① $f = x + y$; $M = \{(x,y);\ g(x,y) = 0\}$

$$g = (x^2+y^2)^2 - 2xy.$$

---

f spojité, M omez. uzavř. $\Rightarrow \exists$ globální extrémy.

? omezenost: $(x^2+y^2)^2 = 2xy \leq x^2+y^2$

$x^2+y^2 \leq 1$: $x, y \in [-1,1]$.

? uzavřenost: $M = g^{-1}(\{0\})$; $g: \mathbb{R}^2 \to \mathbb{R}$ spojité

$\{0\} = [0,0] \subset \mathbb{R}$ uzavřené

---

podezřelé body: (a) f, g nehladké $\emptyset$

(b) $\nabla g = 0$: $g_x = 2x(x^2+y^2)\cdot 2 - 2y = 0$

$g_y = 2y(x^2+y^2)\cdot 2 - 2x = 0$

$$2x(x^2+y^2) = y \quad (1)$$

$$2y(x^2+y^2) = x \quad (2)$$

$(1)x + (2)y = 0$

$2(x^2+y^2)^2 = 2xy$; zároveň $g = 0$

$(x^2+y^2)^2 = 2xy$

$2(x^2+y^2)^2 = (x^2+y^2)^2$

$\Rightarrow (x,y) = (0,0) =: A$; $f(A) = 0$.

1 – dokončení: (c) $\nabla f = \lambda \nabla g$:

$$1 = 2\lambda\left[2x\,(x^2+y^2) - y\right]$$
$$1 = 2\lambda\left[2y\,(x^2+y^2) - x\right]$$

zjevně $\lambda \neq 0$, tedy $2x(x^2+y^2) - y = 2y(x^2+y^2) - x$

$$x\left[2(x^2+y^2)+1\right] = y\left[2(x^2+y^2)+1\right]$$
$$x = y.$$

podmínka $g = 0$: $(2x^2)^2 = 2x^2$;

(α) $x = 0 \rightarrow y = 0$; $A = (0,0)$ již podezíravé, viz (b)

(β) $x \neq 0 \rightarrow 2x^2 = 1$; $x = \pm\frac{1}{\sqrt{2}}$; $B = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$C = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

$f(A) = 0$;

$f(B) = \sqrt{2}$ ... glob. max.

$f(C) = -\sqrt{2}$ glob. min.

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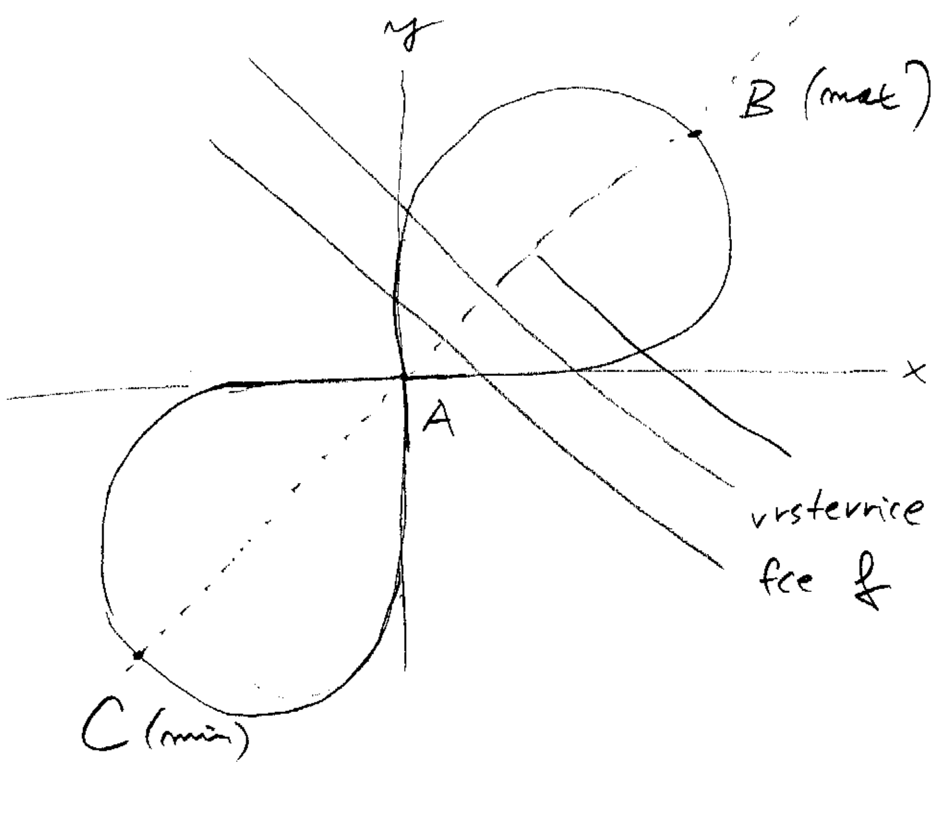
polární souřadnice

