Chapter X

Perturbed Linear Systems

This chapter is concerned with methods for the asymptotic integrations of differential equations $\xi' = E\xi + F(t, \xi)$ which can be considered as perturbations of linear systems with constant coefficients $\xi' = E\xi$.

The first section of the chapter concerns the simple but important case $E = 0$. Since a very easy argument, which has wide applications, gives the desired result in this case, it seems worth isolating it.

One of the most important methods to be used for an arbitrary E is based on a simple topological principle, discussed in §§ 2–3. This principle has wide applications beyond the scope of this chapter. A very different method for obtaining results analogous to those of §§ 13 and 16 is discussed in Part III of Chapter XII.

In this chapter, for convenience and generality, we shall allow the components of ξ to be complex-valued, so that a linear change of coordinates permits the assumption that E is in a suitable normal form; cf. § IV 9. Correspondingly, if ξ_1, ξ_2 are two vectors, then $\xi_1 \cdot \xi_2$ denotes the scalar product $\sum_k \xi_1^k \bar{\xi}_2^k$.

1. The Case $E = 0$

This section concerns the equation

$$\xi' = F(t, \xi), \tag{1.1}$$

where F is "small" in a suitable sense. The main results are the following:

Theorem 1.1. *Let $F(t, \xi)$ be continuous for $t \geqq 0$, $\|\xi\| < \delta\,(\leqq \infty)$ and satisfy*

$$\|F(t, \xi)\| \leqq \psi(t)\, \|\xi\|, \tag{1.2}$$

where $\psi(t)$ is a continuous function for $t \geqq 0$ such that

$$\int_0^\infty \psi(t)\, dt < \infty. \tag{1.3}$$

If $\|\xi_0\|$ *is sufficiently small, say*

$$\|\xi_0\| \exp \int_0^\infty \psi(t)\, dt < \delta, \tag{1.4}$$

then a solution $\xi(t)$ *of* (1.1) *satisfying* $\xi(0) = \xi_0$ *exists for* $t \geqq 0$. *Furthermore, if* $\xi(t)$ *is a solution of* (1.1) *for large* t, *say* $t \geqq t_0$, *then*

$$\xi_\infty = \lim_{t\to\infty} \xi(t) \tag{1.5}$$

exists and $\xi_\infty \neq 0$ *unless* $\xi(t) \equiv 0$.

In other words, the solutions of (1.1) for large t behave like the solutions of $\xi' = 0$, namely, like constants. Theorem 1.1 has the following extension:

Theorem 1.2. *Let* $F(t, \xi)$ *be as in Theorem* 1.1, *and let* ξ_∞ *be an arbitrary vector such that*

$$\|\xi_\infty\| \exp \int_0^\infty \psi(t)\, dt < \delta\ (\leqq \infty). \tag{1.6}$$

Then (1.1) *has at least one solution* $\xi(t)$ *for* $t \geqq 0$ *satisfying* (1.5). *If, in addition,* $F(t, \xi)$ *satisfies the following type of Lipschitz condition*

$$\|F(t, \xi_1) - F(t, \xi_2)\| \leqq \psi(t)\, \|\xi_1 - \xi_2\|, \tag{1.7}$$

then for a given ξ_∞, *there is at most one solution* $\xi(t)$ *of* (1.1) *which exists for large* t *and satisfies* (1.5).

The last part of Theorem 1.2 states that condition (1.7) establishes a one-to-one correspondence between solutions $\xi = \xi_\infty = \text{const.}$ of $\xi' = 0$ and solutions of (1.1), with the understanding that $\|\xi_\infty\|$ is sufficiently small when $\delta < \infty$.

Proof of Theorem 1.1. Multiply (1.1) scalarly by ξ, so that (1.1), (1.2) imply that

$$|\xi \cdot \xi'| \leqq \psi(t)\, \|\xi\|^2. \tag{1.8}$$

Since $d\, \|\xi(t)\|^2/dt = 2 \operatorname{Re} \xi \cdot \xi'$, a quadrature gives

$$\|\xi(t_0)\| \exp - \left| \int_{t_0}^t \psi(s)\, ds \right| \leqq \|\xi(t)\| \leqq \|\xi(t_0)\| \exp \left| \int_{t_0}^t \psi(s)\, ds \right|, \tag{1.9}$$

if $\xi(t)$ exists on a t-interval containing t and t_0, where $t \gtreqless t_0$. In particular, if $t_0 = 0$ and $\xi(0) = \xi_0$ satisfies (1.4), then $\xi(t)$ exists for $t \geqq 0$.

More generally, if $\xi(t)$ exists for $t \geqq t_0$, then it is bounded,

$$\|\xi(t)\| \leqq \|\xi(t_0)\|\, M(t_0), \qquad \text{where} \quad M(t_0) = \exp \int_{t_0}^\infty \psi(t)\, dt. \tag{1.10}$$

Hence, (1.1), (1.2) show that $\|\xi'(t)\| \leqq \psi(t)\, \|\xi(t_0)\|\, M(t_0)$. Consequently,

$\int^{\infty} \xi'(t)\,dt$ is absolutely convergent and so the limit (1.4) exists. In fact

$$\|\xi(t) - \xi_\infty\| \leqq \|\xi(t_0)\|\, M(t_0) \int_t^\infty \psi(s)\,ds \qquad \text{for} \quad t \geqq t_0. \tag{1.11}$$

Note that the inequality (1.9) shows that $\xi(t) \equiv 0$ if and only if $\xi(t)$ vanishes at some point t_0. When $\xi(t) \not\equiv 0$, the first inequality in (1.9) implies that $\xi_\infty \neq 0$. This proves Theorem 1.1.

Proof of Theorem 1.2. Consider first the existence assertion. For a given $t_0 \geqq 0$, let $\xi = \xi(t, t_0)$ be a fixed solution (which is not necessarily unique) of the initial value problem

$$\xi' = F(t, \xi), \qquad \xi(t_0) = \xi_\infty. \tag{1.12}$$

Since (1.9) holds for any t at which $\xi(t, t_0)$ exists, it follows from (1.6) and $\xi(t_0) = \xi_\infty$, that $\xi(t, t_0)$ exists for $t \geqq 0$. Also, $\|\xi(t, t_0)\| \leqq \|\xi_\infty\|\, M(0) < \delta$, where $M(0)$ is defined in (1.10). Hence (1.2) shows that $\|\xi'(t, t_0)\| \leqq \psi(t)\, \|\xi_\infty\|\, M(0)$ for all $t \geqq 0$; thus, for $0 \leqq t \leqq t_0$,

$$\|\xi(t, t_0) - \xi_\infty\| \leqq M(0)\, \|\xi_\infty\| \int_t^{t_0} \psi(s)\,ds. \tag{1.13}$$

In particular, the family of functions $\xi(t, t_0)$ are uniformly bounded and equicontinuous on every bounded t-interval. Hence there exists a sequence $t_1 < t_2 < \ldots$ of t_0-values such that $t_n \to \infty$ as $n \to \infty$, and

$$\xi(t) = \lim \xi(t, t_n)$$

exists uniformly on every bounded t-interval. Furthermore $\xi = \xi(t)$ is a solution of (1.1). Putting $t_0 = t_n$ in (1.13) and letting $n \to \infty$, with t fixed, gives

$$\|\xi(t) - \xi_\infty\| \leqq M(0)\, \|\xi_\infty\| \int_t^\infty \psi(s)\,ds. \tag{1.14}$$

This implies (1.5) and completes the existence proof.

Uniqueness will now be proved under the assumption (1.7). Let $\xi = \xi_1(t), \xi_2(t)$ be two solutions of (1.1) for large t, say $t \geqq T$, satisfying (1.5). Let $\xi(t) = \xi_1(t) - \xi_2(t)$. Then (1.1) and (1.7) give (1.8), hence (1.9) for $t_0 \geqq t \geqq T$. If t is fixed and $t_0 \to \infty$ in (1.9), it follows that $\xi(t) = 0$ since $\xi(t_0) \to 0$ as $t_0 \to \infty$. This completes the proof of Theorem 1.2.

The majorant $\psi(t)\, \|\xi\|$ in (1.2) involving a factor $\|\xi\|$ is convenient in Theorems 1.1 and 1.2 only to assure that certain solutions exist for $t \geqq 0$. A simpler result involving existence for "large t" is given in the following exercise.

Exercise 1.1. Let $F(t, \xi)$ be continuous on a product set $\{t \geqq 0\} \times D$, where D is a bounded open ξ-set. Let F satisfy $\|F(t, \xi)\| \leqq \psi(t)$, $t \geqq 0$

and $\xi \in D$, for some continuous function $\psi(t)$ satisfying (1.3). (*a*) Let $\xi_0 \in D$. Then there exists a number T, depending only on dist $(\xi_0, \partial D)$ and the function $\psi(t)$, such that if $t_0 \geqq T$, then a solution $\xi(t)$ of (1.1) satisfying $\xi(t_0) = \xi_0$ exists for $t \geqq T$. Furthermore, any solution $\xi(t)$ of (1.1) for large t has a limit ξ_∞ as $t \to \infty$. (*b*) Let $\xi_\infty \in D$. Then there exists a number T, depending only on dist $(\xi_\infty, \partial D)$ and the function $\psi(t)$, such that (1.1) has a solution $\xi(t)$ for $t \geqq T$ satisfying (1.5).

