Mechanical oscillators with fully implicit damping terms

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Abstract

We study vibrations of a lumped parameter system, consisting of a linear spring and a dashpot with general implicit relationship between the damping force and the velocity. We prove the global existence result, and discuss conditions that are sufficient for uniqueness as well as some counterexamples.

1 Introduction

The aim of this paper is to study system of equations

$$m\ddot{x} + F_d + kx = F, \tag{1}$$

$$g(\dot{x}, F_d) = 0. \tag{2}$$

Here x is the (unknown) displacement, m is the mass (we henceforth set $m = 1$ for the sake of simplicity), F is the given external forcing, and F_d is the damping force.
We are particularly interested in the cases where the damping law (2) is fully implicit, meaning that it can be (globally) written neither as $F_d = \hat{g}(\dot{x})$ nor as $\dot{x} = \hat{g}(F_d)$. There are numerous physically meaningful examples of such dissipation; the most notable instance being the Coulomb law friction.
Such problems are usually solved by an ad-hoc method of patching together solutions for $\dot{x}$ negative or positive, see for example [3]. The obtained result is physically correct, it is still desirable to put the analysis on a firmer basis, by proving that the problem is well-posed in a suitable, clearly defined sense.
The paper is organized as follows: Section 2 specifies the precise mathematical assumptions, and collects several auxiliary results, in particular concerning maximal monotone relations. Section 3 contains main results: we prove global existence of solution and (under slightly stronger conditions) its uniqueness. Section 4 brings several examples where the solutions can become non-unique.

2 Preliminaries

We will henceforth assume that (2) is specialized to

$$F_d = F_c - \gamma(\dot{x}), \tag{3}$$
$$(F_c, \dot{x}) \in \mathcal{A}. \tag{4}$$

where $\gamma(\cdot)$ is a relaxation function a F_c is Coulomb-like force, being in a general monotone relation with $\dot{x}$. It is assumed that

$$\gamma \text{ is continuous}, \qquad |\gamma(u)| \leq c_1(1 + |u|), \tag{5}$$

and $\mathcal{A}$ is maximal monotone relation, meaning that

$$(F, y),\ (\tilde{F}, \tilde{y}) \in \mathcal{A} \implies (F - \tilde{F})(y - \tilde{y}) \geq 0, \tag{6}$$
$$(F_0 - F)(y_0 - y) \geq 0 \quad \forall (F, y) \in \mathcal{A} \implies (F_0, y_0) \in \mathcal{A}. \tag{7}$$

We will also need some coercivity with respect to force:

$$(F, y) \in \mathcal{A} \implies Fy \geq c_2 F^2 - c_3. \tag{8}$$

One can see that (together with maximality) this implies that $\mathcal{A}$ can be satisfied for arbitrary values the second argument.
A typical example of maximal monotone relationship of this type is the Coulomb law of friction, given by the conditions

$$\begin{aligned} &\dot{x} > 0 \implies F_d = \phi_0, \\ &\dot{x} < 0 \implies F_d = -\phi_0, \\ &\dot{x} = 0 \implies F_d \in [-\phi_0, \phi_0] \quad \text{is arbitrary.} \end{aligned} \tag{9}$$

See Figure 1. Note that such relationship is truly implicit, i.e., one can (globally) write neither F_d as a function of $\dot{x}$, nor $\dot{x}$ as a function of F_d.
Assuming that the relaxation function is bounded, increasing and $\gamma(0) = 0$, we come to a more general model (see Figure 2).
Regarding the analysis, we would like to emphasize that continuity of γ only guarantees existence of solutions; uniqueness requires a stronger assumption, namely that γ is Lipschitz continuous. See also last section for several examples of non well-posed problems.
Our final assumption is that

$$F = F(t) \in L^2(0, T). \tag{10}$$

Let us conclude this section with two results that will be instrumental in handling the monotone relationship between force and velocity. Firstly, we recall a useful "geometric" characterization of maximal monotone graphs.

