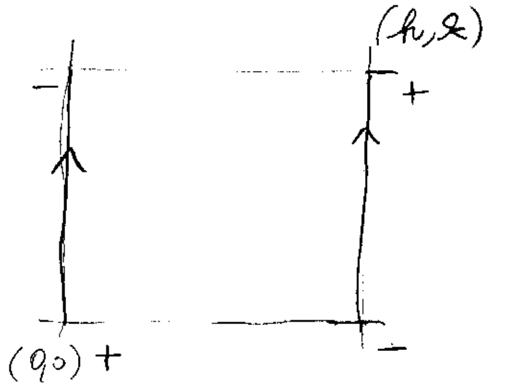
---

dk.: BÚNO $\underset{\sim}{a} = (0,0)$

$$Q(h,k) := \frac{1}{hk}\left[f(h,k) - f(h,0) - f(0,k) + f(0,0)\right]; \quad h, k > 0$$

$$\varphi_k(h) = \frac{1}{k}\left[f(h,k) - f(h,0)\right];$$

$$\psi_h(k) = \frac{1}{h}\left[f(h,k) - f(0,k)\right]$$



$$Q(h,k) = \frac{1}{h}\Bigg[\underbrace{\frac{f(h,k) - f(h,0)}{k}}_{\varphi_k(h)} - \underbrace{\frac{f(0,k) - f(0,0)}{k}}_{\varphi_k(0)}\Bigg]$$

$$= \frac{1}{h}\left[\varphi_k(h) - \varphi_k(0)\right] = \varphi_k'(a_1); \quad a_1 \in (0,h)$$

$$\varphi_k'(a_1) = \frac{\partial}{\partial h}\varphi_k(a_1) = \frac{1}{k}\left[\frac{\partial f}{\partial h}(a_1, \underset{\uparrow}{k}) - \frac{\partial f}{\partial h}(a_1, 0)\right]$$

Lagrange prom $\underline{k}$: $= \frac{\partial}{\partial k}\left(\frac{\partial f}{\partial h}\right)(a_1, b_1) = \frac{\partial^2 f}{\partial k \partial h}(a_1, b_1)$.

$$Q(h,k) = \frac{1}{k}\Bigg[\underbrace{\frac{f(h,k) - f(0,k)}{h}}_{\psi_h(k)} - \underbrace{\frac{f(h,0) - f(0,0)}{h}}_{\psi_h(0)}\Bigg]$$

$$= \frac{1}{k}\left[\psi_h(k) - \psi_h(0)\right] = \psi_h'(d_1)\,; \quad d_1 \in (0, k)$$

$$\psi_h'(c_1) = \frac{\partial}{\partial k}\psi_h(c_1) = \frac{1}{h}\left[\frac{\partial f}{\partial k}(h, d_1) - \frac{\partial f}{\partial k}(0, d_1)\right]$$

Lagrange prom: h

$$= \frac{\partial}{\partial h}\left(\frac{\partial f}{\partial k}\right)(c_1, d_1) = \frac{\partial^2 f}{\partial h \partial k}(c_1, d_1).$$

$$Q(h,k) = \frac{\partial^2 f}{\partial k \partial h}(a_1, b_1) = \frac{\partial^2 f}{\partial h \partial k}(c_1, d_1)$$

$(h, k) \to (0,0)$

$(a_1, b_1) \to (0,0)$

$(c_1, d_1) \to (0,0)$

$$\frac{\partial^2 f}{\partial k \partial h}(0,0) = \frac{\partial^2 f}{\partial h \partial k}(0,0).$$

Důsledek: $f \in C^k(\Omega)$; $\Omega \subset \mathbb{R}^N$ otevř.

$I = (i_1, \dots i_k) \in \{1, \dots N\}$

$J = (j_1, \dots j_k)$ libovolná permutace

$$\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{j_1} \dots \partial x_{j_k}} \quad \text{všude v } \Omega.$$

BÚNO: I se liší transpozicí sousedních indexů & předchozí věta.

$$\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} = \frac{\partial^{\ell}}{\partial x \dots}\left(\frac{\partial^2}{\partial x_i \partial x_j}\right)\frac{\partial^{\ell} f}{\partial x_i \dots \partial}$$

Pozn.: $V = V(x,y) \in C^1$

$\exists$ t.d. $dV(a)$ a je repr. maticí $\nabla V(a) = \left(\frac{\partial V}{\partial x}(a), \frac{\partial V}{\partial y}(a)\right)$.

(skal. součin)

$v \in \mathbb{R}^2$; $\|v\| = 1$

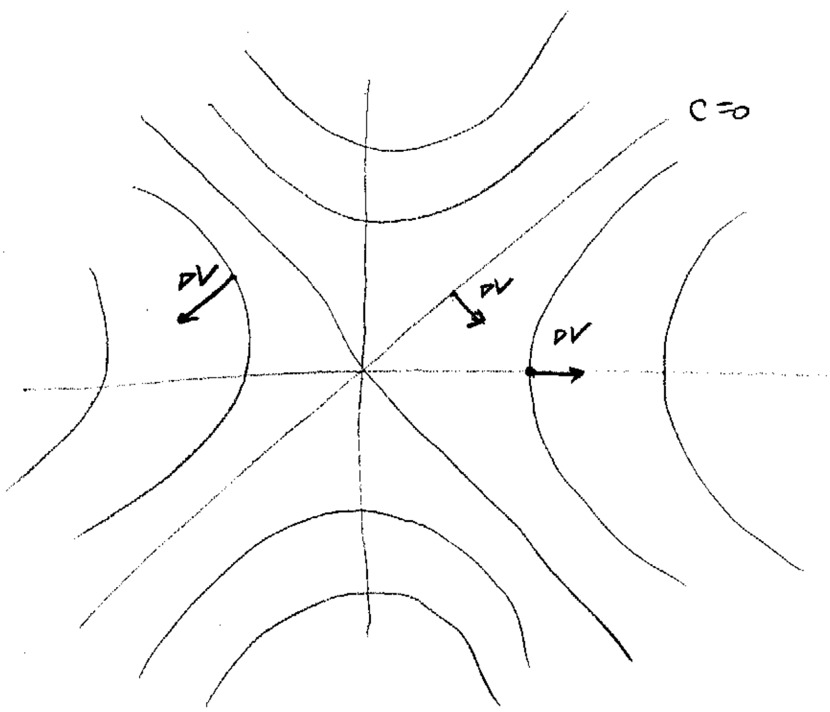
$$\frac{\partial V}{\partial v}(a) = [dV(a)](v) = \nabla V(a)\, v^T = \langle \nabla V(a), v \rangle$$

$= 0$; pokud $v \perp \nabla V(a)$ „vrstevnice"

max/min ; $v \parallel \nabla V(a)$ „spádnice"

Příkl.: $V(x,y) = x^2 - y^2$

$\nabla V = (2x, -2y)$



Def.: Rovnicí ve tvaru totálního diferenciálu rozumíme

(R) $M(x,y)dx + N(x,y)dy = 0$

$M, N: \Omega \to \mathbb{R}$; $\Omega \subset \mathbb{R}^2$ otevř. mn.