Exercise 1.2. Show that Theorem 1.1 and the first part of Theorem 1.2 remain valid if $F(t, \xi)$ is continuous for $t \geqq 0$ and all ξ, if ξ_0 and ξ_∞ are arbitrary, and if (1.2) is replaced by

$$(1.2') \qquad |\xi \cdot F(t, \xi)| \leqq \psi(t)\varphi(\|\xi\|^2),$$

where $\psi(t)$ is as in Theorem 1.1 and $\varphi(r)$ is continuous for $r \geqq 0$ and satisfies

$$(1.3') \qquad \int^\infty \frac{dr}{\varphi(r)} = \infty.$$

Theorem 1.1 and 1.2 have corollaries for the case that (1.1) is replaced by

$$(1.15) \qquad \zeta' = A(t)\zeta + G(t, \zeta),$$

where $A(t)$ is a continuous $d \times d$ matrix. Here, solutions of (1.15) should be compared with

$$(1.16) \qquad \zeta' = A(t)\zeta.$$

Let $Z(t)$ be a fundamental matrix for (1.16), so that the change of variables

$$(1.17) \qquad \zeta = Z(t)\xi$$

transforms (1.15) into

$$(1.18) \qquad \xi' = Z^{-1}(t)G(t, Z(t)\xi).$$

Thus an application of Theorems 1.1 and 1.2 to (1.18) gives

Corollary 1.1. *Let $A(t)$ be continuous for $t \geqq 0$ and $Z(t)$ a fundamental matrix for* (1.16). *Let $G(t, \zeta)$ be continuous for $t \geqq 0$ and all ζ and satisfy*

$$(1.19) \qquad \|Z^{-1}(t)G(t, Z(t)\xi)\| \leqq \psi(t)\, \|\xi\|,$$

where $\psi(t)$ is as in Theorem 1.1. *Let $\zeta(t)$ be a solution of* (1.15) *on some t-interval. Then $\zeta(t)$ exists for $t \geqq 0$,*

$$(1.20) \qquad \xi_\infty = \lim_{t\to\infty} Z^{-1}(t)\zeta(t)$$

exists and $\xi_\infty \neq 0$ unless $\zeta(t) \equiv 0$; conversely, for a given ξ_∞, there is a solution $\zeta(t)$ of (1.15) *satisfying* (1.20).

When $Z(t)$ is bounded for $t \geqq 0$, we can formulate a corresponding result when $G(t, \zeta)$ is only defined for $t \geqq 0$, $\|\zeta\| < \delta < \infty$. In addition, we can obtain an analogue of the uniqueness assertion of Theorem 1.2. When $A(t) \equiv A$ is a constant matrix, Corollary 1.1 takes the following form:

Corollary 1.2. *Let $G(t, \zeta)$ be continuous for $t \geqq 0$ and all ζ and satisfy*

$$(1.21) \qquad \|e^{-At}G(t, e^{At}\xi)\| \leqq \psi(t)\,\|\xi\|,$$

where $\psi(t)$ is as in Theorem 1.1. *Let $\zeta(t)$ be a solution of*

$$(1.22) \qquad \zeta' = A\zeta + G(t, \zeta)$$

on some t-interval. Then $\zeta(t)$ exists for $t \geqq 0$,

$$\xi_\infty = \lim e^{-At}\zeta(t)$$

exists and $\xi_\infty \neq 0$ unless $\zeta(t) \equiv 0$; furthermore, if ξ_∞ is given, there is a solution of (1.22) *for $t \geqq 0$ satisfying* (1.23).

Exercise 1.3. Formulate theorems related to Corollaries 1.1 and 1.2 as Exercises 1.1 and 1.2 are related to Theorems 1.1 and 1.2.

Generally, a result of the type given by Corollary 1.2 is only convenient when $e^{\pm At}$ are bounded for $t \geqq 0$. For example, suppose that $d = 2$ and $A = \text{diag}\,[1, -1]$, so that $e^{At} = \text{diag}\,[e^t, e^{-t}]$. Then, if Corollary 1.2 is applicable, (1.22) has a solution of the form $\zeta = e^t(1 + o(1), o(1))$ as $t \to \infty$, but not necessarily a solution of the form $\zeta = e^{-t}(o(1), 1 + o(1))$ as $t \to \infty$. Furthermore the hypothesis (1.21) can be very severe for the type of conclusion stated in Corollary 1.2. The results obtained in the remainder of this chapter are much better, under less stringent conditions, for the situation just described.

Exercise 1.4. Suppose that (1.1) is a linear homogeneous system, say

$$(1.23) \qquad \xi' = G(t)\xi,$$

where $G(t)$ is a continuous matrix for $t \geqq 0$. The system (1.23) will be said to be of class (*) if (i) every solution $\xi(t)$ of (1.23) has a limit ξ_∞ as $t \to \infty$, and (ii) for every constant vector ξ_∞, there is a solution $\xi(t)$ of (1.23) such that $\xi(t) \to \xi_\infty$ as $t \to \infty$. (*a*) Show that (1.23) is of class (*) if and only if, for one and/or every fundamental matrix $Y(t)$ of (1.23), $Y_\infty = \lim Y(t)$ exists as $t \to \infty$ and is nonsingular (and that this is true if and only if $Y_\infty = \lim Y(t)$ exists as $t \to \infty$ and $\int^\infty \text{tr}\, G(s)\, ds$ converges, possibly conditionally). (*b*) The system (1.23) is of class (*) if and only if the adjoint system $\xi' = -G^*(t)\xi$ is class (*); cf. § IV 7. (*c*) The system

(1.23) is of class (*) if $\int^{\infty} \|G(t)\|\, dt < \infty$ [or, equivalently, if $G(t) = (g_{jk}(t))$ and $\int^{\infty} |g_{jk}(t)|\, dt < \infty$ for $j, k = 1, \ldots, d$]. This is merely a consequence of Theorems 1.1, 1.2. (*d*) Show that (*c*) has the following corollary [which is a refinement of (*c*)]: The system (1.23) is of class (*) if $G_0(t) = \int_t^{\infty} G(s)\, ds$ converges (possibly just conditionally) and either $\int^{\infty} \|G(t)G_0(t)\|\, dt < \infty$ or $\int^{\infty} \|G_0(t)G(t)\|\, dt < \infty$.

2. A Topological Principle

Let y, f be d-dimensional vectors with real- or complex-valued components and $f(t, y)$ a continuous function defined on an open (t, y)-set Ω. Let Ω^0 be an open subset of Ω, $\partial\Omega^0$ the boundary and $\bar{\Omega}^0$ the closure of Ω^0. Recall, from § III 8, that a point $(t_0, y_0) \in \Omega \cap \partial\Omega^0$ is called an egress point of Ω^0, with respect to the system

$$y' = f(t, y), \tag{2.1}$$

if for every solution $y = y(t)$ of (1.1) satisfying the initial condition

$$y(t_0) = y_0, \tag{2.2}$$

there is an $\epsilon > 0$ such that $(t, y(t)) \in \Omega^0$ for $t_0 - \epsilon \leqq t < t_0$. An egress point (t_0, y_0) of Ω^0 is called a strict egress point of Ω^0 if $(t, y(t)) \notin \bar{\Omega}^0$ for $t_0 < t \leqq t_0 + \epsilon$ for a small $\epsilon > 0$. The set of egress points of Ω^0 will be denoted by $\Omega_e{}^0$ and the set of strict egress points by Ω_{se}^0.

If U is a topological space and V a subset of U, a continuous mapping $\pi: U \to V$ defined on all of U is called a *retraction* of U onto V if the restriction $\pi \mid V$ of π to V is the identity; i.e., $\pi(u) \in V$ for all $u \in U$ and $\pi(v) = v$ for all $v \in V$. When there exists a retraction of U onto V, V is called a *retract of* U. This notion can be illustrated by the following examples, which will have applications.

Example 1. Let U be a d-dimensional ball $\|y\| \leqq r$ in the Euclidean y-space and V its boundary sphere $\|y\| = r$. Then V is not a retract of U. For if there exists a retraction $\pi: U \to V$, then there exists a map of U into itself, $y \to -\pi(y)$, without fixed points, which is impossible by the classical fixed point theorem of Brouwer; for the latter, see Hureiwicz and Wallman [1, pp. 40–41].