Lemma 1. *Relation $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}$ is maximal monotone if and only if there exists a 1-Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ such that*

$$(F, y) \in \mathcal{A} \iff F - y = \varphi(F + y). \tag{11}$$

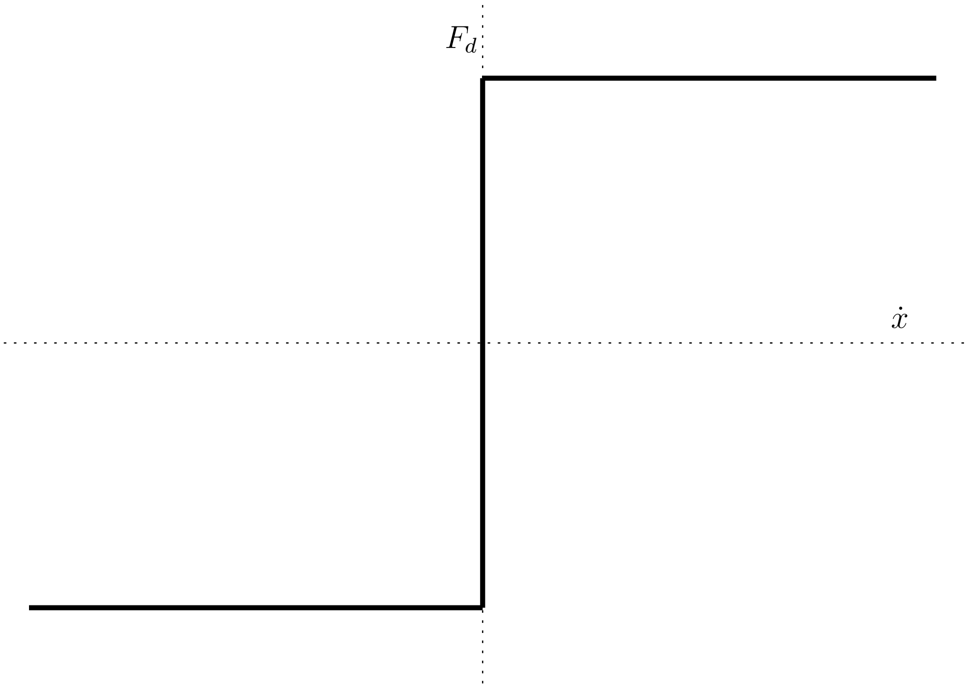

Figure 1: Coulomb friction: force-velocity relationship

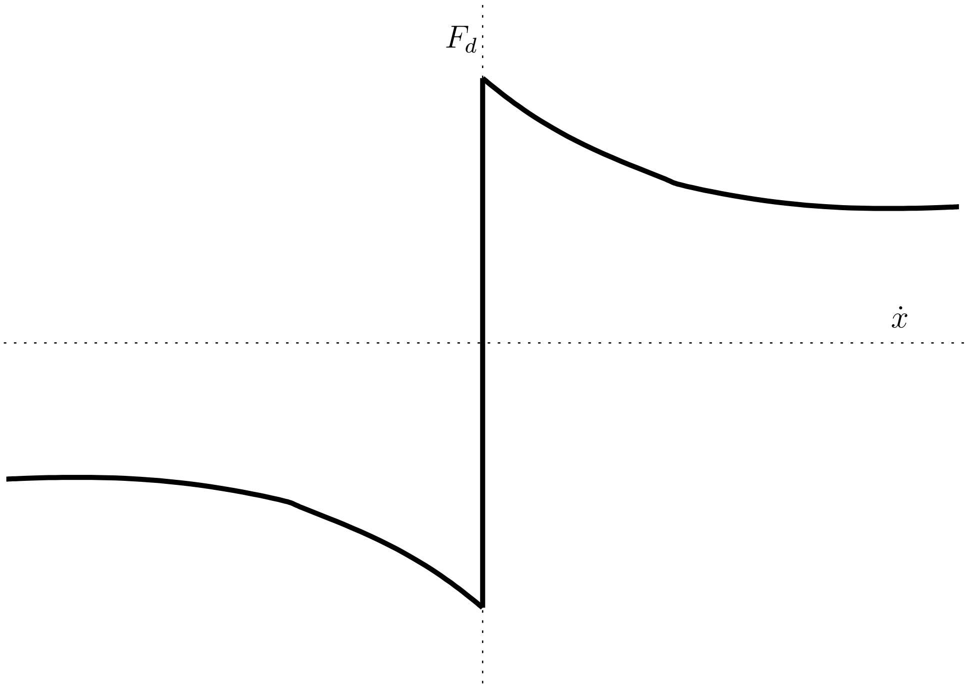

Figure 2: Coulomb friction with relaxation

Proof. See [1, Lemma 2.1]. □

The following result on weak closeness is also well-known and we only give the proof for the sake of reader's convenience.

