A1 $\int x^2 \arcsin x \, dx$; 13.2.2006 (4t)

30.1.2006 (3t)

A2 $\int \frac{1}{x}\sqrt{\frac{x-1}{2-x}}\, dx$

B1 $\int e^{2x} \operatorname{arctg}(e^x+1)\, dx$ 2.2.2009 (3t)

B2 $\int \frac{dx}{(3+\cos x)\sin x}$; 16.1.2006 (1t)

(2) $\int x^2 \arcsin x \, dx$. [8b]

per-partes: $u' = x^2$, $u = \frac{x^3}{3}$

$v = \arcsin x$, $v' = \frac{1}{\sqrt{1-x^2}}$ [2]

$$= \frac{x^3}{3}\arcsin x - \frac{1}{3}\underbrace{\int \frac{x^3}{\sqrt{1-x^2}}\,dx}_{I}$$ [1]

$$I = \int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int \frac{2x\,x^2}{\sqrt{1-x^2}}\,dx \quad \left| \begin{matrix} x^2 = y \\ 2x\,dx = dy \end{matrix} \right| = \frac{1}{2}\int \frac{y\,dy}{\sqrt{1-y}}$$ (2)

$$\left| \begin{matrix} \sqrt{1-y} = t & dy = -2t\,dt \\ 1-y = t^2 & \\ y = 1-t^2 & \end{matrix} \right| = \frac{1}{2}\int \frac{(1-t^2)\cdot(-2t)\,dt}{t} = \int (t^2-1)\,dt$$ [1]

$$= \frac{t^3}{3} - t = \frac{(1-y)^{3/2}}{3} - (1-y)^{\frac{1}{2}} = \frac{(1-x^2)^{3/2}}{3} - (1-x^2)^{\frac{1}{2}}.$$

Výsledek: $\frac{x^3}{3}\arcsin x + \frac{1}{3}(1-x^2)^{\frac{1}{2}} - \frac{1}{9}(1-x^2)^{3/2}$ [1]

platí v $(-1,1)$ [1]

(2) $\displaystyle\int \frac{1}{x}\sqrt{\frac{x-1}{2-x}}\,dx$ [16]

$t = \sqrt{\frac{x-1}{2-x}}$; [1]

$x = \frac{2t^2+1}{t^2+1}$; $dx = \frac{2t}{(t^2+1)^2}\,dt$ [2]

$\int g(t)\,dt$; $g(t) = \frac{2t^2}{(t^2+1)(2t^2+1)} = \frac{At+B}{t^2+1} + \frac{Ct+D}{2t^2+1}$ [1]

$A = C = 0$; $B = 2$; $D = -2$ [2]

$2\int \frac{dt}{t^2+1} - 2\int \frac{dt}{2t^2+1} = 2\,\mathrm{arctg}\, t - \sqrt{2}\,\mathrm{arctg}\,\sqrt{2}\,t$ [2]

výsledek: $2\,\mathrm{arctg}\sqrt{\frac{x-1}{2-x}} - \sqrt{2}\,\mathrm{arctg}\sqrt{\frac{2x-2}{2-x}}$;

$x \in (1,2)$ [1]

[8b]

(2) $\int e^{2x} \cdot \operatorname{arctg}(e^x+1)\,dx$.

subst.: $e^x = y$

$x = \ln y$ $\qquad \int y^2 \cdot \operatorname{arctg}(y+1)\,\frac{dy}{y}$

$dx = \frac{1}{y}\,dy$

$$= \int y \operatorname{arctg}(y+1)\,dy.$$

[2]

per-partes: $\int \underbrace{y}_{u'} \cdot \underbrace{\operatorname{arctg}(y+1)}_{v}\,dy$

$u = \frac{1}{2}y^2 \quad v' = \frac{1}{(y+1)^2+1}$

$$= \frac{1}{2}y^2 \operatorname{arctg}(y+1) - \frac{1}{2}\int \frac{y^2}{y^2+2y+2}\,dy.$$

[2]

dělení polynomů: $\frac{y^2 \pm (2y+2)}{y^2+2y+2} = 1 - \frac{2y+2}{y^2+2y+2}$ [1]

[2]

$\int \frac{2y+2}{y^2+2y+2}\,dy$ $\Bigg|$ 1. věta o subst.: $\varphi(y) = y^2+2y+2$, $\varphi'(y) = 2y+2$ $\Bigg|$ $= \int \frac{\varphi'(y)}{\varphi(y)}\,dy = \ln|\varphi(y)|$.

platí v $\mathbb{R}$: [1]

celkově: $\frac{1}{2} e^{2x} \operatorname{arctg}(e^x+1) - \frac{1}{2}e^x + \frac{1}{2}\ln(e^{2x}+2e^x+2)$.

(2) $$\int \frac{dx}{(3+\cos x)\sin x};$$

1. možnost: $$f(x) = \frac{\sin x}{(3+\cos x)(\sin^2 x)} = \frac{-\sin x}{(3+\cos x)(\cos^2 x - 1)};$$

$$y = \cos x;$$
$$dy = -\sin x \, dx$$

$$I = \int \frac{dy}{(3+y)(y^2-1)}; \qquad \frac{1}{(3+y)(y^2-1)} = \frac{A}{y+3} + \frac{B}{y+1} + \frac{C}{y-1}$$

$$\rightarrow A = \frac{1}{8} = C; \quad B = -\frac{1}{4}$$

$$I = \frac{1}{8}\ln|y+3| - \frac{1}{4}\ln|y+1| + \frac{1}{8}\ln|y-1|;$$

výsledek: $$\frac{1}{8}\ln(\cos x+3) - \frac{1}{4}\ln(\cos x+1) + \frac{1}{8}\ln(1-\cos x)$$
$$x \in (k\pi, (k+1)\pi).$$

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2. možnost: $$t = \operatorname{tg}\frac{x}{2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad \sin x = \frac{2t}{1+t^2};$$
$$dx = \frac{2\,dt}{1+t^2};$$

$$\rightsquigarrow \int \frac{1}{\left(3+\frac{1-t^2}{1+t^2}\right)\frac{\cancel{2}t}{1+t^2}} \cdot \frac{\cancel{2}\,dt}{1+t^2} = \int \frac{(t^2+1)\,dt}{\cancel{(t^2+1)}\cdot t\cdot(2t^2+4)}$$

$$= \frac{1}{2}\int \frac{t^2+1}{t(t^2+2)}\,dt; \qquad \frac{t^2+1}{t(t^2+2)} = \frac{A}{t} + \frac{Bt+C}{t^2+2}; \quad A = \frac{1}{2}, \; B = \frac{1}{2}, \; C = 0.$$

$$= \frac{1}{4}\int \frac{1}{t}\,dt + \frac{1}{8}\int \frac{2t}{t^2+2}\,dt$$
$$= \frac{1}{4}\ln\left|\operatorname{tg}\frac{x}{2}\right| + \frac{1}{8}\ln\left(\operatorname{tg}^2\frac{x}{2}+2\right);$$