Example 2. Let C be the "cylinder" which is the product space of a Euclidean sphere $\|y\| = r$ and a Euclidean u-space, so that $C = \{(y, u)\colon \|y\| = r, u \text{ arbitrary}\}$. Let S be a section of C, say, $S = \{(y, u_0)\colon \|y\| \leqq r, u_0 \text{ fixed}\}$; see Figure 1. Then $S \cap C = \{(y, u_0)\colon \|y\| = r,$

u_0 fixed} is a retract of C [as can be seen by choosing the retraction $\pi(y, u) = (y, u_0)$], but $S \cap C$ is not a retract of S by Example 1.

Theorem 2.1. *Let $f(t, y)$ be continuous on an open (t, y)-set Ω with the property that initial values determine unique solutions of* (2.1). *Let Ω^0 be*

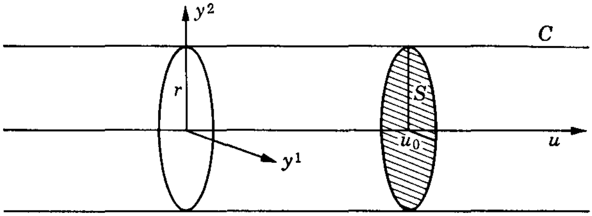

Figure 1.

an open subset of Ω satisfying $\Omega_e^0 = \Omega_{se}^0$; i.e., all egress points of Ω^0 are strict egress points. Let S be a nonempty subset of $\Omega^0 \cup \Omega_e^0$ such that $S \cap \Omega_e^0$ is not a retract of S but is a retract of Ω_e^0. Then there exists at least one point $(t_0, y_0) \in S \cap \Omega^0$ such that the solution arc $(t, y(t))$ of (2.1), (2.2) *is contained in Ω^0 on its right maximal interval of existence.*

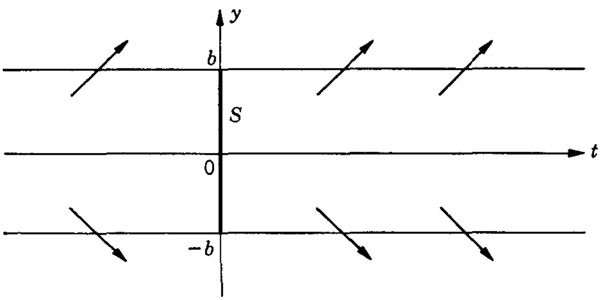

Figure 2.

As an illustration, consider (2.1) where y is a real variable and $f(t, y)$ is continuous on Ω: (t, y) arbitrary. Let Ω^0 be a strip $|y| < b$, $-\infty < t < \infty$; see Figure 2. Thus the part of the boundary of Ω^0 in Ω, i.e., $\partial\Omega^0 \cap \Omega$, consists of the two lines $y = \pm b$. Suppose that $f(t, b) > 0$ and $f(t, -b) < 0$, so that $\Omega_e^0 = \Omega_{se}^0 = \partial\Omega^0 \cap \Omega$. Let S be the line segment $S = \{(t, y)\colon t = 0, |y| \leqq b\}$. Then $S \cap \Omega_e^0$ is the set of two points $(0, \pm b)$ and is a retract of Ω_e^0 but not of S. Thus it follows from Theorem 2.1, that there exists at least one point $(0, y_0)$, $|y_0| < b$, such that a solution of (2.1) determined by $y(0) = y_0$ exists and satisfies $|y(t)| < b$ for $t \geqq 0$.

Proof of Theorem 2.1. Suppose that the theorem is false. Then for $(t_0, y_0) \in S - \Omega_e^0$, there exists a $t_1 = t_1(t_0, y_0)$ such that $t_1 > t_0$ and the solution $y(t)$ of (2.1), (2.2) exists on $t_0 \leqq t \leqq t_1$, $(t, y(t)) \in \Omega^0$ for $t_0 \leqq t < t_1$ and $(t_1, y(t_1)) \in \Omega_e^0$ for $t = t_1$. Define a map π_0: $S \to \Omega_e^0$ as follows: $\pi_0(t_0, y_0) = (t_1, y(t_1))$ if $(t_0, y_0) \in S - \Omega_e^0$ and $\pi_0(t_0, y_0) = (t_0, y_0)$ if $(t_0, y_0) \in S \cap \Omega_e^0$. Since the solutions of (2.1) depend continuously on initial conditions (Theorem V 2.1) and $\Omega_e^0 = \Omega_{se}^0$, it follows that π_0 is continuous. In order to see this, let $y(t) = \eta(t, t^0, y^0)$ be the solution of (2.1) such that $\eta(t^0, t^0, y^0) = y^0$, so that $\eta(t, t^0, y^0)$ is continuous. Suppose that $(t_0, y_0) \in S \cap \Omega^0$, and (t^0, y^0) is near (t_0, y_0), then $\eta(t, t^0, y^0)$ exists on the interval $[t^0, t_1(t_0, y_0) + \epsilon]$ for some ϵ and $(t, \eta(t, t^0, y^0) \in \Omega^0$ on $t^0 \leqq t \leqq t_1(t_0, y_0) - \epsilon$ and $(t, \eta(t, t^0, y^0)) \notin \bar{\Omega}^0$ if $t = t_1(t_0, y_0) + \epsilon$. Thus, $|t_1(t^0, y^0) - t_1(t_0, y_0)| < \epsilon$, and so $(t_1, \eta(t_1(t^0, y^0), t^0, y_0))$ is a continuous function of (t^0, y^0); i.e., π_0 is continuous at (t_0, y_0). A similar argument holds if $(t_0, y_0) \in S \cap \Omega_e^0$.

Let π: $\Omega_e^0 \to S \cap \Omega_e^0$ be a retraction of Ω_e^0 onto $S \cap \Omega_e^0$. Then the composite map $\pi\pi_0$ is a retraction of S onto $S \cap \Omega_e^0$. The existence of such a retraction gives a contradiction and proves the theorem.

Exercise 2.1. Let U be a topological space; V_1, V_2 subsets of U. The set V_1 is called a *quasi-isotopic deformation retract* of V_2 in U if there exists a continuous map π: $V_2 \times \{0 \leqq s \leqq 1\} \to U$ such that (i) $\pi(v_2, 0) = v_2$ for $v_2 \in V_2$; (ii) $\pi(v_1, s) = v_1$ for $v_1 \in V_1$ and $0 \leqq s \leqq 1$; (iii) $\pi(v_2, 1) \in V_1$ for $v_2 \in V_2$; and (iv) for fixed s on $0 \leqq s < 1$, $\pi(v_2, s)$ is a homeomorphism of V_2 onto its image. Let f, Ω, Ω^0 be as in Theorem 2.1; S_1 a subset of Ω_e^0; S a nonempty subset of $\Omega^0 \cup S_1$ such that S_1 is not a quasi-isotopic deformation retract of $S \cup S_1$ in $\Omega^0 \cup S_1$. Then there exists at least one point $(t_0, y_0) \in S \cap \Omega$ such that the solution arc $(t, y(t))$ of (2.1), (2.2) is either in Ω^0 on its right maximal interval of existence or first meets $\partial\Omega^0$ at a point of $S - S_1$.

3. A Theorem of Ważewski

The usefulness of Theorem 2.1 depends on suitable choices of Ω^0. One of the difficulties in the application is the determination of the set of egress points. In some cases to be described, this difficulty can be overcome.

Recall from § III 8 that a real-valued function $u(t, y)$ defined on an open subset of Ω is said to possess a trajectory derivative $\dot{u}(t, y)$ at the point (t_0, y_0) along the solution $y(t)$ of (2.1)–(2.2) if $u(t, y(t))$ has a derivative at $t = t_0$; in this case,

$$\dot{u}(t_0, y_0) = [u(t, y(t))]'_{t=t_0}. \tag{3.1}$$

If y (hence f) has real-valued components and $u(t, y)$ is of class C^1, this trajectory derivative exists and is

$$\dot{u}(t, y) = \partial u/\partial t + (\text{grad}_y\, u) \cdot f, \tag{3.2}$$

where the last term is the scalar product of f and the gradient of u with respect to y.

When y has complex-valued components, a function $u(t, y)$ is said to be of class C^1 if it has continuous partial derivatives with respect to t and the real and imaginary parts of the components of y. Write the kth component y^k of y as $y^k = \sigma^k + i\tau^k$, where σ^k, τ^k are real, so that $\sigma^k = (y^k + \bar{y}^k)/2$, $\tau^k = (y^k - \bar{y}^k)/2i$. This suggests the standard notation, $\partial u/\partial y^k = \frac{1}{2}[\partial u/\partial \sigma^k - i\, \partial u/\partial \tau^k]$ and $\partial u/\partial \bar{y}^k = \frac{1}{2}[\partial u/\partial \sigma^k + i\, \partial u/\partial \tau^k]$. Thus if $\text{grad}_y u = (\partial u/\partial y^1, \ldots, \partial u/\partial y^d)$ and $\text{grad}_{\bar{y}} u = (\partial u/\partial \bar{y}^1, \ldots, \partial u/\partial \bar{y}^d)$, then (3.2) should be replaced by

$$(3.2^*) \qquad \dot{u}(t, y) = \partial u/\partial t + (\text{grad}_y\, u) \cdot \bar{f} + (\text{grad}_{\bar{y}}\, u) \cdot f,$$

as can be seen by writing (2.1) as a system of $2d$ differential equations for $(\sigma, \tau) = (\sigma^1, \ldots, \sigma^d, \tau^1, \ldots, \tau^d)$.