Lemma 2. *Let F^n, y^n converge to F, y weakly in $L^2(0,T)$, and let $F^n y^n \to Fy$ in the sense of distributions. If $(F^n(t), y^n(t)) \in \mathcal{A}$ for a.e. $t \in (0,T)$, where $\mathcal{A}$ is maximal monotone, then also $(F(t), y(t)) \in \mathcal{A}$ for a.e. $t \in (0,T)$.*

Proof. Let w be arbitrary, smooth function, compactly supported in $(0,T)$. We have

$$|F^n - y^n - \varphi(w)|^2 \le |F^n + y^n - w|^2$$

a.e. in $(0,T)$, where φ is provided by Lemma 1. This is equivalent to

$$-2(F^n - y^n)\varphi(w) + |\varphi(w)|^2 \le -2(F^n + y^n)w + |w|^2 + 4F^n y^n.$$

Here we can take the limit $n \to \infty$, and backward manipulation yields

$$|F - y - \varphi(w)|^2 \le |F + y - w|^2.$$

By density argument, we replace w by arbitrary square integrable function; in particular, taking $w = F + y$ concludes the proof, in view of (11). □

Theorem 1 (Uniform contraction theorem.)**.** *Let X, Y be nonempty Banach spaces; let the mapping $G : X \times Y \to X$ satisfies*

$$\|G(x,y) - G(\tilde{x},y)\|_X \le \alpha \|x - \tilde{x}\|_X, \tag{12}$$
$$\|G(x,y) - G(x,\tilde{y})\|_X \le \beta \|y - \tilde{y}\|_Y, \tag{13}$$

for arbitrary x, $\tilde{x} \in X$, y, $\tilde{y} \in Y$ with some constants $\alpha \in (0,1)$ and $\beta > 0$. Then for any $y \in Y$ there exists unique $x \in X$ such that $x = G(x,y)$ and moreover, the mapping $y \mapsto x$ is Lipschitz continuous.

Proof. See [2, §1, Theorem (3.2)] for a proof of more general result. □

3 Main results.

For the sake of definiteness, we start with the definition of the concept of solution.

Definition 1. *By solution we understand a couple $(F_d, x) \in L^2(0,T) \times W^{2,2}(0,T)$ such that* (1)*,* (2) *hold a.e. in $(0,T)$.*

Note that it follows that x and $\dot{x}$ have absolutely continuous representatives; in particular, the problem is naturally equipped with initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = x_1. \tag{14}$$

Theorem 2. *Let the assumptions (5–8) be in force; let moreover the function γ be Lipschitz continuous. The the problem* (1), (3), (4) *has at most one solution, subject to initial conditions* (14).

Proof. Let (x^1, F_d^1) and (x^2, F_d^2) be two solutions with the same initial condition. Setting $z = x^1 - x^2$, we deduce

$$\ddot{z} + F_d^1 - F_d^2 + kz = 0.$$

Multiplying by $2\dot{z}$, and noting that monotonicity of $\mathcal{A}$ implies

$$(F_d^1 - F_d^2)(\dot{x}^1 - \dot{x}^2) \geq -(\gamma(\dot{x}^1) - \gamma(\dot{x}^2))(\dot{x}^1 - \dot{x}^2),$$

we arrive at

$$\frac{d}{dt}\big((\dot{z})^2 + kz^2\big) \leq 2L(\dot{z})^2,$$

where L is the Lipschitz constant of γ. Since $z(0) = \dot{z}(0) = 0$, we conclude by Gronwall's lemma. □

Theorem 3. *Let the assumptions (5–8) be in force. Then the problem* (1), (3), (4) *has at least one solution, subject to initial conditions* (14).

