(1) $\int_0^{\pi} \cos^4 x\,dx = \frac{3}{8}\pi$

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1}{2}(1+\cos 2x)\right)^2 = \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)$$

$$= \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right)$$

$$= \frac{1}{4}+\frac{1}{2}\cos 2x+\frac{1}{8}(1+\cos 4x)$$

$$= \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$$

(2) Č1: 10.34 $\frac{3}{4}\pi^2-6$

(3) $\int_{-1}^{1} \frac{dx}{1-x^2} = +\infty$

(4) $\int_0^{\frac{\pi}{4}} \operatorname{tg} x\,dx = \left[-\ln\cos x\right]_0^{\frac{\pi}{4}} = -\ln\frac{1}{\sqrt{2}} = \frac{1}{2}\ln 2$

(5) $\int_0^{\infty} x^n e^{-x}\,dx = n!$

(6) $\int_0^{\infty} \frac{x}{1+x^4}\,dx = \frac{1}{4}\pi$

(7) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_0 = \frac{\pi}{2}$$

$$I_1 = 1$$

8

$$I = \int_0^\infty e^{-\alpha x} \cos\beta x$$

$$J = \int_0^\infty e^{-\alpha x} \sin\beta x$$

$$K = \int_0^\infty e^{-\lambda x} dx \;; \quad \lambda = \alpha + i\beta$$

$$K = \frac{1}{\lambda} = \frac{\alpha - i\beta}{\alpha^2+\beta^2}$$

$$K = \int_0^\infty e^{-\alpha x - i\beta x} dx = I \mp iJ$$

$$I = \left[-\frac{1}{\alpha} e^{-\alpha x} \cdot \cos\beta x\right]_0^\infty - \frac{\beta}{\alpha}\int_0^\infty e^{-\alpha x} \sin\beta x \, dx$$

$$I = \frac{1}{\alpha} - \frac{\beta}{\alpha} J.$$

$$J = \int_0^\infty e^{-\alpha x} \sin\beta x \, dx = \underbrace{\left[-\frac{1}{\alpha} e^{-\alpha x} \sin\beta x\right]_0^\infty}_{=0} + \frac{\beta}{\alpha} I$$

$$J = \frac{\beta}{\alpha} I = \frac{\beta}{\alpha}\left(\frac{1}{\alpha} - \frac{\beta}{\alpha} J\right)$$

$$J = \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2} J$$

$$J\left(1 + \frac{\beta^2}{\alpha^2}\right) = \frac{\beta}{\alpha^2}$$

$$J \cdot \frac{\alpha^2+\beta^2}{\alpha^2} = \frac{\beta}{\alpha^2}$$

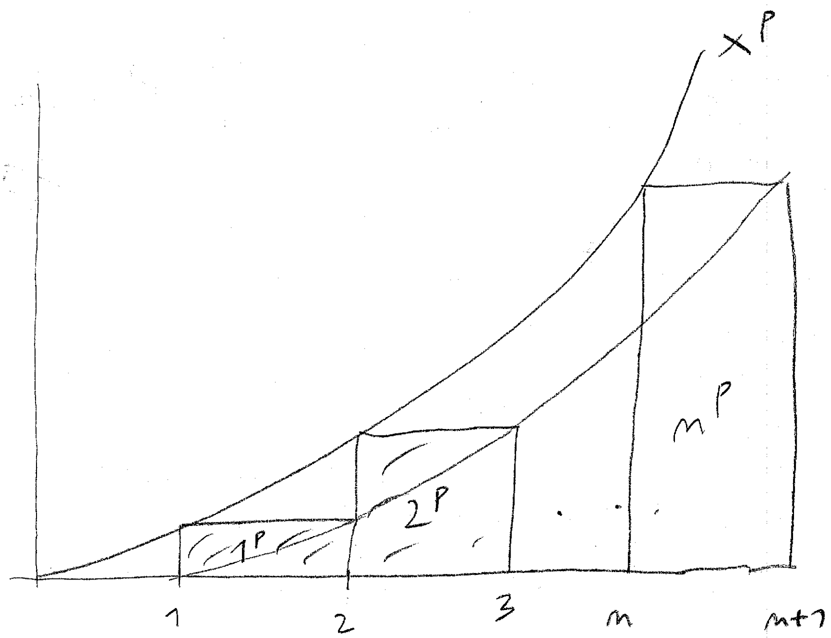
$$\boxed{J = \frac{\beta}{\alpha^2+\beta^2} \qquad I = \frac{\alpha}{\alpha^2+\beta^2}}$$

(a) $\lim\limits_{n\to\infty} \dfrac{1^p+2^p+\dots+n^p}{n^{p+1}} = \,?\quad ; \quad (p>0).$

heuristicky : $s_n = 1^p+2^p+\dots+n^p \approx \int\limits_0^n x^p\,dx = \dfrac{n^{p+1}}{p+1}$