An open subset Ω^0 of Ω will be called a *(u, v)-subset* of Ω with respect to (2.1) if there exists an (arbitrary) number of real-valued continuous functions, $u_1(t, y), \ldots, u_l(t, y)$, $v_1(t, y), \ldots, v_m(t, y)$, on Ω such that

$$(3.3) \qquad \Omega^0 = \{(t, y)\colon u_j(t, y) < 0 \quad \text{and} \quad v_k(t, y) < 0 \text{ for all } j, k\}$$

and if U_α, V_β are the sets

$$(3.4) \qquad \begin{aligned} U_\alpha &= \{(t, y)\colon u_\alpha(t, y) = 0 \text{ and } u_j(t, y) \leqq 0,\ v_k(t, y) \leqq 0 \text{ for all } j, k\}, \\ V_\beta &= \{(t, y)\colon v_\beta(t, y) = 0 \text{ and } u_j(t, y) \leqq 0,\ v_k(t, y) \leqq 0 \text{ for all } j, k\}, \end{aligned}$$

then the trajectory derivatives $\dot{u}_\alpha$, $\dot{v}_\beta$ exist on U_α, V_β and satisfy

$$(3.5) \qquad \dot{u}_\alpha(t, y) > 0 \quad \text{for} \quad (t, y) \in U_\alpha,$$

$$(3.6) \qquad \dot{v}_\beta(t, y) < 0 \quad \text{for} \quad (t, y) \in V_\beta,$$

respectively, along all solutions through (t, y). In this definition, either l or m can be zero.

Lemma 3.1. *Let $f(t, y)$ be continuous to an open (t, y)-set Ω and Ω^0 a (u, v)-subset of Ω with respect to* (2.1). *Then*

$$(3.7) \qquad \Omega_e^0 = \Omega_{se}^0 = \bigcup_{\alpha=1}^{l} U_\alpha - \bigcup_{\beta=1}^{m} V_\beta.$$

Proof. It is clear that $\partial\Omega^0 \cap \Omega \subset (\bigcup U_\alpha) \cup (\bigcup V_\beta)$. In addition, $\Omega_e^0 \cap V_\beta$ is empty, for if $(t_0, y_0) \in V_\beta$ and $y(t)$ is a solution of (2.1), (2.2), then (3.6) shows that $v_\beta(t, y(t) > 0$ for $t_0 - \epsilon \leqq t < t_0$ for small $\epsilon > 0$, so that $(t, y(t)) \notin \Omega^0$. Thus

$$(3.8) \qquad \Omega_{se}^0 \subset \Omega_e^0 \subset (\partial\Omega^0 \cap \Omega) - \bigcup_{\beta=1}^{m} V_\beta \subset \bigcup_{\alpha=1}^{l} U_\alpha - \bigcup_{\beta=1}^{m} V_\beta.$$

Let $(t_0, y_0) \subset \bigcup U_\alpha - \bigcup V_\beta$. Then $u_j(t_0, y_0) \leqq 0$ and $v_k(t_0, y_0) < 0$ for all j, k. By (3.5), there is an $\epsilon > 0$ such that $u_\alpha(t_0, y(t)) < 0$ or > 0 according as $t_0 - \epsilon \leqq t < t_0$ or $t_0 < t \leqq t_0 + \epsilon$ if $(t_0, y_0) \in U_\alpha$, $u_j(t, y(t)) < 0$ for $t_0 - \epsilon \leqq t \leqq t_0 + \epsilon$ if $(t_0, y_0) \notin U_j$, and $v_k(t, y(t)) < 0$ for $t_0 - \epsilon \leqq t \leqq t_0 + \epsilon$ for all k. Hence $(t_0, y_0) \in \Omega^0_{se}$; i.e., $\bigcup U_\alpha - \bigcup V_\beta \subset \Omega^0_{se}$. In view of (3.8), this proves the lemma.

Theorem 3.1. *Let $f(t, y)$ be continuous on an open (t, y)-set Ω with the property that solutions of* (2.1) *are uniquely determined by initial conditions. Let Ω^0 be a (u, v)-subset of Ω with respect to* (2.1). *Let S be a nonempty subset of $\Omega^0 \cup \Omega_e{}^0$ satisfying $S \cap \Omega_e{}^0$ is not a retract of S but is a retract of $\Omega_e{}^0$. Then there exists at least one point $(t_0, y_0) \in S \cap \Omega^0$ such that a solution arc $(t, y(t))$ of* (2.1), (2.2) *is contained in Ω^0 on its right maximal interval of existence.*

This is a corollary of Theorem 2.1 and Lemma 3.1. Sometimes, the requirement of the uniqueness of the solution of (2.1), (2.2) can be omitted:

Corollary 3.1. *Let f, Ω, Ω^0, S be as in Theorem* 3.1. *except that it is not required that solutions of* (2.1) *be uniquely determined by initial conditions. But, in addition, let S be compact and let $u_j(t, y)$, $v_k(t, y)$ be of class C^1 (with respect to t and the real and imaginary parts of the components of y). Then the conclusion of Theorem* 3.1 *is valid.*

Proof. Let $f_1(t, y), f_2(t, y), \ldots$ be a sequence of functions of class C^1 on Ω which tend to $f(t, y)$ uniformly, as $n \to \infty$, on compact subsets of Ω. Let $\Omega_1, \Omega_2, \ldots$ be a sequence of open subsets of Ω, such that $S \subset \Omega_1$, Ω_n has a compact closure $\bar{\Omega}_n \subset \Omega_{n+1}$, and $\Omega = \bigcup \Omega_n$.

By replacing $f_1, f_2, \ldots$ by a subsequence, if necessary, it can be supposed that

$$\partial u_\alpha/\partial t + (\text{grad}_y\, u_\alpha) \cdot \bar{f}_n + (\text{grad}_{\bar{y}}\, u_\alpha) \cdot f_n > 0 \qquad \text{on} \quad U_\alpha \cap \Omega_n,$$

$$\partial v_\beta/\partial t + (\text{grad}_y\, v_\beta) \cdot \bar{f}_n + (\text{grad}_{\bar{y}}\, v_\beta) \cdot f_n < 0 \qquad \text{on} \quad V_\beta \cap \Omega_n.$$

Thus if $\Omega_n{}^0 = \Omega^0 \cap \Omega_n$, then $\Omega_n{}^0$ is a (u, v)-subset of Ω_n with respect to the system

$$y' = f_n(t, y). \tag{3.9}$$

The set of (strict) egress points Ω^0_{ne} of $\Omega_n{}^0$ is $\Omega_e{}^0 \cap \Omega_n$. Hence $\Omega^0_{ne} \cap S = \Omega_e{}^0 \cap S$ is not a retract of S, but $\Omega^0_{ne} \cap S$ is a retract of $\Omega^0_{ne} \subset \Omega_e{}^0$.

Thus by Theorem 2.1 there is a point $(t_n, y_n) \in S$, such that the solution $y = y_n(t)$ of (3.9) satisfying $y_n(t_n) = y_n$ is in $\Omega_n{}^0$ on its right maximal interval of existence $[t_n, \tau_n)$ relative to Ω_n. If (3.9) is considered on Ω, instead of Ω_n, let the right maximal interval of existence of $y_n(t)$ be $[t_n, \omega_n)$, so that $\tau_n \leqq \omega_n \leqq \infty$ and $\tau_n < \omega_n$ implies that $(\tau_n, y_n(\tau_n)) \subset \partial\Omega_n \cap \Omega$.

Since S is compact, there is a point (t_0, y_0) on S which is a cluster point of the sequence of points $(t_1, y_1), (t_2, y_2), \ldots$. By Theorem II 3.2, there exists a solution $y(t)$ of (2.1), (2.2) having a right maximal interval of existence $[t_0, \omega)$ and a sequence of integers $n(1) < n(2) < \cdots$ such that $y_{n(k)}(t) \to y(t)$ uniformly as, $k \to \infty$, on any interval $[t_0, t^*] \subset [t_0, \omega)$. It follows that $(t, y(t)) \subset \bar{\Omega}^0 \cap \Omega$ for $t_0 \leqq t < \omega$. For suppose that there is a t-value t^0, $t_0 < t^0 < \omega$, such that $(t^0, y(t^0)) \notin \bar{\Omega}^0$. Then, for $n = n(k)$ and large k, $(t^0, y_n(t^0)) \notin \bar{\Omega}^0$, so that $(t^0, y_n(t^0)) \notin \bar{\Omega}_n{}^0$. Hence $\tau_n < t^0 < \omega_n$ for $n = n(k)$ and large k. By choosing a subsequence, if necessary, it can be supposed that $\tau = \lim \tau_{n(k)}$ exists as $k \to \infty$, so that $t_0 \leqq \tau \leqq t^0$ and $(\tau_n, y_n(\tau_n)) \to (\tau, y(\tau))$ as $n = n(k) \to \infty$. But this gives a contradiction for $(\tau_n, y_n(\tau_n)) \in \partial\Omega_n \cap \Omega$, where $n = n(k)$, cannot have a limit point $(\tau, y(\tau)) \in \Omega$.