$x = r\cos\varphi$

$y = r\sin\varphi$:

$r^2 = \sin 2\varphi$;



(1b) $f = x - y$; fixed… ; $A = (0,0)$.

(c) $\nabla f = \lambda \nabla g$

$$1 = 2\lambda\,\big(2x(x^2+y^2) - y\big)$$

$$-1 = 2\lambda\,\big(2y(x^2+y^2) - x\big)$$

jistě $\lambda \neq 0$; tedy $2x(x^2+y^2) - y = -2y(x^2+y^2) + x$.

$$(x+y)\left[2(x^2+y^2) - 1\right] = 0.$$

(α) $x = -y$: spolu s rcí $g = 0$:

$(x^2+y^2)^2 + 2x^2 = 0 \Rightarrow (x,y) = (0,0) = A$ již máme.

(β) $\left.\begin{array}{l} x^2+y^2 = \frac{1}{2}\ ; \\ (x^2+y^2)^2 = 2xy \end{array}\right\} \Rightarrow xy = \frac{1}{8}$; tj. $y = \frac{1}{8x}$.

$$x^2 + \frac{1}{64x^2} = \frac{1}{2}\ ;\quad x^2 = R\ (>0$$

$$64R^2 - 32R + 1 = 0\ ;$$

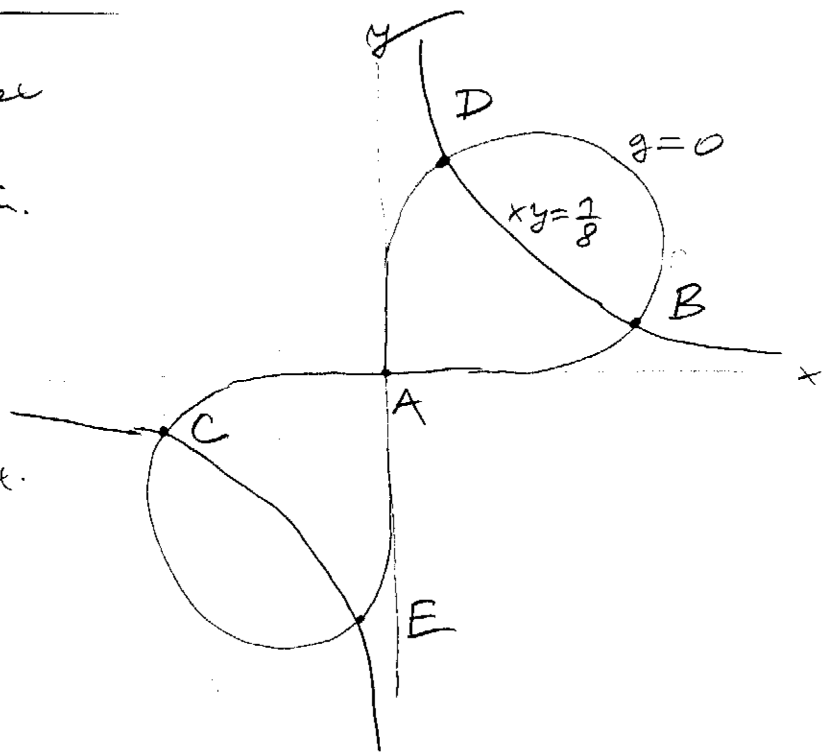
$$R_{1,2} = \frac{2 \pm \sqrt{3}}{8}$$ … obě kladná čísla.