- řešení (R): křivky, jejichž tečna (dx, dy) je kolmá na (M, N)
- pokud $\exists V: \Omega \to \mathbb{R}$; $M = \frac{\partial V}{\partial x}$, $N = \frac{\partial V}{\partial y}$

pak tyto křivky jsou vrstevnicemi V

tj. jsou určeny relací $V(x,y) = c$.

(R) $dV = 0$; $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy$.

• $N \neq 0 \Rightarrow$ (R) je ekvivalentní $M(x,y) + N(x,y)\frac{dy}{dx} = 0$.

$$y' = -\frac{M(x,y)}{N(x,y)}. \quad (RO)$$

Def.: Rovnice (R) se nazve exaktní, jestliže

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \qquad (E)$$

Pozn.: • pokud $\exists V: \Omega \to \mathbb{R}$; $\nabla V = (M, N)$; a navíc $V \in C^2$

$\Rightarrow$ platí (E): $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right)$; $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$

viz dle V. 14.5.

• obráceně: (E) $\Rightarrow \exists$ (aspoň lokálně) V takové, že $\nabla V = (M, N)$.

Postup řešení: 1. platí-li (E), dopočteme V

2. neplatí-li (E), nalezneme vhodnou fci („integrační faktor"), aby (E) platila.

Lemma 14.1. $V = V(x,y)$ je C^1 na okolí $(x_0, y_0) \in \mathbb{R}^2$, $\nabla V = (M, N)$; $N(x_0, y_0) \neq 0$.

Potom pro fci $y(x) \in C^1(\mathcal{U}(x_0))$, splňující $y(x_0) = y_0$, je ekvivalentní:

(1) y řeší (RO) na nějakém okolí x_0.

(2) $V(x, y(x)) \equiv c$ na nějakém okolí x_0, kde $c = V(x_0, y_0)$

Dk.: $V(x, y(x)) \equiv c \Leftrightarrow \frac{d}{dx} V(x, y(x)) = 0$

$$\frac{\partial V}{\partial x}(x, y(x)) \cdot \frac{\partial x}{\partial x} + \frac{\partial V}{\partial y}(x, y(x)) \frac{\partial y(x)}{\partial x} = 0$$

BÚNO:

$$M(x, y(x)) + N(x, y(x))\, y'(x) = 0$$

$V(x_0, y(x_0)) = V(x_0, y_0) = c$

$$y'(x) = -\frac{M(x, y(x))}{N(x, y(x))}.$$

• $N(x, y(x)) \neq 0$ $x = x_0$

spojitost.

Taylor – 1. proměnná: $\varphi: \mathcal{U}(a) \to \mathbb{R}$; $a \in \mathbb{R}$; C^{m+1}.

$$\varphi(a+t) = \varphi(a) + \varphi'(a)t + \varphi''(a)\frac{t^2}{2} + \dots$$

$$= \sum_{k=0}^{m} \frac{\varphi^{(k)}(a)}{k!} t^k + R_{m+1}(t); \quad \text{kde } R_{m+1}(t) = \frac{\varphi^{(m+1)}(\theta)}{(m+1)!} t^{m+1}$$

$$\exists \theta \in (0,t).$$

<u>Věta 14.6</u>: $f(\underset{\sim}{x}): \mathcal{U}(\underset{\sim}{a}) \to \mathbb{R}$; C^3.

Pak pro $\forall \underset{\sim}{h} \in \mathcal{U}(\underset{\sim}{a})$ $\exists\, \underset{\sim}{\theta} \in (0, \underset{\sim}{h})$ tak, že

$$f(\underset{\sim}{h}) = f(\underset{\sim}{a}) + \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{a}) h_i + \frac{1}{2!}\sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underset{\sim}{a}) h_i h_j + R_{3,1}(\underset{\sim}{h}),$$

$$\text{kde } R_{3,1}(\underset{\sim}{h}) = \frac{1}{3!}\sum_{i,j,k=1}^{N} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\underset{\sim}{\theta}) h_i h_j h_k.$$

dk: $\varphi(t) := f(\underset{\sim}{a} + t\underset{\sim}{h})$,

$$\varphi(1) = \varphi(0) + \varphi'(0)1 + \frac{1}{2}\varphi''(0)1^2 + R_3$$

Taylor:

$$R_3 = \frac{1}{3!}\varphi'''(\theta)1^3$$

$$\varphi'(t) = \frac{\partial}{\partial t}\underbrace{f(\underset{\sim}{a} + t\underset{\sim}{h})}_{(f\circ\underset{\sim}{\psi})(t)} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{\psi}(t)) \frac{\partial \psi_i}{\partial t}(t) \quad = h_i$$

$\psi: t \mapsto \underset{\sim}{a} + t\underset{\sim}{h} \in \mathbb{R}^N$

$$\psi_i = a_i + t h_i$$

$$= \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{a} + t\underset{\sim}{h}) h_i;$$

tj.: 2. člen: $\varphi'(0)$.

$$\varphi''(t) = \frac{\partial}{\partial t} \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{x}(t))\, h_i = \sum_{i=1}^{N} \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x_i}(\underset{\sim}{x}(t))\right] h_i$$

$$= \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)}_{\frac{\partial^2 f}{\partial x_j \partial x_i}} (\underset{\sim}{x}(t)) \underbrace{\frac{\partial x}{\partial x_j}(t)}_{h_j} \right) h_i$$

obecně: $\varphi^{(m)}(t) = \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \cdots \sum_{i_m=1}^{N} \frac{\partial^m f}{\partial x_{i_1} \cdots \partial x_{i_m}} (\underset{\sim}{a} + t \underset{\sim}{h})\, h_{i_1} \cdots h_{i_m}$

Lepší zápis & opakování členů.