Since $(t_0, y_0) \in S \subset \Omega^0 \cap \Omega_e{}^0$ and $\Omega_e{}^0 = \Omega^0_{se}$, it is seen that $(t_0, y_0) \in \Omega^0$, otherwise $(t, y(t)) \notin \Omega^0$ for $t_0 < t \leqq t_0 + \epsilon$ for some $\epsilon > 0$. By the same argument, $(t, y(t)) \subset \Omega^0$ for $t_0 \leqq t < \omega$. This proves the corollary.

4. Preliminary Lemmas

The theorems of § 3 will be illustrated by using them to obtain results about the asymptotic integrations of

$$\xi' = E\xi + F(t, \xi), \tag{4.1}$$

where E is a constant matrix and $F(t, \xi)$ is "small", say,

$$\|F(t, \xi)\| \leqq \psi(t) \, \|\xi\| \tag{4.2}$$

and $\psi(t)$ is "small" for large t. In this section, we state the basic Lemmas 4.1, 4.2, 4.3. Their proofs are given in §§ 5–7 using the results of § 3. Theorems on the asymptotic integration of (4.1) are stated in §§ 8, 11, 13 and 16 and are deduced, respectively, from Lemma 4.1 in §§ 9-10, from Lemmas 4.1–4.2 in § 12, and from Lemmas 4.1–4.3 in §§ 14–15.

If E has at least two eigenvalues with distinct real parts, we can suppose, after a linear change of variables with constant coefficients, that $E = \operatorname{diag}[P, Q]$, $\xi = (y, z)$, $E\xi = (Py, Qz)$, where $\dim y + \dim z = \dim \xi$, the real parts of the eigenvalues $p_1, p_2, \ldots, q_1, q_2, \ldots$ of P, Q satisfy

$$\operatorname{Re} p_j \leqq \mu, \quad \operatorname{Re} q_k > \mu \tag{4.3}$$

for some number μ. We can also assume that P, Q are in a suitable normal form (cf. § IV 9), so that for an arbitrarily fixed $\epsilon > 0$ and some c,

$$0 < \epsilon < c, \tag{4.4}$$

we have the inequalities

(4.5) $\quad \operatorname{Re} y \cdot Py \leqq (\mu + \epsilon) \|y\|^2, \qquad \operatorname{Re} z \cdot Qz \geqq (\mu + c) \|z\|^2.$

Correspondingly, write (4.1) in the form

(4.6) $\quad y' = Py + F_1(t, y, z), \qquad z' = Qz + F_2(t, y, z),$

where $F = (F_1, F_2)$. The initial conditions will be of the form

(4.7) $\quad y(t_0) = y_0, \qquad z(t_0) = z_0.$

When (4.2) holds, (4.5) and (4.6) give

(4.8)
$$\operatorname{Re} y \cdot y' \leqq (\mu + \epsilon) \|y\|^2 + \psi(t) \|\xi\| \cdot \|y\|$$
$$\operatorname{Re} z \cdot z' \geqq (\mu + c) \|z\|^2 - \psi(t) \|\xi\| \cdot \|z\|.$$

Sometimes, it will be convenient to suppose that $E = \operatorname{diag}(A_1, A_2, A_3)$, $\xi = (x, y, z)$, $E\xi = (A_1x, A_2y, A_3z)$, where the eigenvalues $\alpha_{j1}, \alpha_{j2}, \ldots$ of A_j satisfy

(4.9) $\quad \operatorname{Re} \alpha_{1k} < \mu - c, \qquad \operatorname{Re} \alpha_{2k} = \mu, \qquad \operatorname{Re} \alpha_{3k} > \mu + c,$

where (4.4) holds. Correspondingly, it will be supposed that

(4.10)
$$\operatorname{Re} x \cdot A_1x \leqq (\mu - c) \|x\|^2, \qquad \operatorname{Re} z \cdot A_3z \geqq (\mu + c) \|z\|^2,$$
$$|\operatorname{Re} y \cdot Ay - \mu \|y\|^2| \leqq \epsilon \|y\|^2.$$

The initial value problem to be considered is

(4.11) $\quad x' = A_1x + F_1, \qquad y' = A_2y + F_2, \qquad z' = A_3z + F_3,$

where $F(t, \xi) = (F_1, F_2, F_3)$, and

(4.12) $\quad x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0.$

When (4.2) holds, (4.10) and (4.11) imply that

(4.13)
$$\operatorname{Re} x \cdot x' \leqq (\mu - c) \|x\|^2 + \psi(t) \|\xi\| \cdot \|x\|,$$
$$|\operatorname{Re} y \cdot y' - \mu \|y\|^2| \leqq \epsilon \|y\|^2 + \psi(t) \|\xi\| \cdot \|y\|,$$
$$\operatorname{Re} z \cdot z' \geqq (\mu + c) \|z\|^2 - \psi(t) \|\xi\| \cdot \|z\|.$$

In what follows, x, y, z are (real or complex) Euclidean vectors; $\xi = (y, z)$ or $\xi = (x, y, z)$ and $F = (F_1, F_2)$ or $F = (F_1, F_2, F_3)$ are Euclidean vectors in the corresponding product space. The first lemma refers to (4.6) and (4.7); the last two, to (4.11) and (4.12).

Lemma 4.1. *Let μ, ϵ, c be constants and P, Q constant matrices satisfying* (4.4)–(4.5). *Let $F(t, \xi) = (F_1, F_2)$ be continuous for $t \geqq 0$ and $\|y\|$,*

$\|z\| < \delta\ (\leqq \infty)$ *and satisfy* (4.2), *where* $\psi(t) > 0$ *is continuous for* $t \geqq 0$, *and*

(4.14) $$\tau(t) = \int_t^\infty \psi(s) e^{-(c-\epsilon)(s-t)}\, ds$$

converges, so that there exists a $T \geqq 0$ *such that*

(4.15) $$5\tau(t) \leqq 1 \quad \text{if} \quad t \geqq T.$$

Let $t_0 > T$ *and* $0 < \|y_0\| < \delta$. *Then there exists at least one* z_0, $\|z_0\| < \delta$, *such that* (4.6)–(4.7) *has a solution* $y(t)$, $z(t)$ *satisfying*

(4.16) $$\|z(t)\| < 5\tau(t)\, \|y(t)\|,$$

(4.17) $$\|y(t)\| \leqq \|y_0\| \exp \int_{t_0}^t [\mu + \epsilon + 2\psi(s)]\, ds$$

on its right maximal interval $t_0 \leqq t < \omega\ (\leqq \infty)$. *In particular, if the right side of* (4.17) *is less than* δ *for* $t \geqq t_0$, *then* $\omega = \infty$.

The last assertion is a consequence of Corollary I 3.1. The other parts of Lemma 4.1 will be proved in § 5.

Lemma 4.2. *Let* μ, ϵ, c *be constants and* A_1, A_2, A_3 *constant matrices satisfying* (4.4) *and* (4.10). *Let* $F(t, \xi) = (F_1, F_2, F_3)$ *be continuous for* $t \geqq 0$ *and* $\|x\|$, $\|y\|$, $\|z\| < \delta\ (\leqq \infty)$ *and satisfy* (4.2). *Let* $\psi(t) > 0$ *be continuous for* $t \geqq 0$,

(4.18) $$\sigma(t) = \int_0^t \psi(s) e^{-(c-\epsilon)(t-s)}\, ds \quad \text{and} \quad \tau(t) = \int_t^\infty \psi(s) e^{-(c-\epsilon)(s-t)}\, ds$$

converges, and let there exist a $T \geqq 0$ *such that*

(4.19) $$7\sigma(t) \leqq 1 \quad \text{and} \quad 7\tau(t) \leqq 1 \quad \text{if} \quad t \geqq T.$$

Let $t_0 > T$, $\|x_0\| < 7\sigma(t_0)\, \|y_0\|$, $0 < \|y_0\| < \delta$. *Then there exists at least one* z_0, $\|z_0\| < \delta$, *such that* (4.11)–(4.12) *has a solution* $x(t)$, $y(t)$, $z(t)$ *satisfying*

(4.20) $$\|x(t)\| < 7\sigma(t)\, \|y(t)\|, \qquad \|z(t)\| < 7\tau(t)\, \|y(t)\|,$$

(4.21) $$\|y_0\| \exp \int_{t_0}^t (\mu - \epsilon - 3\psi)\, ds \leqq \|y(t)\| \leqq \|y_0\| \exp \int_{t_0}^t (\mu + \epsilon + 3\psi)\, ds.$$

on its right maximal interval of existence $t_0 \leqq t < \omega\ (\leqq \infty)$. *In particular, if the right side of* (4.21) *is less than* δ *for* $t \geqq t_0$, *then* $\omega = \infty$.

