A1 [5b]

podílové kritérium

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)!\,(k+1)^3}{p(p+1)\cdots(p+k)(p+k+1)} \cdot \frac{p(p+1)\cdots(p+k)}{k!\,k^3}$$

$$= \frac{(k+1)^4}{(p+k+1)\,k^3} = \frac{\left(1+\frac{1}{k}\right)^4}{\left(\frac{p+1}{k}+1\right)} \to 1 \quad \text{nic neříká}$$

[2]

Raabe: $$k\left(\frac{a_k}{a_{k+1}} - 1\right) = \frac{1}{1/k} \cdot \left(\frac{x(p+1)+1}{(x+1)^4} - 1\right)$$ [1]

stačí počítat $\lim\limits_{x\to 0}$; $\frac{1}{k} = t$.

$$\underbrace{\frac{1}{x}\cdot\left[x(p+1)-1-(x+1)^4\right]}_{0}\cdot\underbrace{\frac{1}{(x+1)^4}}_{\to 1}$$ [1]

l'Hospital „$\frac{0}{0}$":

$$\frac{1}{1}\cdot\left[p+1-4(x+1)^3\right] \to p-3$$

diskuse: $p-3>1$

$p>4$: konverguje

$p<4$: diverguje. [1]

? $p=4$: Raabe neumí rozhodnout.

(A2) $\displaystyle\sum_{k=1}^{\infty} \frac{k+2}{2k-1} \cdot (-1)^{\frac{k(k+1)}{2}} \cdot \frac{1}{\sqrt{k^2+k}}$ [5b]

$$(-1)^{\frac{k(k+1)}{2}} = -1, -1, 1, 1, -1, -1, 1, 1, \dots$$

$\displaystyle\sum_{k=1}^{\infty} (-1)^{\frac{k(k+1)}{2}}$ .. omezené částečné součty. [1]

$\dfrac{1}{\sqrt{k^2+k}} \to 0$; zjevně monotónně.

Dirichlet: $\displaystyle\sum_{k=1}^{\infty} (-1)^{\frac{k(k+1)}{2}} \frac{1}{\sqrt{k^2+k}}$ konverguje. [1]

$$c_k = \frac{k+2}{2k-1} = \frac{1+2/k}{2-1/k} \to \frac{1}{2}$$; tudíž je omezené

? monotónní: $f(x) = \dfrac{x+2}{2x-1}$; $f'(x) = \dfrac{1}{(2x-1)^2} \cdot \left[2x-1-2(x+2)\right]$

c_k .. klesá; $= \dfrac{-5}{(2x-1)^2} < 0$

$\Rightarrow$ Abel: původní řada konverguje. [1]

? absolutní konvergence: konverguje neabsolutně.

$$|a_k| = \frac{k+2}{2k-1} \cdot \frac{1}{\sqrt{k^2+k}} \sim \frac{1}{k}$$ $\Rightarrow$ diverguje: srovnávací krit.

& $\sum \frac{1}{k}$ div.

dk: $$\frac{|a_k|}{1/k} = k|a_k| = \underbrace{\frac{k+2}{2k-1}}_{\to \frac{1}{2}} \cdot \underbrace{\frac{k}{\sqrt{k^2+k}}}_{\to 1} \to \frac{1}{2} \in \mathbb{R} \setminus \{0\}.$$ [2]

(A3) $f_n(t) = \operatorname{arctg}\left(\frac{n^2x^2}{n^4+x^4}\right)$. [5b]

$$\frac{n^2x^2}{n^4+x^4} = \begin{cases} 0; & x=0 \\ \dfrac{x^2}{n^2+x^4/n^2} \to \dfrac{x^2}{\infty+0} = 0; & x\neq 0. \end{cases}$$

tedy: $f_n(t) \to 0 =: f(t)$; bodová limita. [1]

$I = [0,10]$.

$$\sigma_n = \sup_{x\in[0,10]} |f_n(t)-f(t)| = \sup_{x\in[0,10]} \operatorname{arctg}\left(\frac{n^2x^2}{n^4+x^4}\right).$$

odhad: $\operatorname{arctg} y \le y$; $\forall y \in [0,\infty)$.

$$\Rightarrow \sigma_n \le \sup_{x\in[0,10]} \frac{n^2x^2}{n^4+x^4} \le \sup \frac{x^2}{n^2+\frac{x^4}{n^2}}$$

$$\le \sup \frac{x^2}{n^2} \le \frac{100}{n^2}.$$

tedy: $0 \le \sigma_n \le \frac{100}{n^2} \to 0$; $\sigma_n \to 0$.

