A1/ $\dfrac{a_{k+1}}{a_k} = \dfrac{2^{k+1}}{(k+1)!} \cdot \dfrac{k!}{2^k} = \dfrac{2}{k+1} \to 0 < 1$; konverguje (podílové krit.)

A2/ $\dfrac{a_{k+1}}{a_k} = \dfrac{[(k+1)!]^2}{[2(k+1)]!} \cdot \dfrac{[2k]!}{[k!]^2} = \dfrac{(k+1)^2}{(2k+2)(2k+1)} = \dfrac{k+1}{2(2k+1)}$

$= \dfrac{1+\frac{1}{k}}{2\left(2+\frac{1}{k}\right)} \to \dfrac{1}{4} < 1$; konv. (podílové krit.)

A3/ $\dfrac{a_{k+1}}{a_k} = \dfrac{(k+1)^{a(k+1)}}{(k+1)!} \cdot \dfrac{k!}{k^{ak}} = \left(\dfrac{k+1}{k}\right)^{ak} (k+1)^{a-1}$

$\left(\dfrac{k+1}{k}\right)^{ak} = \left(1+\dfrac{1}{k}\right)^{ka} \to e^a$; $(k+1)^{a-1} \to \begin{cases} +\infty; & a>1 \\ 1; & a=1 \\ 0; & a<1. \end{cases}$

$\dfrac{a_{k+1}}{a_k} \to \begin{cases} +\infty, & a>1 \\ e; & a=1 \\ 0; & a<1 \end{cases}$; $\sum a_k$ konv $\iff a<1$ (podílové krit.)

A4/ $\dfrac{a_{k+1}}{a_k} = \dots = \dfrac{(k+1)^2}{2^{2k+1}} \le \left(\dfrac{k+1}{2^k}\right)^2 \to 0$; konv. (podílové krit.)

výpočet limity: $\dfrac{k+1}{2^k} = \dfrac{k+1}{\exp(k \underbrace{\ln 2}_{>0})}$; $e^{\gamma x}$ silnější než x^β $\forall \gamma>0$; $\beta \in \mathbb{R}$.

A5/ $\sqrt[k]{a_k} = \dfrac{k^{2/k}}{\frac{\pi}{3} + \frac{1}{k}} \to \dfrac{1}{\frac{\pi}{3}+0} = \dfrac{3}{\pi} < 1$; konv. (odm. krit.)

$k^{\frac{2}{k}} = \exp\left(\dfrac{2}{k} \ln k\right) \to \exp(0) = 1$; neboť $\dfrac{\ln k}{k} \to 0, k \to \infty$

A6/ $$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1+p}} \cdot \frac{k^{k+p}}{k!} = \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{k}{k+1}\right)^p \to \frac{1}{e} < 1$$

konv. (podíl. krit.)

A7/ $$\sqrt[k]{a_k} = \frac{k}{(2k^2+k+1)^{1/2}} = \frac{1}{\left(2+\frac{1}{k}+\frac{1}{k^2}\right)^{1/2}} \to \frac{1}{\sqrt{2}} < 1;$$

konv. (odm. krit.)

A8/ $$\frac{a_{k+1}}{a_k} = \frac{3(k+1)-1}{4(k+1)-3} = \frac{3k+2}{4k+1} = \frac{3+\frac{2}{k}}{4+\frac{1}{k}} \to \frac{3}{4} < 1;$$

konv. (podíl. krit.)

A9/ $$\frac{a_{k+1}}{a_k} = \dots = \left(\frac{k}{k+1}\right)^q \cdot \left(\frac{p+k}{k+1}\right) \to 1 \quad (\forall q \in \mathbb{R})$$

podílové krit.: neříká nic.

Raabe: $$k\left(\frac{a_k}{a_{k+1}} - 1\right) = k\left(\left(\frac{k+1}{k}\right)^q \left(\frac{k+1}{k+p}\right) - 1\right) = f\left(\frac{1}{k}\right)$$

$$f(x) = \frac{1}{x}\left(\frac{(1+x)^{q+1}}{1+px} - 1\right) = \underbrace{\frac{1}{1+px}}_{\to 1} \cdot \underbrace{\frac{(1+x)^{q+1} - (1+px)}{x}}_{\to q+1-p}$$

pro $x \to 0$: $\to 1$, $\to q+1-p$ (použ. l'Hosp. $\frac{0}{0}$)

Heineho věta: $f\left(\frac{1}{k}\right) \to q-p+1$.

$\sum a_k$ konv pro $q-p+1 > 1 \iff q > p$

div. $q < p$

$q = p$ .. dále nevíme.

A10) $\frac{a_{k+1}}{a_k} = \left(\frac{2k+1}{2k+2}\right)^p \to 1$ $(\forall p)$; podílové neříká nic.