In applications of Lemmas 4.1 and 4.2, it is convenient to know when

(4.22) $$\sigma(t), \quad \tau(t) \to 0 \quad \text{as} \quad t \to \infty.$$

This is the case if

$$(4.23)\qquad \psi(t) \to 0 \quad \text{as} \quad t \to \infty \qquad \text{or} \qquad \int^{\infty} \psi(t)\, dt < \infty;$$

or, more generally, if

$$(4.24)\qquad \sup_{s \geqq t} (1 + s - t)^{-1} \int_t^s \psi(r)\, dr \to 0 \qquad \text{as} \quad t \to \infty.$$

Hölder's inequality shows that a sufficient condition for (4.24) [hence, for (4.22)] is that

$$(4.25)\qquad \int^{\infty} |\psi(t)|^p\, dt < \infty \qquad \text{for some} \quad p \geqq 1.$$

Actually, the next exercise states that if $\psi \geqq 0$ and $c - \epsilon > 0$, then (4.24) is necessary and sufficient for (4.22) to hold.

Exercise 4.1. Let $\psi(t) \geqq 0$ be continuous for $t \geqq 0$ and $c - \epsilon > 0$. (*a*) Show that (4.24) implies (4.22). In fact, if $\delta(t)$ denotes the "sup" in (4.24), then

$$\sigma(t) \leqq e^{-(c-\epsilon)(t-T)} \int_0^t \psi(s)\, ds + [1 + (c - \epsilon)^{-1}]\, \delta(T),$$
$$\tau(t) \leqq [1 + (c - \epsilon)^{-1}]\, \delta(t) \qquad \text{for} \quad 0 \leqq T \leqq t.$$

(*b*) Conversely, show that if either $\sigma(t) \to 0$ or $\tau(t) \to 0$ as $t \to \infty$, then (4.24) holds.

Exercise 4.2. Let ψ, $c - \epsilon$ be as in Exercise 4.1 and let (4.25) hold for some p, $1 \leqq p \leqq 2$. (*a*) Show that $\int^{\infty} \sigma^p(t)\, dt < \infty$, $\int^{\infty} \tau^p(t)\, dt < \infty$. (*b*) Conclude that $\int^{\infty} \psi(t)[\sigma(t) + \tau(t)]\, dt < \infty$.

Exercise 4.3. Show that Lemma 4.2 remains valid if $\epsilon = \epsilon(t)$, $c = c(t)$, $\mu = \mu(t)$ are continuous functions of t for $t \geqq 0$ satisfying (4.4), (4.10) and if (4.18) is replaced by

$$\sigma(t) = \int_0^t \psi(s) \exp\left\{-\int_s^t [c(r) - \epsilon(r)]\, dr\right\} ds,$$
$$\tau(t) = \int_t^{\infty} \psi(s) \exp\left\{-\int_t^s [c(r) - \epsilon(r)]\, dr\right\} ds,$$

where it is assumed that the last integral converges and (4.19) holds for some T.

Lemma 4.3. *In addition to the assumptions of Lemma* 4.2, *assume that* $\psi(t)$ *satisfies* (4.24) [*so that* (4.22) *holds*]; *that* $\mu = 0$, $\epsilon = 0$ *in* (4.13); *that an equality of the form*

$$(4.26)\qquad \|y'\| \leqq \psi_0(t)\, \|\xi\|, \qquad \text{i.e., } A_2 = 0, \quad \|F_2\| \leqq \psi_0(t)\, \|\xi\|,$$

holds, where $\psi_0(t)$ is a continuous function for $t \geqq 0$ satisfying

$$\int^{\infty} \psi_0(t)\, dt < \infty; \tag{4.27}$$

finally, that

$$\|y_0\| \exp 3\int_{t_0}^{\infty} \psi_0(s)\, ds < \delta. \tag{4.28}$$

Then $\omega = \infty$ in the assertion of Lemma 4.2,

$$x(t),\ z(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad y_\infty = \lim_{t\to\infty} y(t) \tag{4.29}$$

exists, and $y_\infty \neq 0$. Furthermore there exist $\delta_1 > 0$, $T \geqq 0$ and, for every $t_0 > T$, a positive constant $\delta_2(t_0)$, such that if $y_\infty \neq 0$, x_0 are given vectors and $\|y_\infty\| < \delta_1$, $\|x_0\| < \delta_2 \|y_\infty\|$, then there exist y_0 and z_0 such that (4.11)–(4.12) *has a solution for $t \geqq t_0$ satisfying* (4.20) *and* (4.29). [*When $\delta = \infty$, δ_1 can be taken to be ∞.*]

Remark 1. The proof of this lemma will show that there exists a constant C depending only on the integral of $\psi_0(t)$ over $t_0 \leqq t < \infty$ such that the solutions mentioned satisfy

$$\|y(t) - y_\infty\| \leqq C \|y_*\| \int_t^{\infty} \psi_0(s)\, ds,$$

$$\|x(t)\| \leqq C \|y_*\| \sigma(t), \qquad \|z(t)\| \leqq C \|y_*\| \tau(t),$$

where $t \geqq t_0$ and y_* can be either y_0 or y_∞.

Remark 2. In the proof of the first part of Lemma 4.3, the inequalities (4.13) with $\mu = \epsilon = 0$ and (4.26) need not hold for all $\|\xi\| < \delta$. For in view of (4.21), the proof will involve only y satisfying $c_1 \|y_0\| < \|y\| < c_2 \|y_0\|$, hence

$$c_1 \|y_0\| < \|\xi\| < 3c_2 \|y_0\|, \tag{4.30}$$

by (4.20), where

$$c_j = \exp 3(-1)^j \int_T^{\infty} \psi_0(s)\, ds \qquad \text{for} \quad j = 1, 2. \tag{4.31}$$

Correspondingly, (4.13) and (4.26) need only be assumed when (4.30) holds. In the second part of Lemma 4.3, the same remains true if (4.30) is replaced by

$$c_1 \|y_\infty\| < \|\xi\| < 3c_2 \|y_\infty\|. \tag{4.32}$$

These assertions permit the replacement of the assumptions (4.2) and (4.26) in the derivation of (4.13) by another type of hypothesis: For a pair of numbers r, R satisfying $0 < r < R$ ($\leqq \infty$), let there exist a continuous function $\varphi_{rR}(t) > 0$ for $t \geqq 0$ such that

(4.33) $$\|F(t, \xi)\| \leqq \varphi_{rR}(t) \quad \text{if} \quad r < \|\xi\| < R,$$

(4.34) $$\int^{\infty} \varphi_{rR}(t)\, dt < \infty.$$

Then (4.33) implies that

(4.35) $$\|F(t, \xi)\| \leqq r^{-1}\varphi_{rR}(t)\, \|\xi\| \quad \text{for} \quad r < \|\xi\| < R,$$

which is the analogue of (4.2) with

(4.36) $$\psi(t) = r^{-1}\varphi_{rR}(t).$$

Notice that with this choice of $\psi(t)$ and $\psi_0(t) = \psi(t)$, (4.31) shows that $c_j \to 1$ as $T \to \infty$. Hence if $r < \|y_0\| < R/3$ [or $r < \|y_\infty\| < R/3$], then the first part [or last part] of Lemma 4.3 remains valid.

The case $\mu \neq 0$ can be reduced to $\mu = 0$ by the change of variables $\xi = e^{\mu t}\zeta$ [when $\mu > 0$, it is necessary to assume that $F(t, \xi)$ is defined for $t \geqq 0$ and *all* ξ]:

Corollary 4.1. *In addition to the assumptions of Lemma* 4.2, *assume that $\psi(t)$ satisfies* (4.24) [*so that* (4.22) *holds*] *and that an inequality of the form*

(4.37) $$\|y' - \mu y\| \leqq \psi_0(t)\, \|\xi\|; \qquad i.e., \quad A_2 = \mu I, \quad \|F_2\| \leqq \psi_0(t)\, \|\xi\|,$$

holds, where $\psi_0(t)$ is a continuous function for $t \geqq 0$ satisfying (4.27). *If $\mu > 0$, assume that $\delta = \infty$ (so that F is defined for $t \geqq 0$ and all ξ). Then the assertions of Lemma* 4.3 *remain valid if* (4.29) *is replaced by*

(4.38) $$e^{-\mu t}x(t), \quad e^{-\mu t}z(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad y_\infty = \lim_{t\to\infty} e^{-\mu t}y(t).$$

Exercise 4.4. Verify Corollary 4.1.