$\Rightarrow \quad a_n \to \dfrac{1}{p+1}$

rigorózně:



$$s_n \leq \int\limits_1^{n+1} x^p\,dx = \left[\frac{x^{p+1}}{p+1}\right]_1^{n+1} = \frac{1}{p+1}\left((n+1)^{p+1}-1\right)$$

$$s_n \leq \int\limits_0^{n} x^p\,dx = \frac{1}{p+1}\cdot n^{p+1}.$$

$$\frac{1}{p+1} \leq a_n \leq \frac{1}{p+1}\cdot\left(\left(1+\frac{1}{n}\right)^{p+1} - \left(\frac{1}{n}\right)^{p+1}\right) \to \frac{1}{p+1}$$

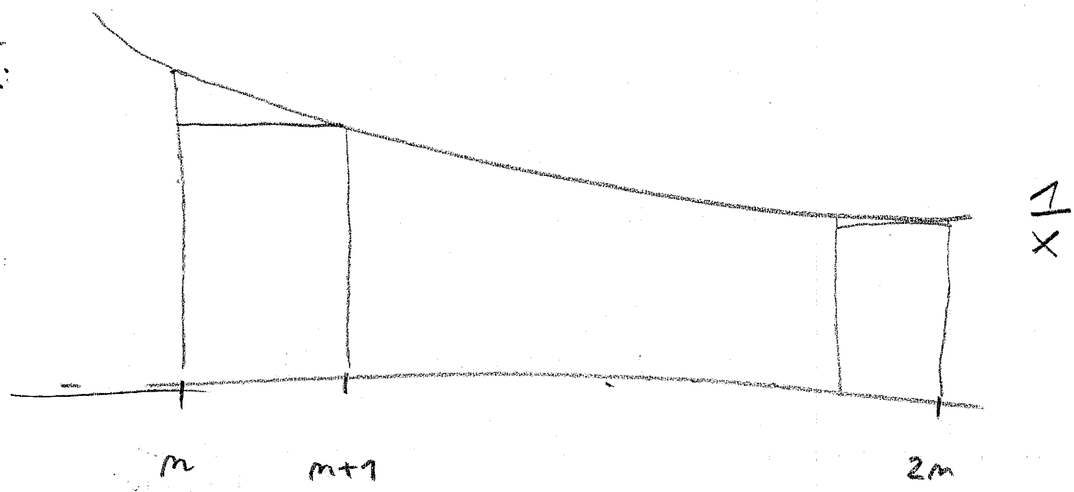
věta „o dvou policajtech" : $a_n \to \dfrac{1}{p+1}$

(b) $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} = \sum_{k=1}^{n} \frac{1}{n+k}$

heuristicky: $a_n \approx \int_1^n \frac{dx}{n+x} = \left[\ln(n+x)\right]_1^n = \ln\left(\frac{2n}{n+1}\right)$

$\to \ln 2$.

rigorózně:



$$a_n \le \int_n^{2n} \frac{1}{x}\,dx = \left[\ln x\right]_n^{2n} = \ln 2$$

$$a_n \ge \int_{n+1}^{2n+1} \frac{1}{x}\,dx = \left[\ln x\right]_{n+1}^{2n+1} = \ln \frac{2n+1}{n+1} \to \ln 2$$

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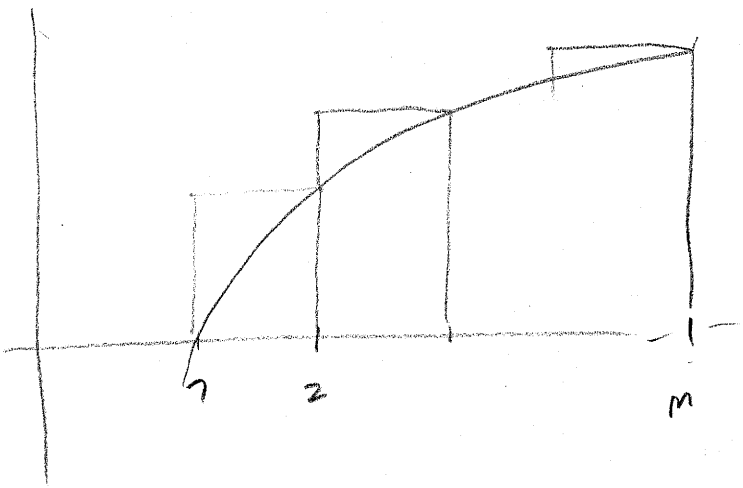
věta „o dvou policajtech": $a_n \to \ln 2$

(c) $\lim\limits_{n\to\infty} \frac{1}{n} \ln(n!)$

$$\ln(n!) = \ln n\cdot(n-1)\ldots 2\cdot 1 = \ln n + \ln(n-1) + \ldots + \ln 2 + \ln 1$$

$$= \sum_{k=1}^{n} \ln k = \sum_{k=2}^{n} \ln k \geq \int_1^n \ln x \, dx = \Big[(x-1)\ln x\Big]_1^n$$



$= (n-1)\ln n$;

$\Rightarrow \quad a_n \geq \frac{n-1}{n}\cdot \ln n \to +\infty$; $\quad a_n \to +\infty$.