$f_n \rightrightarrows 0$ v $[0,10]$. [2]

$I = [0, \infty)$.

$$\sigma_n = \sup_{x \in \mathbb{R}} \operatorname{arctg}\left(\frac{n^2 x^2}{n^4 + x^4}\right) \geq \operatorname{arctg}\left(\frac{n^2 x^2}{n^4 + x^4}\right)\Bigg|_{x=n}$$

$$= \operatorname{arctg}\left(\frac{n^4}{2n^4}\right) = \operatorname{arctg}\frac{1}{2}.$$

$\sigma_n \geq \operatorname{arctg}\frac{1}{2} > 0$; tedy $\sigma_n \not\to 0$; $f_n \not\rightrightarrows 0$ v $\mathbb{R}$.

[2]

Pozn.: $\dfrac{n^2 x^2}{n^4 + x^4}$ – má extrémy v $x = \pm n$

-- lze zjistit derivováním.

(B1) $\displaystyle\sum_{k=1}^{\infty} \frac{\ln\left(1+\frac{1}{k}\right)}{(k+1)^{q+1/k}} \cdot \frac{1}{\operatorname{arctg} k}$. [5b]

$$a_k = \frac{\ln\left(1+\frac{1}{k}\right)}{(k+1)^q} \cdot \underbrace{\frac{1}{\operatorname{arctg} k}}_{\to \frac{2}{\pi}} \cdot \underbrace{\frac{1}{(k+1)^{1/k}}}_{\to 1} > 0.$$

$$(k+1)^{\frac{1}{k}} = \exp\Big[\underbrace{\tfrac{1}{k}\ln(k+1)}_{\to 0}\Big] \to 1.$$

[2]

tedy $a_k \sim \dfrac{\ln\left(1+\frac{1}{k}\right)}{(k+1)^q}$;

dále: $\ln\left(1+\frac{1}{k}\right) \sim \frac{1}{k}$; $(k+1)^q \sim k^q$; celkem $a_k \sim \dfrac{1}{k^{q+1}}$ [2]

$\sum a_k$ konverguje

$\iff q+1 > 1$

$\underline{q > 0}$. [1]

(B2) $\sum a_k x^k$; [5b]

$R = ?$: $\left|\frac{a_{k+1}}{a_k}\right| = \frac{3^{k+1}(k+1)!}{(3+\pi)\cdots(3k+3+\pi)} \cdot \frac{(3+\pi)\cdots(3k+\pi)}{3^k k!}$

$$= \frac{3(k+1)}{3k+3+\pi} = \frac{3(1+\frac{1}{k})}{3+\frac{3+\pi}{k}} \to 1.$$

tedy $R = 1$. [1]

$x = -R = -1$: $\sum a_k (-1)^k = \sum \underbrace{\frac{3^k k!}{(3+\pi)\cdots(3k+\pi)}}_{b_k > 0}$.

$\frac{b_{k+1}}{b_k} = \left|\frac{a_{k+1}}{a_k}\right| = \frac{3(1+\frac{1}{k})}{3+\frac{3+\pi}{k}} \to 1$; podílové krit. neříká nic. [2]

Raabe: $k\left(\frac{b_k}{b_{k+1}} - 1\right) = \frac{1}{1/k}\left(\frac{3+\frac{3+\pi}{k}}{3(1+\frac{1}{k})} - 1\right)$

subst. $\frac{1}{k} = x \to 0$ $= \underbrace{\frac{1}{x}\cdot\left[3 + x(3+\pi) - 3(1+x)\right]}_{\text{l'Hosp. } „\frac{0}{0}"} \cdot \underbrace{\frac{1}{3(1+x)}}_{\to 3}$

$\frac{1}{1}\cdot\left[3+\pi-3\right] \to \frac{\pi}{3} > 1$: konverguje.

[2]

B3 $\quad f_n(x) = \sin\left(\dfrac{\pi}{nx^2+2}\right).$

$$\frac{\pi}{nx^2+2} = \begin{cases} \frac{\pi}{2}\,; & x=0 \\ \to \dfrac{\pi}{\infty\cdot x^2+2} = \dfrac{\pi}{\infty} = 0\,; & x\neq 0. \end{cases}$$

sudčíz: $f_n(x) \to f(x) = \begin{cases} 1\,; & x=0 \\ 0\,; & x\neq 0. \end{cases}$ [1]

a)

$f(x)$ nespojitá v $x=0$; $f_n(x)$ spojité. $\Rightarrow$ $f_n \not\rightrightarrows f(x)$

na žádném okolí 0, tedy ani v $[0,10]$. (2)

b) $x\in[10,\infty)$.

$$\sigma_n = \sup_{x\geq 10} |f_n(x) - \underbrace{f(x)}_{\substack{=0 \\ \text{pro } x\geq 10}}| = \sup_{x\geq 10} \left|\sin\left(\frac{\pi}{nx^2+2}\right)\right|$$

odhad: $|\sin y| \leq |y| \;\forall y\in\mathbb{R}$;

tedy: $\sigma_n \leq \sup\limits_{x\geq 10} \left|\dfrac{\pi}{nx^2+2}\right| \leq \dfrac{\pi}{100n} \to 0.$

$\sigma_n \to 0$; tedy $f_n \rightrightarrows 0$ v $[10,\infty)$.

[2]