The condition $A_2 = \mu I$ can be replaced by the assumption that A_2 is a diagonal matrix (or has simple elementary divisors) and that all of its eigenvalues have the same real part μ:

Corollary 4.2. *Let the assumptions of Corollary* 4.1 *hold except that* $A_2 = \operatorname{diag}[\mu + i\gamma_1, \mu + i\gamma_2, \ldots]$, *where $\gamma_1, \gamma_2, \ldots$ are real numbers, and* (4.37) *is replaced by*

(4.39) $$\|y' - A_2 y\| \leqq \psi_0(t)\, \|\zeta\|.$$

Then the assertions of Lemma 4.3 *remain valid if* (4.29) *is replaced by*

(4.40) $$e^{-\mu t}x(t), \quad e^{-\mu t}z(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad y_\infty = \lim_{t\to\infty} e^{-A_2 t}y(t).$$

Note that the last part of (4.40) means that the kth component $y^k(t)$ of $y(t)$ satisfies $e^{-(\mu+i\gamma_k)t}y^k(t) \to y_\infty{}^k$ as $t \to \infty$.

Exercise 4.5. Reduce Corollary 4.2 to Lemma 4.3 by the change of variables $(x, y, z) \to (u, v, w)$ given by $x = e^{\mu t}u$, $y = e^{A_2 t}v$, $z = e^{\mu t}w$.

Exercise 4.6. Let E be a constant matrix with eigenvalues $\lambda_1, \ldots, \lambda_d$ such that $\lambda_1, \ldots, \lambda_k$ are simple eigenvalues with $\operatorname{Re} \lambda_j = 0$ for $j = 1, \ldots, k$ for some k, $1 \leqq k \leqq d$. Of the eigenvalues $\lambda_{k+1}, \ldots, \lambda_d$, let m have positive real parts, n negative real parts, where $0 \leqq m, n \leqq d - k$ and $m + n = d - k$. Let $G(t)$ be a continuous matrix for $t \geqq 0$ such that $G(t) \to 0$ as $t \to \infty$ and the elements $g_{ij}(t)$ are of bounded variation for $t \geqq 0$ (i.e., $\int^\infty |dg_{ij}(t)| < \infty$). For example, if $G(t)$ is continuously differentiable, let $G(t) \to 0$ as $t \to \infty$ and $\int^\infty \|G'(t)\|\, dt < \infty$. For large t, the matrix $E + G(t)$ has k simple continuous eigenvalues $\lambda_1(t), \ldots, \lambda_k(t)$ such that $\lambda_j(t) \to \lambda_j$ as $t \to \infty$; cf. Exercise IV 9.1. (*a*) Show that the linear system $\xi' = [E + G(t)]\xi$ has n linearly independent exponentially small solutions as $t \to \infty$. (*b*) If $\operatorname{Re} \lambda_j(t) \leqq 0$ for $j = 1, \ldots, k$, then $\xi' = [E + G(t)]\xi$ has $n + k$ bounded solutions as $t \to \infty$. (*c*) If $k = 1$, then there exists a vector $c \neq 0$ such that $\xi' = [E + G(t)]\xi$ has a solution of the form $\xi = (c + o(1)) \exp \int^t \lambda_1(s)\, ds$ as $t \to \infty$. (*d*) If $\int^t \operatorname{Re} \lambda_j(s)\, ds$ is bounded for $j = 1, \ldots, k$, then there exist linearly independent vectors $c_1, \ldots, c_k$ such that $\xi' = [E + G(t)]\xi$ has solutions of the form

$$\xi = [c_j + o(1)] \exp \int^t \lambda_j(s)\, ds \quad \text{as} \quad t \to \infty \quad \text{for} \quad j = 1, \ldots, k.$$

For applications of the corollaries of Lemma 4.3, see the exercises in § VIII 3. Further applications and extensions of Lemma 4.3 and its corollaries are given in §§ 13–16

Exercise IX 5.4 gives an analogue of Lemma 4.1 for difference equations. Exercises 4.8 and 4.9 to follow give analogues of Lemmas 4.2 and 4.3.

Exercise 4.7. Let $R = R^d$ be the $\xi = (\xi^1, \ldots, \xi^d)$-space. For $n = 1, 2, \ldots$, let S_n be a map of R into itself and $T_n = S_n \circ S_{n-1} \circ \cdots \circ S_1$. Let S be a compact and $K_0, K_1, K_2 \ldots, K_{00}, K_{10}, K_{20}, \ldots$ closed sets of R such that $S \subset K_0 \cap K_{00}$, $S_n(R - K_{n-1}) \subset R - K_n$, $S_n(K_{n-1} \cap K_{n-1,0}) \subset K_{n0}$, and $K_n \cap T_n(S)$ is not empty for $n = 1, 2, \ldots$. Then there exists a point $\xi_0 \in S$ such that $T_n \xi_0 \subset K_n \cap K_{n0}$ for $n = 1, 2, \ldots$.

Exercise 4.8. Let A, B, C be square matrices satisfying

$$(4.41) \quad \|Ax\| \leqq (\mu - c)\, \|x\|, \ (\mu - \epsilon)\, \|y\| \leqq \|By\| \leqq (\mu + \epsilon\, \|y\|,$$
$$\|Cz\| \geqq (\mu + c)\, \|z\|,$$

where $\mu > 0$ and $0 < \epsilon < c$. For $n = 1, 2, \ldots$, let X_n, Y_n, Z_n be continuous, vector-valued functions defined for all (x, y, z) which vanish for large $\|x\| + \|y\| + \|z\|$. Let $T_n = S_n \circ S_{n-1} \circ \cdots \circ S_1$, where S_n is the map

$$(4.42) \quad S_n:\ x_1 = Ax + X_n(x, y, z),\ y_1 = By + Y_n(x, y, z),$$
$$z_1 = Cz + Z_n(x, y, z).$$

(*a*) Let $0 < \theta < 1$ and $\|x_0\| \leqq \theta\, \|y_0\|$. Show that if $0 \leqq \delta \leqq (c - \epsilon)\theta/6$ and

$$(4.43) \qquad \|X_n\|,\ \|Y_n\|,\ \|Z_n\| \leqq (\|x\| + \|y\| + \|z\|)\delta,$$

then there exists a z_0 such that $(x_n, y_n, z_n) = T_n(x_0, y_0, z_0)$ satisfies

$$(4.44) \qquad \|x_n\| \leqq \theta\, \|y_n\|,\ \|z_n\| \leqq \theta\, \|y_n\| \quad \text{for} \quad n = 0, 1, \ldots.$$

(*b*) Show that if $0 < \mu < 1$ and

$$\|X_n\|,\ \|Y_n\|,\ \|Z_n\| = o(\|x\| + \|y\| + \|z\|)$$

as $(n, x, y, z) \to (\infty, 0, 0, 0)$, then (4.44) implies that $\|x_n\|/\|y_n\| \to 0$ and $\|z_n\|/\|y_n\| \to 0$ as $n \to \infty$.

Exercise 4.9. Let A, C be matrices satisfying

$$\|Ax\| \leqq a\, \|x\| \text{ and } \|Cz\| \geqq c\, \|z\|, \quad \text{where} \quad 0 < a < 1 < c.$$

Let θ, X_n, Y_n, Z_n be as in Exercise 4.8, with $\delta \leqq (c - 1)\theta/6$ in (4.43). Let $T_n = S_n \circ S_{n-1} \circ \cdots \circ S_1$, where S_n is given by (4.42) with $B = I$. Let $\psi_0(1) + \psi_0(2) + \cdots$ be convergent, and

$$\|Y_n(x, y, z)\| \leqq \psi_0(n)(\|x\| + \|y\| + \|z\|).$$

(*a*) Let (x_0, y_0, z_0) be such that $(x_n, y_n, z_n) = T_n(x_0, y_0, z_0)$ satisfies (4.44). Show that

$$(4.45) \qquad y_\infty = \lim_{n \to \infty} y_n$$

exists. (*b*) In addition, assume that $\delta < (1 - a)\theta/3$ and that $0 \leqq 3\psi_0(n) < 1$. Let x_0, y_∞ be given and satisfy $\|x_0\| \leqq \theta\, \|y_\infty\|$. Show that there exists a (y_0, z_0) such that $(x_n, y_n, z_n) = T_n(x_0, y_0, z_0)$ satisfies (4.44) and (4.45). (*c*) Formulate analogues of parts (*a*) and (*b*) when the matrix B in (4.42) is μI, $\mu \neq 0$ (instead of I).