$\Rightarrow B = \left(\sqrt{R_1}, \frac{1}{8\sqrt{R_1}}\right)$ -- max

$C = \left(-\sqrt{R_1}, \frac{-1}{8\sqrt{R_1}}\right)$ … min.

$D = \left(\sqrt{R_2}, \frac{1}{8\sqrt{R_2}}\right)$ min.

$E = \left(-\sqrt{R_2}, \frac{-1}{8\sqrt{R_2}}\right)$ … max.



(3) $f = 2x + y^2$; $M = \{xy = 1\} \cap \{x > 0\}$.

$\{g = 0\}$; kde $g = xy - 1$.

Existence extrémů? (M není omezené, např. $(n, \frac{1}{n}) \in M$).

podezřelé body?

(a) f, g nehladké $\emptyset$

(b) $\nabla g = 0$: $g_x = y = 0$ .. nikdy v M .. $\emptyset$

(c) $\nabla f = \lambda \nabla g$: $2 = \lambda y$ ... potřebné $x, y, \lambda \neq 0$.

$2y = \lambda x$ 2.rce : 1.rce $\Rightarrow$ $y = \frac{x}{y}$

$\underline{y^2 = x.}$

$xy = 1$

$y^3 = 1$.

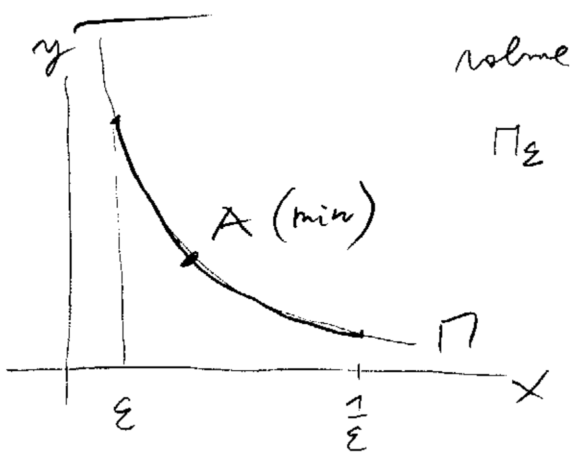
$y = 1$

$A = (1,1)$ -- jediný podezřelý bod $\longleftarrow$

$f(A) = 3$.

? $\exists$ extrémy : $f(n, \frac{1}{n}) = 2n + \frac{1}{n^2} \to +\infty$, $n \to \infty$

$\to$ f shora neomezené. $\nexists$ maximum.



volme $M_\varepsilon := M \cap \{x \geq \varepsilon, y \geq \varepsilon\}$.

M_ε omezená, uzavřená : $\exists$ min f na M_ε.

$(x, y) \in M \setminus M_\varepsilon$: buď $x \geq \frac{1}{\varepsilon}$ nebo $y \geq \frac{1}{\varepsilon}$

tedy $f \geq \frac{2}{\varepsilon}$ nebo $f \geq \frac{1}{\varepsilon^2}$.

ε malé: $\frac{2}{\varepsilon}, \frac{1}{\varepsilon^2} > 3 = f(A)$;

$A \in M_\varepsilon \Rightarrow \exists$ min f na M

$\longrightarrow$ musí jít o bod A.

④ $f = 2y + x^2$; $M = \{g = 0\} \cap \{x > 0\}$;

$$g = x^2 y - 1.$$

podezřelé body:

(a) f, g nehladké ... $\emptyset$

(b) $\nabla g = 0$: $g_y = x^2 \neq 0$ na M ... $\emptyset$.