Def: Multiindex je N-tice $\alpha = (\alpha_1, \dots \alpha_N)$; $\alpha_i \geq 0$ celé

stupeň multiindexu $|\alpha| = \alpha_1 + \dots \alpha_N$

~~$f: \mathbb{R}^N \to \mathbb{R}$~~ $\quad D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_N}\right)^{\alpha_N}$

$\underset{\sim}{x} \in \mathbb{R}^N$: $\quad \underset{\sim}{x}^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$.

tvrdím: $\varphi^{(m)}(t) = \sum_{|\alpha| = m} \binom{m}{\alpha} D^{\alpha} f(\underset{\sim}{a} + t\underset{\sim}{h})\, \underset{\sim}{h}^{\alpha}$

$\binom{m}{\alpha} = \frac{m!}{\alpha_1! \cdots \alpha_N!}$ zobecněné kombinační číslo...

$m = |\alpha|$

„kolika způsoby rozdělím m prvků do N přihrádek, α_i – počet v i-té přihrádce."

bi-nomické věte: $(h_1+h_2)^n = \sum_{k=0}^{n} \binom{n}{k} h_1^k h_2^{n-k}$

$$= \sum_{|\alpha|=n} \binom{n}{\alpha} \underset{\sim}{h}^{\alpha}$$

$\underset{\sim}{h} = (h_1, h_2)$

$\underset{\sim}{\alpha} = (k, n-k)$;

$k = 0, \dots n.$

N-nomické věte:

$$(h_1 + \dots + h_N)^n = \sum_{|\alpha|=n} \binom{n}{\alpha} \underset{\sim}{h}^{\alpha} \qquad \alpha = (\alpha_1, \dots \alpha_N)$$

$(h_1 + \dots + h_N) \cdot \dots \cdot (h_1 + \dots + h_N) \dots N^n$ členů

$$= \sum_{i_1=1}^{N} \sum_{i_2=1}^{N} \dots \sum_{i_n=1}^{N} h_{i_1} \dots h_{i_n}$$

$$\varphi^{(n)}(t) = \left(\frac{\partial f}{\partial x_1} h_1 + \dots + \frac{\partial f}{\partial x_N} h_N \right)^n$$

formálně:

Pozn.: obecný polynom stupně n:

$$p(\underset{\sim}{x}) = \sum_{|\alpha| \le n} c_\alpha \underset{\sim}{x}^{\alpha}; \quad \text{přičemž } \exists \alpha;\ |\alpha| = n,\ c_\alpha \neq 0.$$

$$\sum_{|\alpha|=n} \binom{n}{\alpha} = N^n.$$

obecný tvar věty: $f \in C^{m+1}(\mathcal{U}(\underset{\sim}{a}))$;

$$f(\underset{\sim}{h}) = \sum_{m=0}^{m} \left(\sum_{|\alpha|=m} \frac{1}{m!} \binom{m}{\alpha} D^{\alpha} f(\underset{\sim}{a}) \underset{\sim}{h}^{\alpha} \right) + R_{m+1}(\underset{\sim}{h});$$

$$R_{m+1}(\underset{\sim}{h}) = \frac{1}{(m+1)!} \sum_{|\alpha|=m+1} \binom{m+1}{\alpha} D^{\alpha} f(\underset{\sim}{\theta}) \underset{\sim}{h}^{\alpha}$$

$$\underset{\sim}{\theta} \in (0; \underset{\sim}{h}).$$

$$m=1: \quad \sum_{|\alpha|=1} \frac{1}{1!} \binom{m}{\alpha} D^{\alpha} f(\underset{\sim}{a}) \underset{\sim}{h}^{\alpha} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{a}) h_i$$

$|\alpha|=1 \ldots (0, \ldots 1, \ldots)$ $\quad = \langle Df(\underset{\sim}{a}), \underset{\sim}{h} \rangle$

diferenciál 1. řádu. (lineární fce)

$$m=2: \quad \frac{1}{2!} \left(\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underset{\sim}{a}) h_i h_j \right)$$

diferenciál 2. řádu. (bilineární fce).

$$D^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{N} \quad \ldots \text{ Hessova matice}$$

(symetrická díky V.).

$$h \cdot D^2 f(\underline{a}) \cdot h^T = \langle \underset{\sim}{h}, D^2 f(\underline{a}) \cdot h^T \rangle$$

Diferenciál vyššího řádu:

$$g(t) = f(t\underset{\sim}{h} + \underset{\sim}{a}); \quad t \in [0,1] \qquad \varphi(t) = \underset{\sim}{a} + t\underset{\sim}{h}$$

$$g'(t) = \nabla f(t\underset{\sim}{h} + \underset{\sim}{a}) h^T = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\;) h_i \qquad \nabla\varphi = \underset{\sim}{h}^T$$

$$g''(t) = \frac{d}{dt}\left(\sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(t\underset{\sim}{h} + \underset{\sim}{a}) h_i\right)$$

$$\sum_{j=1}^{N}\sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(t\underset{\sim}{h} + \underset{\sim}{a}) h_i h_j$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial x_i}(t\underset{\sim}{h} + \underset{\sim}{a}) h_i\right) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(t\underset{\sim}{h} + \underset{\sim}{a}) h_i h_j$$

---

podobně $\displaystyle g'''(t) = \sum_{i,j,k=1}^{N} \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}(t\underset{\sim}{h} + \underset{\sim}{a}) h_i h_j h_k$.