Proof. Assume first that γ is a globally Lipschitz function. The relation between F_d and $\dot{x}$ can be written as

$$(F_d + \gamma(\dot{x}), \dot{x}) \in \mathcal{A}.$$

In view of Lemma 1, this is equivalent to

$$F_d = \dot{x} - \gamma(\dot{x}) + \varphi(F_d + \gamma(\dot{x}) + \dot{x}).$$

Replacing $\dot{x}$ by $\dot{x} - F_d/n$, we obtain a sequence of approximating problems

$$\big(1 + \tfrac{1}{n}\big)\, F_d = \dot{x} - \gamma(\dot{x} - \tfrac{1}{n}F_d) + \varphi\big((1 - \tfrac{1}{n})F_d + \gamma(\dot{x} - \tfrac{1}{n}F_d) + \dot{x}\big). \tag{15}$$

For n large enough (depending on the Lipschitz constant of γ), the right-hand side is a uniform contraction in F_d. It follows from Theorem 1 that last formula is equivalent to

$$F_d = g_n(\dot{x}),$$

and g_n is globally Lipschitz function. The corresponding ODE reads

$$\ddot{x} + g_n(\dot{x}) + kx = F, \tag{16}$$

and is globally solvable by the standard theory – see [4, Theorem 2.4.5], for example. We need a priori estimates that are independent of n. Now (15) is equivalent to

$$\big(F_d + \gamma(\dot{x} - \tfrac{1}{n}F_d), \dot{x} - \tfrac{1}{n}F_d\big) \in \mathcal{A}.$$

By coercivity assumption (8),

$$\big(F_d + \gamma(\dot{x} - \tfrac{1}{n}F_d)\big)\big(\dot{x} - \tfrac{1}{n}F_d\big) \geq c_1\big|F_d + \gamma(\dot{x} - \tfrac{1}{n}F_d)\big|^2 - c_2.$$

For n large enough it follows that

$$F_d\dot{x} \geq c_3|F_d|^2 - c_4(\dot{x})^2 - c_5. \tag{17}$$

Hence, multiplying (16) by $2\dot{x}$ yields

$$\frac{d}{dt}\big((\dot{x})^2 + kx^2\big) + c_6F_d^2 \leq c_7\big((\dot{x})^2 + F(t)^2 + 1\big).$$

Consequently, the sequence $(F_d^n, \dot{x}^n)$ of solutions to (16) is, at least for n large enough, bounded in $L^2(0,T) \times W^{2,2}(0,T)$, and there is a subsequence (not relabelled) such that

$$F_d^n \to F_d,\ \ddot{x}^n \to \ddot{x} \qquad \text{weakly in } L^2(0,T), \tag{18}$$

$$\dot{x}^n \to \dot{x},\ x^n \to x \qquad \text{uniformly on } [0,T]. \tag{19}$$

Obviously, the limiting couple satisfies (1); it remains to verify the relationship between F_d and $\dot{x}$. But this is straightforward: we have for each n

$$\big(F^n, y^n\big) \in \mathcal{A}, \tag{20}$$

$$F^n = F_d^n + \gamma(\dot{x}^n - \tfrac{1}{n}F_d^n), \tag{21}$$

$$y^n = \dot{x}^n - \tfrac{1}{n}F_d^n. \tag{22}$$

Obviously, $F^n \to F_d + \gamma(\dot{x})$ weakly, and $y^n \to \dot{x}$ strongly in $L^2(0,T)$, and thus $(F_d + \gamma(\dot{x}), \dot{x}) \in \mathcal{A}$ by Lemma 2.
Finally, if γ is a general continuous function, we can approximate it by a locally uniformly converging sequence of Lipschitz functions γ_n, which can be chosen to satisfy (5) with the same constant c_1. The corresponding a priori estimates and passage limit are analogous to the above and we omit further details.

□

4 Some examples of nonuniqueness

The aim of this Section is to look at some examples showing that our problem can lack the property of uniqueness, should one go beyond the assumptions of Theorem 2.
Consider, as a first example, a generalized Coulomb law of friction, given by

$$\begin{aligned} \dot{x} > 0 &\implies F_d = \phi_0, \\ \dot{x} < 0 &\implies F_d = -\phi_0, \\ \dot{x} = 0 &\implies F_d \in [-\phi_1, \phi_1] \quad \text{is arbitrary,} \end{aligned} \tag{23}$$