(c) $\nabla f = \lambda \nabla g$: $2x = \lambda 2xy$; nutně $x, y, \lambda \neq 0$

$$2 = \lambda x^2$$

podíl rovnic : $\frac{\cancel{2}x}{\cancel{2}} = \frac{2xy}{x^2}$

$$x^3 = 2xy$$

$$x^2 = 2y \rightarrow \text{do: } x^2 y = 1$$

$$2y^2 = 1;$$

$$y = \pm \frac{1}{\sqrt{2}}$$

$A = \left(\sqrt[4]{2}, \frac{1}{\sqrt{2}}\right)$.

leč: $x^2 y = 1$; $x > 0$

$\rightarrow$ nutně $y = \frac{1}{\sqrt{2}}$;

$$x = \frac{1}{\sqrt{y}} = \sqrt[4]{2}$$

Podobně jako u č. ③ se ukáže, že

A je globální minimum, maximum neexistuje.

⑤ $f = x^2 + xy + y^2$; $M = \{g = 0\}$; $g = x^2 + 4y^2 = 4$.

$\exists$ extrémy: f spojité, M omez. uzavř.

podezřelé body

(a) f, g nehladké … $\emptyset$

(b) $\nabla g = 0$: $\left.\begin{array}{l} 2x = 0 \\ 4y = 0 \end{array}\right\}$ neleží na M … $\emptyset$

(c) $\nabla f = \lambda \nabla g$: $2x + y = \lambda 2x$;

$$x + 2y = \lambda 4y$$

zřejmě $\lambda \neq 0$; jinak $x = y = 0 \notin M$

podobně $x \neq 0$, $y \neq 0$.

1. rce: $\lambda = 1 + \frac{y}{2x}$ $\qquad 1 + \frac{k}{2} = \frac{1}{4k} + \frac{1}{2}$;

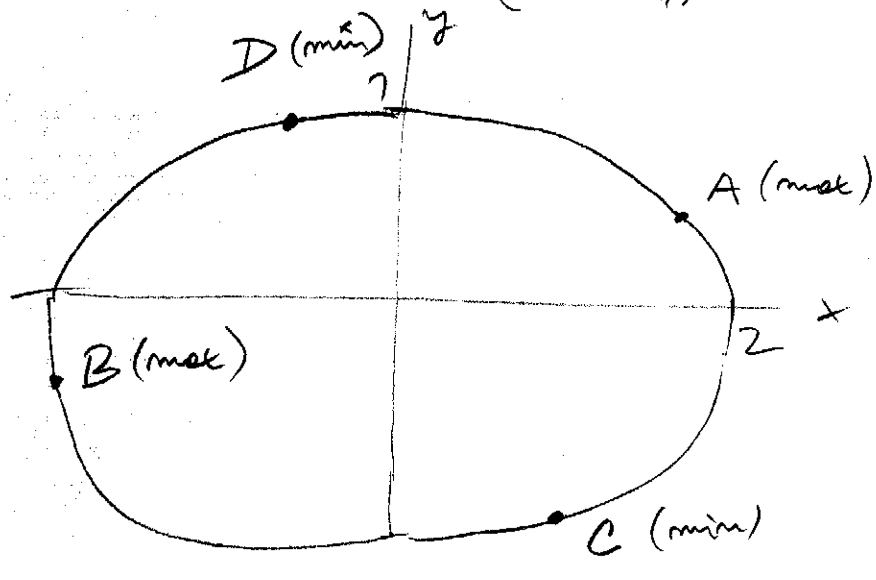
2. rce: $\lambda = \frac{x}{4y} + \frac{1}{2}$. $\qquad$ kde $k = \frac{y}{x}$.

$$\Rightarrow \quad 2k^2 + 2k - 1 = 0$$

$$k_{1,2} = \frac{-1 \pm \sqrt{3}}{2}$$

dopočteme x, y: $x^2 + 4(k_{1,2}x)^2 = 4$

$$x^2(1 + 4k_{1,2}^2) = 4; \quad x_{1,2} = \pm \frac{2}{\sqrt{1 + 4k_{1,2}^2}}$$



⑥ $f = x - y$; $M = \{ g_1 = 0 \ \& \ g_2 = 0 \}$;

$$g_1 = x^2 + y^2 + z^2 - 1;$$
$$g_2 = x + y + z - 1 \quad ;$$

f spojité; M omezené, uzavřené (a neprázdné) $\Rightarrow \exists$ glob. extrémy.