Taylor: $g(1) = g(0) + g'(0) + \frac{1}{2} g''(0) + \frac{1}{3!} g'''(t_0); \quad t_0 \in (0,1)$

$$f(\underset{\sim}{a} + \underset{\sim}{h}) \stackrel{!}{=} f(\underset{\sim}{a}) + \underbrace{\sum_{i=1}^{N} \frac{\partial f}{\partial x_i}(\underset{\sim}{a}) h_i}_{\nabla f(\underset{\sim}{a}) h^T} + \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_j \partial x_i}(\underset{\sim}{a}) h_i h_j + R(\underset{\sim}{h})$$

$$R(\underset{\sim}{h}) = \frac{1}{3!}\sum_{i,j,k=1}^{N} \frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}(\underset{\sim}{a}) h_i h_j h_k$$

$$\nabla^2 f(\underset{\sim}{a}) = \left\{\frac{\partial^2 f}{\partial x_i \partial x_j}\right\}_{i,j=1}^{N}$$

$$h \, \nabla^2 f(\underset{\sim}{a}) h^T = \langle \nabla^2 f(\underset{\sim}{a}) h^T, h \rangle$$

$$(h_1, \dots h_N)\left(\nabla^2 f(\underset{\sim}{a})\right)\begin{pmatrix} h_1 \\ \vdots \\ h_N \end{pmatrix}.$$

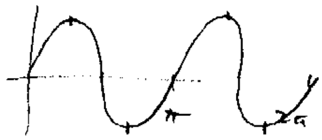
Def.: x_0 … stac. bod; pokud $\nabla f(x_0) = \underline{0}$.

V. 14.7: $x_0 \in \text{int}\, M$; x_0 extrém $\Rightarrow$ x_0 je stac. bod (pokud $\nabla f(x_0)$ $\exists$).

Příklad: $f(x,y) = xy$; $M = \{x^2+y^2 \le 1\}$.

$\nabla f = (0,0) \iff (x,y) = (0,0)$. int

hranice: parametrizuj...



---

postačující podmínky

R: $f(a+h) = f(a) + h f'(a) + \frac{1}{2} f''(a) h^2 + o(h^2)$

$=0$ (nutná podmínka)

rozhoduje znaménko f'':

Opakování: lin. algebra:

$A \in \mathbb{R}^{N\times N}$; symetrická.

Kvadratická forma, určená maticí A

$$Q(h) = \langle A\cdot h^T, h\rangle = h\cdot A\cdot h^T = \sum_{i,j=1}^{N} A_{ij} h_i h_j .$$

$(\cdots)\left(\quad\right)\left(\vdots\right)$

Forma se nazve:

1) pozitivně definitní $\iff$

2) negativně def. $\iff$

3) indefinitní $\iff$

Věta: 14.8. $f \in C^3(\mathcal{U}(\underline{a}))$; $\underline{a} \in \mathbb{R}^N$; nechť $\nabla f(\underline{a}) = \underline{0}$;

nechť $Q(h)$ je kvadr. forma určená maticí $\nabla^2 f(\underline{a})$.

Potom: 1) je-li $Q(h)$ poz. def $\Rightarrow$

2) neg. def.

3) ... ne $\Rightarrow$ není lok. extrém.

Důk.: Věta 14.6: $\underline{h} \in \mathcal{U}(\underline{a}, \delta_1)$

$$f(\underline{a}+\underline{h}) = f(\underline{a}) + \langle \nabla f(\underline{a}), \underline{h} \rangle + \frac{1}{2} Q(\underline{h}) + R_3(\underline{h}),$$

$$R_3(h) = \sum_{|\alpha|=3} \binom{3}{\alpha} D^\alpha f(\underline{\theta})\, \underline{h}^\alpha$$

$$\underline{\theta} \in (\underline{a}; \underline{a}+\underline{h})$$

spojitost $\Rightarrow$ $D^\alpha f$ omezené na $\mathcal{U}(a, \delta_1)$...

$$|\underline{h}^\alpha| = |h_1^{\alpha_1} \cdots h_N^{\alpha_N}| \le \|h\|^{\alpha_1 + \dots + \alpha_N} = \|h\|^{|\alpha|} = \|h\|^3$$

$$|h_i| \le \|\underline{h}\|$$

celkem: $|R_3(\underline{h})| \le C_3 \|\underline{h}\|^3$; $\underline{h} \in \mathcal{U}(\underline{a}, \delta)$.

1: $Q(h)$ je poz. def: $\exists C_1 > 0$; $Q(h) \ge C_1 \|h\|^2$

$$\begin{aligned} f(\underline{a}+\underline{h}) &= f(\underline{a}) + \tfrac{1}{2} Q(\underline{h}) + R_3(\underline{h}) \\ &\ge f(\underline{a}) + \tfrac{C_1}{2} \|h\|^2 - |R_3(\underline{h})| \\ &\ge f(\underline{a}) + \tfrac{C_1}{2} \|h\|^2 - C_3 \|\underline{h}\|^3 \\ &= f(\underline{a}) + \|\underline{h}\|^2 \underbrace{\left(\tfrac{C_1}{2} - C_3 \|\underline{h}\|\right)}_{>0 \text{ pro } \|\underline{h}\| < \frac{C_1}{2C_3} = \delta_2} \end{aligned}$$

$0 < \|\underline{h}\| < \min\{\delta_1, \delta_2\} \Rightarrow f(\underline{a}+h) > f(\underline{a})$..

2. analogicky: $Q(\underset{\sim}{h})$ neg. def. $\exists c_2>0;\ Q(\underset{\sim}{h}) \le -c_2 \|\underset{\sim}{h}\|^2$

$$f(\underset{\sim}{a}+\underset{\sim}{h}) = f(\underset{\sim}{a}) + \tfrac{1}{2}Q(\underset{\sim}{h}) + R_3(\underset{\sim}{h})$$

$$\le f(\underset{\sim}{a}) - \frac{c_2}{2}\|\underset{\sim}{h}\|^2 + c_3\|\underset{\sim}{h}\|^3$$

$$= f(\underset{\sim}{a}) + \|h\|^2\left(-\frac{c_2}{2} + c_3\|\underset{\sim}{h}\|\right)$$

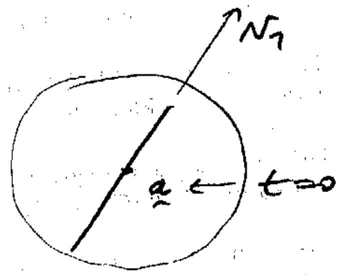
<0 pro: $\|\underset{\sim}{h}\| < \dfrac{c_2}{2c_3} = \delta_3$.