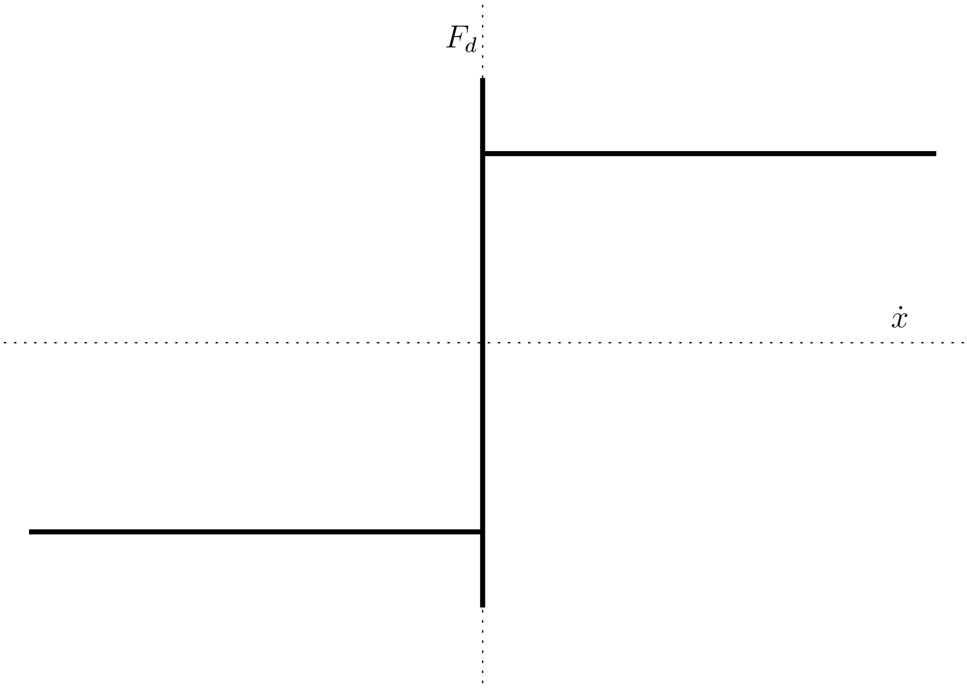

Figure 3: Generalized Coulomb law

where $0 < \phi_0 < \phi_1$. This relationship between F_d and $\dot{x}$ is not monotone (see Figure 3). It corresponds to a physically meaningful situation, where the static friction is larger than a dynamics friction. However, a possible drop of the friction force at the onset of motion means that the problem is not uniquely solvable.
Consider the data

$$\begin{aligned} x(0) = \dot{x}(0) = 0, \\ k = 1,\ F(t) \equiv \phi_1. \end{aligned} \tag{24}$$

Clearly, $F_d \equiv \phi_1$, $x \equiv 0$ is a solution on arbitrary interval $[0, T]$. On the other hand, $F_d \equiv \phi_0$ and

$$x(t) = (\phi_1 - \phi_0)(1 - \cos t),$$

is also a solution, subject to the same data, at least on the interval $[0, \pi]$ where $\dot{x} \geq 0$.
As a second – and perhaps a more interesting – example we consider constitutive relations (3), (4), where $\mathcal{A}$ is the standard Coulomb law (9), and for the relaxation function we take

$$\gamma(u) = 3\sqrt[3]{u^2}\,\mathrm{sgn}\,u.$$

Note that γ grows steeply close to 0 (see Figure 4). This lack of Lipschitz continuity gives rise to nonuniqueness of solutions, as we will presently see. Consider the following data:

$$\begin{aligned} x(0) = \dot{x}(0) = 0, \\ k = 1,\ F(t) \equiv \phi_0. \end{aligned} \tag{25}$$