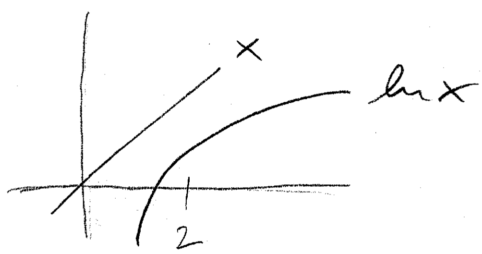
Raabe: $k\left(\frac{a_k}{a_{k+1}} - 1\right) = k\left(\left(\frac{2k+2}{2k+1}\right)^p - 1\right) = f\left(\frac{1}{k}\right)$

$$f(x) = \frac{1}{x}\left(\left(\frac{1+x}{1+\frac{x}{2}}\right)^p - 1\right) \to \frac{p}{2}\ ;\ x \to 0$$

tedy $f\left(\frac{1}{k}\right) \to \frac{p}{2}$; $\sum a_k$ konv pro $p>2$

div $p<2$

B1) $0 < \ln k < k$; $k \geq 2$; $\frac{1}{\ln k} > \frac{1}{k}$; $\sum \frac{1}{k}$ div.



$\Rightarrow \sum \frac{1}{\ln k}$ div. (srovnávací kr.)

B2) $a_k = \frac{k^2}{k^3+1} \sim \frac{1}{k}$; $\sum \frac{1}{k}$ div $\Rightarrow \sum a_k$ div.

ověření $\boxed{\sim}$: $\frac{a_k}{1/k} = k a_k = \frac{k^3}{k^3+1} = \frac{1}{1+\frac{1}{k^3}} \to 1 \in \mathbb{R}\setminus\{0\}$

B3) $a_k = \frac{2k^2+3k+4}{(2k^2+5)^2} \sim \frac{1}{k^2}$; $\sum \frac{1}{k^2}$ konv. $\Rightarrow \sum a_k$ konv.

ověření $\boxed{\sim}$: $\frac{a_k}{1/k^2} = \frac{k^2(2k^2+3k+4)}{(2k^2+5)^2} = \frac{2+\frac{3}{k}+\frac{4}{k^2}}{\left(2+\frac{5}{k^2}\right)^2}$

$\to \frac{2}{2^2} \in \mathbb{R}\setminus\{0\}$

B4/ $a_k = \frac{1}{\sqrt{2k+1}\cdot\sqrt{2k+3}} \sim \frac{1}{k}$; ergo $\sum a_k$ div.

ověření $\boxed{\sim}$: $\frac{a_k}{1/k} = \frac{k}{\sqrt{(2k+1)(2k+3)}} = \frac{1}{\sqrt{(2+\frac{1}{k})(2+\frac{3}{k})}} \to \frac{1}{2}$.

B5/ $a_k = \frac{\sqrt{k+2}-\sqrt{k-2}}{k^a} \sim \frac{1}{k^{a+\frac{1}{2}}}$; tj. $\sum a_k$ konv.

$\iff a+\frac{1}{2} > 1$

$a > \frac{1}{2}$.

ověření $\boxed{\sim}$: $\sqrt{\alpha}-\sqrt{\beta} = \frac{\alpha-\beta}{\sqrt{\alpha}+\sqrt{\beta}}$

$\frac{a_k}{1/k^{a+\frac{1}{2}}} = \frac{4\sqrt{k}}{\sqrt{k+2}+\sqrt{k-2}} = \frac{4}{\sqrt{1+\frac{2}{k}}+\sqrt{1-\frac{2}{k}}} \to \frac{4}{2}$.

B6/ $a_k = (k^{(k^a)} - 1) = \exp(k^a\cdot\ln k) - 1$

(i) $a \geq 0$: $k^a\cdot\ln k \to +\infty$; $\sum a_k$ div.

(ii) $a<0$: $k^a\cdot\ln k = \frac{\ln k}{k^{-a}} \to 0$; neboť $-a>0$.

$a_k \sim \underbrace{\frac{\ln k}{k^{-a}}}_{c_k}$; tedy $\sum a_k$ konv. $\iff -a > 1$

$a < 1$

(viz Příklad B9, $b=-1$)

ověření $\boxed{\sim}$: $\frac{e^x-1}{x} \to 1$; $x\to 0$;

Heineho věta: $\frac{\exp(k^a\cdot\ln k)-1}{k^a\cdot\ln k} = \frac{a_k}{c_k} \to 1$,

neboť $c_k \to 0$; $c_k \neq 0$ $\forall k \geq 2$.

B7 $a_k = \exp\left(\frac{\ln k}{k^2+1}\right) - 1 \sim c_k$; $c_k = \frac{\ln k}{k^2}$.

$\sum c_k$ konv. (viz B9); tedy $\sum a_k$ konv.

ověření $\boxed{\sim}$: $$\frac{a_k}{c_k} = \frac{\exp\left(\frac{\ln k}{k^2+1}\right) - 1}{\frac{\ln k}{k^2+1}} \cdot \frac{k^2}{k^2+1} \to 1 \cdot 1$$

1. část: Heineho věta: $\frac{e^x - 1}{x} \to 1$; $x \to 0$

$\frac{\ln k}{k^2+1} \to 0$; $\neq 0$ $\forall k$.