$0 < \|\underset{\sim}{h}\| < \min\{\delta_2, \delta_3\}$

3.: $Q(\underline{h})$ indefinitní: $\exists v_1 \neq 0;\ Q(v_1) > 0$

$\exists \hat{v} \neq 0;\ Q(\hat{v}) < 0$.

$$\varphi(t) = f(\underset{\sim}{a} + t\underset{\sim}{v_1})\ ;$$



V. 14.6

$$\varphi'(0) = \langle \underbrace{\nabla f(\underset{\sim}{a})}_{=0}, v_1\rangle = 0$$

$$\varphi''(0) = \frac{1}{2}\sum_{i,j=1}^{N} \frac{\partial^2 f(a)}{\partial x_i \partial x_j} v_i v_j = \varphi(\underset{\sim}{v}) > 0$$

– lokální minimum na přímce $\underset{\sim}{a} + t\underset{\sim}{v}$.

---

podobně:

$$\hat{\varphi}(t) = f(\underset{\sim}{a} + t\hat{v})\ ;$$

$$\hat{\varphi}'(0) = 0$$

$$\hat{\varphi}''(0) = \tfrac{1}{2}\varphi(\hat{\underset{\sim}{v}}) < 0 \text{ - lok. max}$$

„sedlový bod".

Věta 14.10. [Existence glob. extrémů]

(1) $f: M \to \mathbb{R}$ spojité; $M \subset \mathbb{R}^N$ omezené a uzavřené.

Pak f má v M globální maximum a minimum.

(2) $f: \mathbb{R}^N \to \mathbb{R}$ spojité; $\lim_{\|\underline{x}\| \to \infty} f(\underline{x}) = +\infty$.

Pak f má v $\mathbb{R}^N$ globální minimum.

(2') $f: \mathbb{R}^N \to \mathbb{R}$ spojité; $\lim_{\|\underline{x}\| \to \infty} f(\underline{x}) = -\infty$.

Pak f má v $\mathbb{R}^N$ glob. max.

(3) $f: \mathbb{R}^N \to \mathbb{R}$ spojité; $\lim_{\|\underline{x}\| \to \infty} f(\underline{x}) = 0$.

$f(\underline{a}) > 0$ pro nějaké $\underline{a} \Rightarrow f$ má glob. max.

$f(\underline{b}) < 0$ pro nějaké $\underline{b} \Rightarrow f$ má glob. min.

Dk: (1) $M \subset \mathbb{R}^N$ omez. & uzavř. $\Rightarrow M$ je kompaktní (V.13.13)

$\exists$ glob. extrémy (V.13.10)

(2) $\exists R > 0$; $\|f(\underline{x})\| > f(\underline{0})$ $\forall \|\underline{x}\| > R$.

$M := \{\underline{x} \in \mathbb{R}^N; \|\underline{x}\| \le R\}$

M omezené

uzavřené:

$$M = \{\underline{x};\ \underbrace{x_1^2 + \dots + x_N^2}_{\varphi(\underline{x})} \le R^2\} = \varphi^{-1}((-\infty, R])$$

$\uparrow$ spojité fce $\qquad\qquad$ $\uparrow$ uzavřené množina

V.13: $\varphi: \mathbb{R}^N \to \mathbb{R}$ spojité

$I \subset \mathbb{R}$ uzavř. $\Rightarrow \varphi^{-1}(I) \subset \mathbb{R}^N$ uzavř.!

bod (1): $\exists\ \underline{a} \in M$; glob. min. f na M.

$f(\underset{\sim}{a}) \le f(\underset{\sim}{e}) < f(\underset{\sim}{x})$; $\underset{\sim}{x} \in \mathbb{R}^N \setminus \Pi$.

$\Rightarrow \underset{\sim}{a}$ je glob. min. na $\mathbb{R}^N$.

(2') analogicky

(3) $\exists \underset{\sim}{b}$; $f(\underset{\sim}{b}) < 0$. ... nebo $f(\underset{\sim}{x}) \to 0$; $\|\underset{\sim}{x}\| \to \infty$

$\exists R > 0$; $f(\underset{\sim}{x}) > f(\underset{\sim}{b})$ $\forall \|\underset{\sim}{x}\| > R$

$\Pi := \{\underset{\sim}{x}; \|\underset{\sim}{x}\| \le R\}$ kompakt.

$\exists \underset{\sim}{c} \in \Pi$ glob. min f na Π.

$\underset{\sim}{x} \in \mathbb{R}^N \setminus \Pi$: $f(\underset{\sim}{x}) > f(\underset{\sim}{b}) \ge f(\underset{\sim}{c})$

$\Rightarrow \underset{\sim}{c}$ je glob. min. na $\mathbb{R}^N$.

Pozn.: • $M \subset \mathbb{R}^N$ omezené: $\exists C > 0$; $\|\underset{\sim}{x}\| \le C$ $\forall \underset{\sim}{x} \in M$.

$x_1^2 + \dots + x_N^2 \le C^2$

$\Leftrightarrow \exists \tilde{C} > 0$; $|x_j| \le C$ $\forall j = 1, \dots, N$

$\forall \underset{\sim}{x} = (x_1, \dots, x_N) \in \Pi$.

• uzavřenost: $\varphi: \mathbb{R}^N \to \mathbb{R}$ spojité;

$\{\underset{\sim}{x} \in \mathbb{R}^N; \varphi(\underset{\sim}{x}) \le c\} = \varphi^{-1}((-\infty, c])$

$\{\underset{\sim}{x} \in \mathbb{R}^N; \varphi(\underset{\sim}{x}) \ge c\} = \varphi^{-1}([c; +\infty))$

vzory uzavř. množin.

heslo: spojité fce & neostré nerovnost

$\Rightarrow$ uz. množiny.

• $\nexists$ extr.