? omez.: $g_1 = 0 \Rightarrow |x|, |y|, |z| \leq 1$

? uzavř.: $M = g_1^{-1}(\{0\}) \cap g_2^{-1}(\{0\}) \dots$

… spojitý vzor uzavřené množiny $\{0\} \subset \mathbb{R}$

& průnik dvou uzavřených množin

---

podezřelé body:

(a) f, g_1, g_2 nehladké … $\emptyset$

(b) $h\left(\frac{\partial g_1, g_2}{\partial x y z}\right) < 2$: $\begin{pmatrix} 2x, & 2y, & 2z \\ 1, & 1, & 1 \end{pmatrix}$ … $\emptyset$

LZ $\Leftrightarrow x = y = z$ … nelze zároveň splnit $g_1 = 0, g_2 = 0$.

(c) $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$

$$1 = 2\lambda x + \mu$$
$$-1 = 2\lambda y + \mu$$
$$0 = 2\lambda z + \mu$$

---

nutno řešit zároveň s podmínkou

$$x^2 + y^2 + z^2 = 1$$
$$x + y + z = 1$$

Pozn.: M je sféra proťnuté rovinou, tj. kružnice.

6 – dokončení!

1.+2. rce: $\quad 0 = 2\lambda(x+y) + 2\mu$

2×3. rce: $\quad 0 = 4\lambda z + 2\mu$

$$0 = 2\lambda\left(x+y-2z\right).$$

(α) $\lambda = 0$: $\mu = 0$; ↯

(β) $\lambda \neq 0$: $\quad x+y-2z = 0. \quad (-1)$

$$x+y+z = 1$$

$$3z = 1;\ z = \frac{1}{3};$$

$$x^2+y^2 = 1 - \frac{1}{9} = \frac{8}{9}$$

$$x+y = \frac{2}{3}$$

$\cancel{x} = \frac{2}{3} \cancel{-y}$: $\quad x^2 + \left(\frac{2}{3}-x\right)^2 = \frac{8}{9}$

$y = \frac{2}{3} - x$

řeš.: $\quad x = \dfrac{1 \pm \sqrt{3}}{3}$

$\Rightarrow$: $A = \left(\frac{1+\sqrt{3}}{3}, \frac{1-\sqrt{3}}{3}, \frac{1}{3}\right) \qquad f = \frac{2}{\sqrt{3}}$ ... max

$B = \left(\frac{1-\sqrt{3}}{3}, \frac{1+\sqrt{3}}{3}, \frac{1}{3}\right) \qquad f = \frac{-2}{\sqrt{3}}$ ... min.

(7) $f = x+y$; $M = \{x^2+y^2+z^2=1\} \cap \{x+y-2z=0\}$

∃ extrému: podobně jako č. 6 výš. $g_1 = x^2+y^2+z^2-1$

$g_2 = x+y-2z$

podezřelé body:

(a) f, g_1, g_2 nehladké ... $\emptyset$

(b) $h\left(\frac{\partial(g_1,g_2)}{\partial(x,y,z)}\right) < 2 \ldots \begin{pmatrix} 2x, & 2y, & 2z \\ 1, & 1, & -2 \end{pmatrix} \ldots \emptyset$.

... muselo by výt: $x = y$; ... odtud: $(g_2=0)$: $x=y=z=0$

$4y = -2z$

$\to g_1 \neq 0$; spor

(c) $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$:

(1) $1 = 2\lambda x + \mu$

(2) $1 = 2\lambda y + \mu$

(3) $0 = 2\lambda z - 2\mu$

$(1)x + (2)y + (3)z \to$

$x+y = 2\lambda(x^2+y^2+z^2) + \mu(x+y-2z)$

$x+y = 2\lambda$.

$(1)+(2)+(3) \to$

$2 = 2\lambda(x+y+z)$

$(1)-(2) \to$

$0 = 2\lambda(x-y)$.

(α) $\lambda = 0$: 3 rce: $\mu = 0$; spor s 1. rcí

(β) $\lambda \neq 0$: $x = y$; \ dále: $2z = x+y = 2x$ $(g_2=0)$

tj: $z = x = y$.