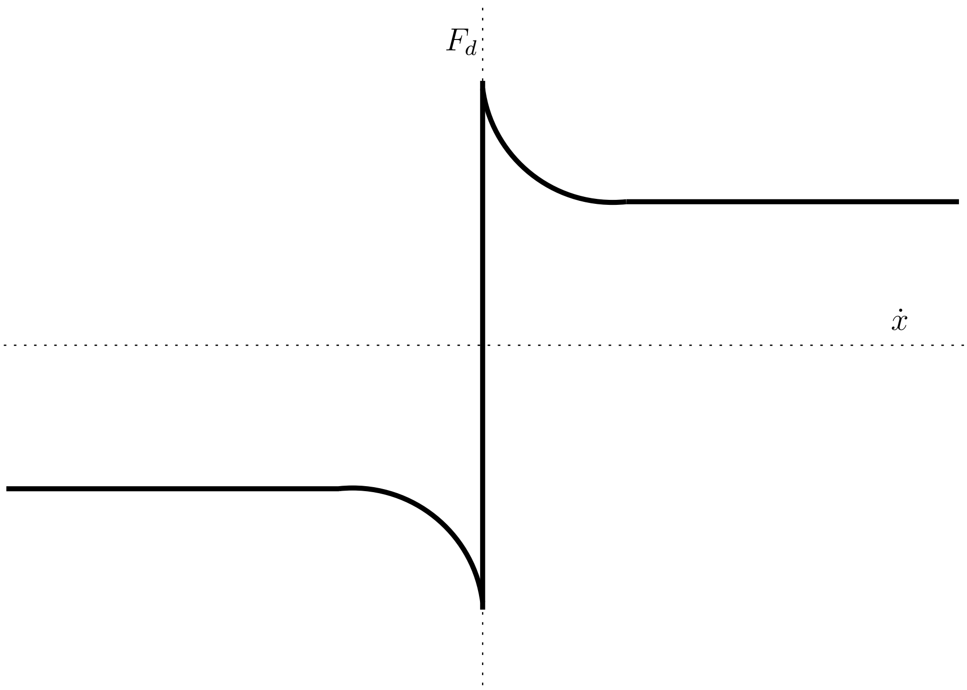

Figure 4: Coulomb law with steep relaxation

Then the function $F_d \equiv \phi_0$, coupled to any x satisfying

$$\begin{aligned} \ddot{x} + x &= 3\sqrt[3]{(\dot{x})^2}, \quad \dot{x} \geq 0, \\ x(0) &= \dot{x}(0) = 0, \end{aligned} \tag{26}$$

gives rise to a solution to our problem. Thus, it remains to consider the reduced problem (26). Obviously, $x \equiv 0$ is solution on arbitrary interval $[0, T]$.
We will now construct a second, nontrivial solution by the means of a fixed point argument. Set

$$\mathcal{Y} = \left\{ x \in C^1([0, \delta]);\ x(0) = \dot{x}(0) = 0,\ at^3 \leq \dot{x}(t) \leq t^3\ \forall t \in [0, \delta] \right\}$$

The constants a, $\delta > 0$ will be determined below. We introduce operator

$$\begin{aligned} \mathcal{T} : \mathcal{Y} &\to C^1([0, \delta]) \\ \left[\mathcal{T}x\right](t) &= y(t),\ t \in [0, \delta] \end{aligned}$$

where y solves

$$\begin{aligned} \ddot{y} &= 3\sqrt[3]{(\dot{x})^2} - x, \\ y(0) &= \dot{y}(0) = 0. \end{aligned}$$

Let us show that an appropriate choice of constants a and δ implies $\mathcal{T}(\mathcal{Y}) \subset \mathcal{Y}$. Indeed, for any $x \in \mathcal{Y}$ one has

$$at^4/4 \leq x(t) \leq t^4/4, \quad t \in [0, \delta],$$

hence

$$\dot{y}(t) = \int_0^t \ddot{y}(s)\, ds = \int_0^t 3(\dot{x}(s))^{2/3} - x(s) ds \leq \int_0^t 3s^2 ds = t^3.$$

Similarly,

$$\dot{y}(t) \geq \int_0^t 3a^{2/3}s^2 - s^4/4ds \geq \int_0^t \big(3a^{2/3} - \delta^2/4)s^2 ds = \big(a^{2/3} - \delta^2/12\big)t^3, \qquad t \in [0, \delta].$$

So it is enough to have

$$a^{2/3} - \delta^2/12 \geq a,$$

which can be satisfied for arbitrary $a \in (0, 1)$, taking a small enough $\delta > 0$.
Finally, we observe that $\mathcal{T}$ is compact in virtue of a bounded second derivative of y, and the set $\mathcal{Y}$ is convex. By Schauder's theorem ([2, §6, Theorem (3.2)]), $\mathcal{T}$ has a fixed point $y \in \mathcal{Y}$, which is the desired nontrivial solution to (26).

References

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