V. 14.12: $F(\underset{\sim}{x}, y) : \mathbb{R}^{N+1} \to \mathbb{R}$; $(\underset{\sim}{a}, b) \in \mathbb{R}^{N+1}$

Nechť: $F(\underset{\sim}{a}, b) = 0$

F je C^1 na okolí $(\underset{\sim}{a}, b)$

klíčový předp.: $\frac{\partial F}{\partial y}(\underset{\sim}{a}, b) \neq 0$

$\Rightarrow \exists\, \Delta, \delta > 0$ a C^1 funkce $Y(\underset{\sim}{x}) : \mathcal{U}(\underset{\sim}{a}, \delta) \to \mathcal{U}(b, \Delta)$

tak, že $\forall (\underset{\sim}{x}, y) \in \mathcal{U}(\underset{\sim}{a}, \delta) \times \mathcal{U}(b, \Delta)$ platí

$$F(\underset{\sim}{x}, y) = 0 \iff y = Y(\underset{\sim}{x}).$$

(tj.: $\{F = 0\} \cap \Omega = \text{graf } Y$.)

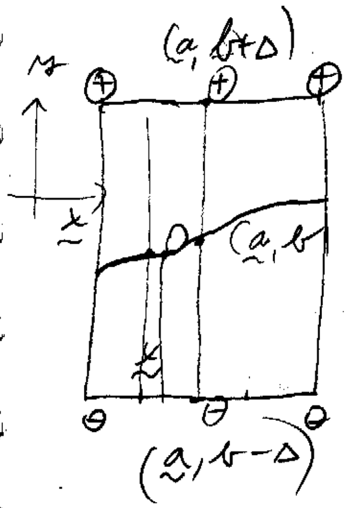
---

dk.: 1. BÚNO: $\frac{\partial F}{\partial y}(\underset{\sim}{a}, b) > 0$;

spojitost ; δ, Δ malé : $\frac{\partial F}{\partial y} > 0$ na Ω

navíc: $F(\underset{\sim}{a}, b+\Delta) > 0 > F(\underset{\sim}{a}, b-\Delta)$

a tedy dále: $F(\underset{\sim}{x}, \Delta) > 0$
$F(\underset{\sim}{x}, -\Delta) < 0$ $\quad \underset{\sim}{x} \in \mathcal{U}(\underset{\sim}{a}, \delta)$.



+ $\underset{\sim}{x} \in \mathcal{U}(\underset{\sim}{a}, \delta)$ libovolné:

$\varphi : y \longmapsto F(\underset{\sim}{x}, y)$ .. spojitá

$\varphi(b-\Delta) < 0$

$\varphi(b+\Delta) > 0$

$\varphi' > 0$ $\quad \mathcal{U}(b, \Delta)$;

Darboux: $\exists!\, y$ (jediné) ; $F(\underset{\sim}{x}, y) = 0$

označíme $Y(\underset{\sim}{x})$.

Navíc: $\left|\frac{\partial F}{\partial x_i}\right| \leq C$; $\frac{\partial F}{\partial y} \geq \varepsilon > 0$ na Ω

2. $\frac{\partial Y}{\partial x_i} = ?$ $\quad \underset{\sim}{e}^i = (0, \ldots \textcircled{1}\ 0 \ldots)$ i-ho'ho' pozice

$\underset{\sim}{x} \in \mathcal{U}(\underset{\sim}{a}, \delta)$ pevné;

$t \in \mathcal{U}(0) \ldots \underset{\sim}{x} + t\underset{\sim}{e}^i \in \mathcal{U}(\underset{\sim}{a}, \delta)$.

$$0 = F(\underset{\sim}{x}, Y(\underset{\sim}{x})) - F(\underset{\sim}{x} + t\underset{\sim}{e}^i, Y(\underset{\sim}{x} + t\underset{\sim}{e}^i)) \pm F(\underset{\sim}{x} + t\underset{\sim}{e}^i, Y(\underset{\sim}{x}))$$

$$= F(\underbrace{\underset{\sim}{x}, Y(\underset{\sim}{x})}_{A}) - F(\underbrace{\underset{\sim}{x} + t\underset{\sim}{e}^i, Y(\underset{\sim}{x})}_{B})$$
$$+ F(\underset{\sim}{x} + t\underset{\sim}{e}^i, Y(\underset{\sim}{x})) - F(\underset{\sim}{x} + t\underset{\sim}{e}^i, Y(\underset{\sim}{x} + t\underset{\sim}{e}_i))$$

$= \underline{D1 + D2}$.

Věta o při. hodu:

$$F(\underset{\sim}{A}) - F(\underset{\sim}{B}) = \langle \nabla F(\underset{\sim}{C}), A - B \rangle; \quad \underset{\sim}{C} \text{ mezi } \underset{\sim}{A}, \underset{\sim}{B}.$$

$\underline{D1}$: $\langle \nabla_{\underset{\sim}{x}} F(\underset{\sim}{C}), -t e_i \rangle$ $\quad \underset{\sim}{c}(t)$ mezi $\underset{\sim}{x}, \underset{\sim}{x} + t\underset{\sim}{e}^i$

$$= -t \frac{\partial F}{\partial x_i}(c(t), Y(\underset{\sim}{x}))$$

D2: $\frac{\partial F}{\partial y}(Y(\underset{\sim}{x} + t\underset{\sim}{e}^i), d(t))(Y(\underset{\sim}{x}) - Y(\underset{\sim}{x} + t\underset{\sim}{e}^i))$; $d(t)$ mezi $Y(\underset{\sim}{x}), Y(\underset{\sim}{x} + te_i)$.

$$0 = -t \frac{\partial F}{\partial x_i}(c(t)) - \frac{\partial F}{\partial y}(d(t))\left(Y(\underset{\sim}{x}) - Y(\underset{\sim}{x} + t\underset{\sim}{e}^i)\right)$$

$$\frac{1}{t}\left[Y(\underset{\sim}{x} + t\underset{\sim}{e}^i) - Y(\underset{\sim}{x})\right] = -\frac{\frac{\partial F}{\partial x_i}(c(t))}{\underbrace{\frac{\partial F}{\partial y}(d(t))}_{\text{omezené}}} = Q(t)$$

$t \to 0$: $c(t) \to \underset{\sim}{x}$

$d(t) \to$

$$Q(t) = -\frac{\frac{\partial F}{\partial x_i}(\underset{\sim}{c}(t), Y(\underset{\sim}{x}))}{\frac{\partial F}{\partial y}(Y(\underset{\sim}{x} + t\underset{\sim}{e}^i), d(t))}$$

$$Y(\underset{\sim}{x}+t\underset{\sim}{e}_i)-Y(\underset{\sim}{x}) = t\cdot \underset{\sim}{Q(t)}$$