$g_1 = 0$: $3x^2 = 1$; $x = \pm\frac{1}{\sqrt{3}}$: $A = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ -- max

$B = -A$ -- min.

(9) $f = x^2+y^2+z^2$; $M = \{xyz = 1\}$; BÚNO : $x,y,z > 0$.

M neomezené ... $\left(n^2, \frac{1}{n}, \frac{1}{n}\right) \in M$;

$f\left(n^2, \frac{1}{n}, \frac{1}{n}\right) \geq n^4$; tj. $\nexists$ glob. maxim.

podezřelé body:

(a) f, g nehladké ... $\emptyset$

(b) $\nabla g = 0$: $g_x = yz = 0$ → musí $x=0$ nebo $y=0$

$g_y = xz = 0$ .. $\nexists$ takové body v M.

$g_z = xy = 0$

(c) $\nabla f = \lambda \nabla g$: $2x = \lambda yz$

$2y = \lambda xz$

$2z = \lambda xy$

musí $x, y, z, \lambda \neq 0$: 1. rce / 2. rce : $\frac{x}{y} = \frac{y}{x} \Rightarrow x = y$

podobně: $\boxed{x = y = z}$.

$A = (1,1,1)$ .. $f(A) = 3$.

volme $M_\varepsilon = M \cap \{x \geq \varepsilon, y \geq \varepsilon, z \geq \varepsilon\}$, ...

$\exists$ min f na M_ε ; BÚNO $\frac{1}{\varepsilon} > 3 = f(A)$; $A \in M_\varepsilon$.

$(x,y,z) \in M \setminus M_\varepsilon$: nechť např. $x \leq \varepsilon$; pak $yz = \frac{1}{x} \geq \frac{1}{\varepsilon}$;

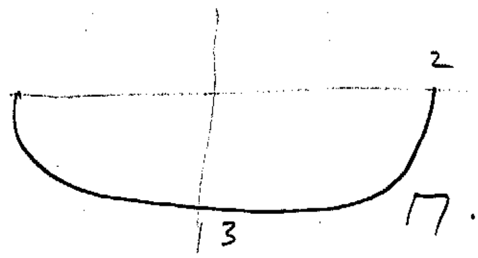
musí buď $y \geq \frac{1}{\sqrt{\varepsilon}}$ nebo $z \geq \frac{1}{\sqrt{\varepsilon}}$; (jinak $yz < \frac{1}{\varepsilon}$; spor)

tedy $f \geq \frac{1}{\varepsilon}$ mimo M_ε ; → $\exists$ glob. min. na M.

(10) $f = x^2 - y^2$; $\Pi = \{\frac{x^2}{4} + \frac{y^2}{9} = 1\} \cap \{y \le 0\}$.

f spojité; Π omez. uzavř.

$\Rightarrow \exists$ glob. extrémy.



podezřelé body:

(a) f, g nehladké ... $\emptyset$ $\qquad g = \frac{x^2}{4} + \frac{y^2}{9} - 1$;

(b) $\nabla g = 0$ ... nikde v Π

(c) krajní body: $A = (2,0)$ ... $f = 4$

$B = (-2,0)$ ... $f = 4$.

(d) $\nabla f = \lambda \nabla g$: $2x = 2\lambda \cdot \frac{x}{4}$

$2y = 2\lambda \cdot \frac{2y}{9}$

1. rce: (α) $x = 0$: — $C = (0,3)$ ... $f = -9$

(β) $x \neq 0$: $2 = \frac{2\lambda}{4}$; $2\lambda = 8$.

2. rce: $2y = 8 \cdot \frac{2y}{9}$ ... nutně $y = 0$ (již máme).

Závěr: A, B ... max; C ... min.

Jiný postup: parametrizace: $x = 2 \cdot \cos t$

$\varphi : t \to (x,y)$ $\quad y = 3 \sin t$; $\quad t \in [\pi, 2\pi]$.

$f(\varphi(t)) = 4\cos^2 t - 9\sin^2 t = g(t)$

$= 4 - 13 \sin^2 t$