B8 $a_k = \exp\left(-k^{1/3}\right)$;

podílové krit.: $$\frac{a_{k+1}}{a_k} = \exp\left(k^{1/3} - (k+1)^{1/3}\right) \to 1$$

nedává nic.

mimochodem: $$k^{1/3}\left(1 - \left(1+\frac{1}{k}\right)^{1/3}\right) = k^{1/3} \cdot \left(-\frac{1}{3k} + o\left(\frac{1}{k}\right)\right)$$
$$= \frac{1}{k^{2/3}}\left(-\frac{1}{3} + o(1)\right) \to 0.$$

Raabe: již víme, že $c_k = k^{1/3} - (k+1)^{1/3} \sim k^{-2/3} \to 0$

ergo: $$k\left(\frac{a_k}{a_{k+1}} - 1\right) = \frac{\exp(-c_k) - 1}{\frac{1}{k}} = \underbrace{\frac{\exp(-c_k) - 1}{-c_k}}_{\to 1} \cdot \underbrace{k(-c_k)}_{\to +\infty}$$

tedy $\sum a_k$ konv.

B9| $a_k = \dfrac{1}{k^a \ln^b k}$; víme: $\sum \dfrac{1}{k^a}$ konv. $\Leftrightarrow$ $a>1$

rel. lim: $\dfrac{\ln^\delta k}{k^\mu} \to 0$, $k \to \infty$

$\forall \mu > 0$; $\delta \in \mathbb{R}$.

(i) $\underline{a>1}$: volme $\varepsilon > 0$; $a-\varepsilon > 1$; tj: $\sum \dfrac{1}{k^{a-\varepsilon}}$ konv.

$$a_k = \frac{1}{k^{a-\varepsilon}} \cdot \underbrace{\frac{1}{k^\varepsilon \ln^b k}}_{\to 0;\ \text{tedy } \leq 1 \text{ pro } k \geq n_0} \leq \frac{1}{k^{a-\varepsilon}}; \ k > n_0, \ \Rightarrow \sum a_k \text{ konv.}$$

(ii) $\underline{a<1}$: volme $\varepsilon > 0$; $a+\varepsilon < 1$; tj: $\sum \dfrac{1}{k^{a+\varepsilon}}$ div.

$$a_k = \frac{1}{k^{a+\varepsilon}} \cdot \underbrace{\frac{k^\varepsilon}{\ln^\beta k}}_{\to +\infty;\ \text{tedy } \geq 1 \text{ pro } k \geq n_0} \geq \frac{1}{k^{a+\varepsilon}}; \ k > n_0 \ \overset{*}{\Rightarrow} \sum a_k \text{ div}$$

(iii) $a=1$: $a_k = \dfrac{1}{k \ln^b k} = f(k)$; $f(x) = \dfrac{1}{x \ln^b x}$;

int. krit: $\sum\limits_{k=2}^{\infty} a_k$ konv $\Leftrightarrow \int\limits_2^\infty f(x)\,dx < \infty$.

$$\int_2^\infty \frac{dx}{x \ln^b x} \left| \begin{array}{l} \ln x = y \\ \frac{dx}{x} = dy \end{array} \right| = \int_{\ln 2}^\infty \frac{dy}{y^b} ; < +\infty \Leftrightarrow b > 1.$$

řešení: $\sum a_k$ konv. $\Leftrightarrow$ $a>1$ nebo $a=1$; $b>1$.

B10)

$a_k = k^3 \left(1 - \cos\frac{1}{k}\right) \sim k$; $\Rightarrow \sum a_k$ div.

ověření $\boxed{\sim}$: $\dfrac{a_k}{k} = \dfrac{1-\cos\frac{1}{k}}{\frac{1}{k^2}} \to \frac{1}{2}$; $k \to \infty$

Heineho věta : ; $\dfrac{1-\cos y}{y^2} \to \frac{1}{2}$, $y \to 0$

$\frac{1}{k} \to 0$;

$\neq 0$ pro $\forall k$ (základní limita)

B11)

$a_k = \sin\frac{1}{k} \cdot \ln\frac{k+1}{k} = \sin\frac{1}{k} \cdot \ln\left(1+\frac{1}{k}\right) \sim \frac{1}{k^2}$

ověření $\boxed{\sim}$: viz výše; $\dfrac{\sin y \cdot \ln(1+y)}{y^2} \to 1$; $y \to 0$.

Heine.

tedy $\sum a_k$ konv.

B12)

$a_k = \sin\dfrac{k}{k^2+1} \sim \dfrac{1}{k}$; tedy $\sum a_k$ div.

ověření $\boxed{\sim}$:

$$\frac{\sin\frac{k}{k^2+1}}{\frac{1}{k}} = \frac{\sin\frac{k}{k^2+1}}{\frac{k}{k^2+1}} \cdot \frac{k^2}{k^2+1} =$$

$$= \frac{\sin\frac{1}{k+\frac{1}{k}}}{\frac{1}{k+\frac{1}{k}}} \cdot \frac{1}{1+\frac{1}{k^2}} \to 1 \cdot 1$$, dle VoAL

$\dfrac{\sin y}{y} \to 1$, $y \to 0$

$\dfrac{1}{k+\frac{1}{k}} \to 0$, $k \to \infty$

$\neq 0$; $\forall k$.