$t\to 0:$ $|Q(t)| \leq \frac{C}{\varepsilon}$ omezené

$$Y(\underset{\sim}{x}+t\underset{\sim}{e}^i)\to Y(\underset{\sim}{x}).$$

$$\Rightarrow d(t)\to \underset{\sim}{x}$$

$$\frac{1}{t}\left[Y(\underset{\sim}{x}+t\underset{\sim}{e}^i)-Y(\underset{\sim}{x})\right] = Q(t) \to -\frac{\frac{\partial F}{\partial x_i}(\underset{\sim}{x}, Y(\underset{\sim}{x}))}{\frac{\partial F}{\partial y}(\underset{\sim}{x}, Y(\underset{\sim}{x}))}$$

$\exists \frac{\partial Y}{\partial x_i}$ a jsou omezené v Ω

V. 14. 1.

$\Rightarrow$ $Y(\underset{\sim}{x})$ spojité

$$\frac{\partial Y}{\partial x_i}(\underset{\sim}{x}) = -\frac{\frac{\partial F}{\partial x_i}(\underset{\sim}{x}, Y(\underset{\sim}{x}))}{\frac{\partial F}{\partial y}(\underset{\sim}{x}, Y(\underset{\sim}{x}))}$$

$\frac{\partial Y}{\partial x_i}$ -- spojité. Y, DF spojité

$$F\in C^1; Y\in C^1 \Rightarrow Y\in C^2$$

$F\in C^2, Y\in C^1 \Rightarrow$ PS $C^1 \Rightarrow$ LS $C^1 \Rightarrow Y\in C^2$

etc..

Věta 14.13 [VIF – obecné verze].

$\underset{\sim}{F}: \mathbb{R}^{N+M} \to \mathbb{R}^M$

$F_j(\underset{\sim}{x}, \underset{\sim}{y})$; $j = 1, \dots M$

$\underset{\sim}{x} \in \mathbb{R}^N$; $\underset{\sim}{y} \in \mathbb{R}^M$. ; $(\underset{\sim}{a}, \underset{\sim}{b}) \in \mathbb{R}^{N+M}$

Nechť: $\underset{\sim}{F}(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{0}$ (M rovnic)

$\underset{\sim}{F}$ C^1 na okolí $(\underset{\sim}{a}, \underset{\sim}{b})$

klíčový před.: $\left\{ \frac{\partial F_j}{\partial y_i}(\underset{\sim}{a}, \underset{\sim}{b}) \right\}_{i,j=1}^{M}$ je regulární.

Potom: $\exists\, \delta, \Delta > 0$ a C^1 funkce $\underset{\sim}{Y}: \mathcal{U}(\underset{\sim}{a}, \delta) \to \mathcal{U}(\underset{\sim}{b}, \Delta)$
$\mathbb{R}^N \to \mathbb{R}^M$

takovou, že pro $(\underset{\sim}{x}, \underset{\sim}{y}) \in \mathcal{U}(\underset{\sim}{a}, \delta) \times \mathcal{U}(\underset{\sim}{b}, \Delta) = \Omega$

platí $\underset{\sim}{F}(\underset{\sim}{x}, \underset{\sim}{y}) = \underset{\sim}{0} \iff \underset{\sim}{y} = \underset{\sim}{Y}(\underset{\sim}{x})$.

Tj: $\{\underset{\sim}{F} = \underset{\sim}{0}\} \cap \Omega$ = graf $\underset{\sim}{Y}$.

---

dk.:: $\underset{\sim}{F}(\underset{\sim}{a}, \underset{\sim}{b}) = \underset{\sim}{0}$

$\underset{\sim}{F}(\underset{\sim}{a} + \underset{\sim}{h}, \underset{\sim}{b} + \underset{\sim}{\eta}) = \underset{\sim}{0}$

$\underset{\sim}{h}$ malé $\exists\, \underset{\sim}{\eta}$ malé;

musí regulární matice..

LS: $\nabla_x F(\underset{\sim}{a}, \underset{\sim}{b}) \cdot \underset{\sim}{h}^T + \boxed{\nabla_y F(\underset{\sim}{a}, \underset{\sim}{b})} \cdot \eta^T + \sout{R(\underset{\sim}{h}, \underset{\sim}{\eta})} = 0$

$\eta^T = -\left[\nabla_y F(a, b)\right]^{-1} \cdot \left[\nabla_x F\right] \cdot h^T$

důkaz V. 14.9.

$f, g : \mathbb{R}^N \to \mathbb{R}$; $\underset{\sim}{a} \in \mathbb{R}^N$.

$\underset{\sim}{a} \in \mathbb{R}^N$ extrém f vzhl. $\Gamma = \{\underset{\sim}{x} ; g(\underset{\sim}{x}) = 0\}$.

f, g C^1 na okolí $\underset{\sim}{a}$

$\nabla g(\underset{\sim}{a}) \neq \underset{\sim}{0}$.

$\Rightarrow \exists \lambda \in \mathbb{R}$; $\nabla f(\underset{\sim}{a}) = \lambda \nabla g(\underset{\sim}{a})$.

Dk: $g = g(x_1, \dots x_N)$ $\qquad \underset{\sim}{a} = (\underset{\sim}{\alpha}, a_N)$

$(x_1, \dots x_N) = (\underset{\sim}{x}, y)$, $\underset{\sim}{x} \in \mathbb{R}^{N-1}$, $y \in \mathbb{R}$.

$\nabla g(\underset{\sim}{a}) \neq \underset{\sim}{0}$: BÚNO: $\frac{\partial g}{\partial y}(\underset{\sim}{a}) \neq 0$.

Věta 14.12: $\qquad \varphi : \mathcal{U}(\underset{\sim}{\alpha}, \delta) \to \mathcal{U}(a_N, \Delta)$

$\Gamma \cap \mathcal{U}(\underset{\sim}{a}) \sim$ graf fce φ.

$\psi : \mathcal{U}(\underset{\sim}{\alpha}, \delta) \to \mathbb{R}$

$\underset{\sim}{x} \longmapsto f(\underset{\sim}{x}, \varphi(\underset{\sim}{x}))$

ψ má v bodě $\underset{\sim}{x} = \underset{\sim}{\alpha}$ lok. extrém vzhl. $\mathcal{U}(\underset{\sim}{\alpha}, \delta)$ (v $\mathbb{R}^{N-1}$)

V. 14.7. $\quad \frac{\partial \psi}{\partial x_i}(\underset{\sim}{\alpha}) = 0$; $\quad i = 1, \dots N-1$.

$$\frac{\partial \psi}{\partial x_i}(\underset{\sim}{x}) = \frac{\partial}{\partial x_i}\left[f(\underset{\sim}{x}, \varphi(\underset{\sim}{x}))\right] = \frac{\partial f}{\partial x_i}(\underset{\sim}{x}, \varphi(\underset{\sim}{x})) + \frac{\partial f}{\partial x_N}(\underset{\sim}{x}, \varphi(\underset{\sim}{x}))\frac{\partial \varphi}{\partial x_i}$$

V. 2.1: $\quad \frac{\partial \varphi}{\partial x_i}(\underset{\sim}{\alpha}) = -\dfrac{\frac{\partial g}{\partial x_i}(\underset{\sim}{a})}{\frac{\partial g}{\partial x_N}(\underset{\sim}{a})}$ — λ

$$\lambda = \frac{\frac{\partial f}{\partial x_N}(\underline{a})}{\frac{\partial g}{\partial x_N}(\underline{a})}$$ -2.6

a'l: ~~Pz~~

modum $\frac{\partial f}{\partial x_i}(\underline{a}) = \lambda \frac{\partial g}{\partial x_i}(\underline{a})$.

$i = N$ jare:

$$0 = \frac{\partial f}{\partial x_i}(\underline{a}) + \frac{\partial f}{\partial x_N}(\underline{a}) \cdot \frac{\frac{\partial g}{\partial x_i}(\underline{a})}{\frac{\partial g}{\partial x_N}(\underline{a})}$$

$= \lambda$.

Věta 2.3. [O inverzní funkci.]

Nechť $\underset{\sim}{F}(\underset{\sim}{x}) : \mathbb{R}^n \to \mathbb{R}^n$ (tj. $\underset{\sim}{F} = (F_1, \dots F_n)$, $F_i = F_i(x_1, \dots x_n)$ $i = 1 \dots n$)

je C^1 na okolí $\underset{\sim}{a} \in \mathbb{R}^n$.

Nechť $\nabla \underset{\sim}{F}(\underset{\sim}{a}) = \left(\frac{\partial F_i}{\partial x_j}(\underset{\sim}{a}) \right)_{i,j=1}^{n}$ je regulární matice.

Pak:

① existuje U okolí bodu $\underset{\sim}{a}$ takové, že $\underset{\sim}{F}|_U$ je prosté.

② označíme-li
$$\underset{\sim}{b} = \underset{\sim}{F}(\underset{\sim}{a})$$
$$V = \underset{\sim}{F}(U)$$
a $\underset{\sim}{G} := \left(\underset{\sim}{F}|_U\right)_{-1}$; $\underset{\sim}{G} = (G_1, \dots G_n)$, $G_i = G_i(y_1, \dots y_n)$

je $V \subset \mathbb{R}^n$ otevřené ok. $\underset{\sim}{b}$.

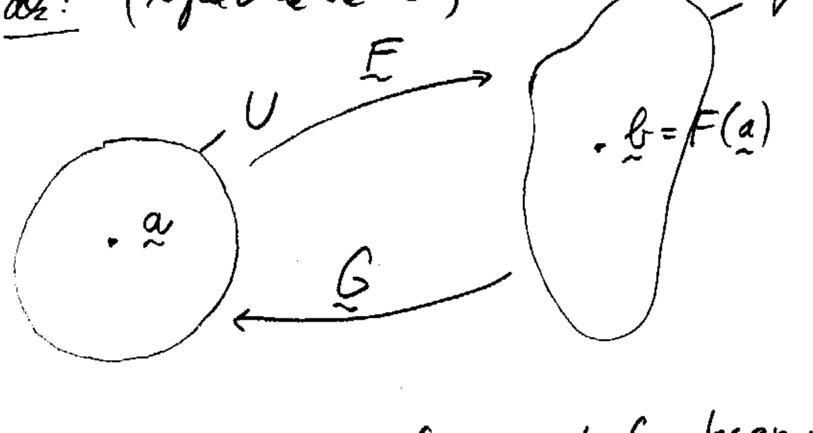
je $\underset{\sim}{G}(\underset{\sim}{y}) \in C^1(V)$, a platí

$$\nabla \underset{\sim}{G}(\underset{\sim}{y}) = \left[\nabla \underset{\sim}{F}\left(\underset{\sim}{G}(\underset{\sim}{y})\right) \right]^{-1}.$$

③ je-li $\underset{\sim}{F}(\underset{\sim}{x}) \in C^k(U)$, je též $\underset{\sim}{G}(\underset{\sim}{y}) \in C^k(V)$.

---

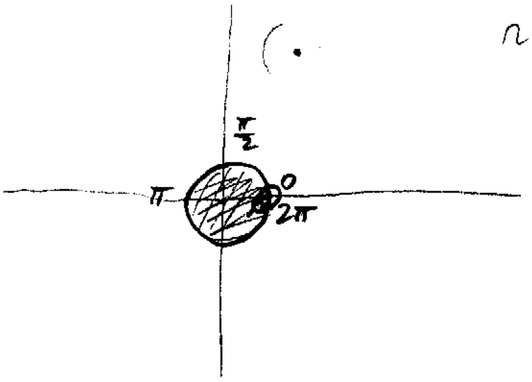
dk: (vynecháváme)



pozn.: Věta příbuzná s impl. funkcemi.

Příklad: $x = r \cdot \cos\varphi$, $y = r \cdot \sin\varphi$ | $(r,\varphi) \overset{F}{\longmapsto} (x,y)$, $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\nabla F(r,\varphi) = \begin{pmatrix} \cos\varphi, & -r\sin\varphi \\ \sin\varphi, & r\cos\varphi \end{pmatrix} \dots \det = \underline{\underline{r}}.$$



$r \neq 0$: tj. $(x,y) \neq (0,0)$ lze lokálně přecházet mezi kart. a polárními.

v okolí $(0,0)$ to nejde .. úhel.