5. Proof of Lemma 4.1

In order to apply Corollary 3.1, let

$$\Omega = \{(t, y, z)\colon t > T;\ \|y\|, \|z\| < \delta;\ (y, z) \neq 0\},$$
$$\Omega^0 = \{(t, y, z) \in \Omega\colon \|z\| < 5\tau(t)\, \|y\|\}.$$

It will be verified that Ω^0 is a (u, v)-subset of Ω determined by the one function $u = \|z\|^2 - 25\tau^2(t)\|y\|^2$. Let

$$U = \{(t, y, z) \in \Omega : u = 0\}.$$

Since $\dot{u} = 2(\operatorname{Re} z \cdot z' - 25\tau^2 \operatorname{Re} y \cdot y' - 25\tau\tau' \|y\|^2)$, it follows from (4.8) that on U, where $5\tau(t) \leqq 1$, $\|z\| = 5\tau(t)\|y\| \leqq \|y\|$, and $\|\xi\| \leqq 2\|y\|$, we have

$$\dot{u} \geqq 50\tau \|y\|^2 [(c - \epsilon)\tau - \tfrac{4}{5}\psi - \tau'].$$

The last factor is positive since τ satisfies the differential equation

$$(c - \epsilon)\tau - \psi - \tau' = 0,$$

and $\psi > 0$. Thus Ω^0 is a (u, v)-subset of Ω and $U = \Omega_e{}^0 = \Omega_{se}^0$.

Note that, by the definition of Ω, the point $(y, z) = (0, 0)$ is not in Ω; hence $(y, z) \in \Omega_e{}^0$ implies that $y \neq 0$.

Let $S = \{(t_0, y_0, z) : \|z\| \leqq 5\tau(t_0)\|y_0\|\}$. Thus $S \cap \Omega_e{}^0 = \{(t_0, y_0, z) : \|z\| = 5\tau(t_0)\|y_0\|\}$. S is a ball, $\|z\| \leqq 5\tau(t_0)\|y_0\|$, and $S \cap \Omega_e{}^0$ is its boundary and is not a retract of S. Since $U = \Omega_e{}^0$, the map $\pi : \Omega_e{}^0 \to S \cap \Omega_e{}^0$ given by $\pi(t, y, z) = (t_0, y_0, z\tau(t_0)\|y_0\|/\tau(t)\|y(t)\|)$ is continuous [since $y \neq 0$ on $\Omega_e{}^0$ and $\tau(t) > 0$] and hence is a retraction of $\Omega_e{}^0$ onto $S \cap \Omega_e{}^0$. The existence of z_0 and a solution $y(t), z(t)$ of (4.6)–(4.7) satisfying (4.16) follows from Corollary 3.1.

Since (4.15), (4.16) imply that $\|z(t)\| \leqq \|y(t)\|$, hence $\|\xi(t)\| \leqq 2\|y(t)\|$ the inequality (4.17) is a consequence of (4.8). This proves Lemma 4.1.

6. Proof of Lemma 4.2

This proof is similar. It depends on the choices

$$\Omega = \{(t, x, y, z) : t > T;\ \|x\|, \|y\|, \|z\| < \delta;\ (x, y, z) \neq 0\}$$
$$\Omega^0 = \{(t, x, y, z) \in \Omega : u < 0,\ v < 0\},$$

where

$$u = \|z\|^2 - 49\tau^2(t)\|y\|^2, \qquad v = \|x\|^2 - 49\sigma^2(t)\|y\|^2.$$

Define the sets

$$U = \{(t, x, y, z) \in \Omega : u = 0,\ v \leqq 0\},$$
$$V = \{(t, x, y, z) \in \Omega : u \leqq 0,\ v = 0\}.$$

It is readily verified that (4.13), (4.19), and (4.20) imply that $\|\xi\| \leqq 3\|y\|$ and

$$\dot{u} > 0 \quad \text{on } U, \qquad \dot{v} < 0 \quad \text{on } V.$$

Hence Ω^0 is a (u, v)-subset of Ω and $\Omega_e{}^0 = U - V = \{(t, x, y, z) \in \Omega : u = 0, v < 0\}$.

Choose S to be the set $\{(t_0, x_0, y_0, z): \|z\| \leqq 7\tau(t_0)\ \|y_0\|\}$. As above, it is seen that $S \cap \Omega_e^0$ is not a retract of S but is a retract of Ω_e^0. Thus Lemma 4.2 follows from Corollary 3.1.

7. Proof of Lemma 4.3

Let (x_0, y_0, z_0) and $\xi(t) = (x(t), y(t), z(t))$ be as in Lemma 4.2. By (4.19) and (4.20), $\|\xi(t)\| \leqq 3\ \|y(t)\|$. Hence (4.26) gives $\|y'\| \leqq 3\psi_0(t)\ \|y\|$. It follows from (4.19), (4.20), and (4.28) that $\xi(t)$ exists for $t \geqq t_0$. The first part of (4.29) follows from (4.20) and (4.22). The inequality $\|y'\| \leqq 3\psi_0(t)\ \|y\|$ implies the existence of the limit y_∞ and $y_\infty \neq 0$ as in the proof of Theorem 1.1.

The last part of Lemma 4.3 will not be deduced from Lemma 4.2 but will be obtained from another application of Corollary 3.1. Let

$$\Omega = \{(t, x, y, z): t > t_0;\ \|x\|, \|y\|, \|z\| < \delta;\ (x, y, z) \neq 0\},$$
$$\Omega^0 = \{(t, x, y, z) \in \Omega: u_1 < 0,\ u_2 < 0,\ v < 0\},$$

where u_1, u_2, v are defined by

$$(7.1)\qquad \begin{aligned} u_1 &= \|y - y_\infty\|^2 - 49\ \|y_\infty\|^2\left(\int_t^\infty \psi_0\, ds\right)^2, \\ u_2 &= \|z\|^2 - 49\tau^2(t)\ \|y\|^2, \qquad v = \|x\|^2 - 49\sigma^2(t)\ \|y\|^2, \end{aligned}$$

and t_0 is a positive constant to be specified. Let U_α be the subset of Ω where $u_\alpha = 0$ and $u_j \leqq 0$, $v \leqq 0$, and V the subset of Ω where $v = 0$ and $u_1, u_2 \leqq 0$. Then, as in the last section, $\dot{u}_2 > 0$ on U_2, $\dot{v} < 0$ on V. When $u_1, u_2, v \leqq 0$, then $\|\xi\| \leqq 3\|y\|$ and

$$(7.2)\qquad \|y - y_\infty\| \leqq 7\ \|y_\infty\| \int_t^\infty \psi_0\, ds,$$

$$(7.3)\qquad \|y_\infty\|\left(1 - 7\int_t^\infty \psi\, ds\right) \leqq \|y\| \leqq \|y_\infty\|\left(1 + 7\int_t^\infty \psi\, ds\right).$$

Since

$$\dot{u}_1 = 2\left[\operatorname{Re}(y - y_\infty)\cdot y' + 49\ \|y_\infty\|^2\ \psi_0(t)\int_t^\infty \psi_0\, ds\right],$$

a simple calculation shows that on U_1,

$$\dot{u}_1 \geqq 14\ \|y_\infty\|^2\ \psi_0(t)\int_t^\infty \psi_0\, ds\left[-3\left(1 + 7\int_t^\infty \psi_0\, ds\right) + 7\right].$$

Let T be so large that

$$8\int_t^\infty \psi_0\, ds < 1 \qquad \text{for} \quad t \leqq T.$$

Thus, if $t_0 > T$, it follows that $\dot{u}_1 > 0$ on U_1. In this case, Ω^0 is a (u, v)-subset of Ω and $\Omega_e^0 = \Omega_{se}^0$ is the subset of Ω where $u_1, u_2 \leqq 0$, $v < 0$ and either $u_1 = 0$ or $u_2 = 0$.

Choose $\delta_2(t_0)$ to be

$$\delta_2(t_0) = \sigma(t_0)\left(1 - 7\int_{t_0}^{\infty} \psi_0 \, ds\right). \tag{7.4}$$

Thus, by (7.3), $\|x_0\| < \delta_2(t_0) \|y_\infty\|$ implies that $\|x_0\| < \sigma(t_0) \|y\|$ for $y \in \Omega^0$. Let $S = \{(t_0, x_0, y, z): \|z\| \leqq 7\tau(t_0) \|y\|, \|y - y_\infty\| \leqq 7 \|y_\infty\| \int_{t_0}^{\infty} \psi \, ds\}$, so that $S \subset \Omega^0 \cup \Omega_e^0$. Topologically, S is a ball in the (y, z)-space. (If y, z

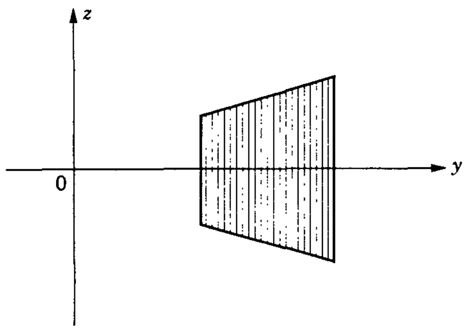

Figure 3.

are 1-dimensional, then S appears as the shaded area in Figure 3.) It is clear that $S \cap \Omega_e^0$ is the subset of S on which $u_1 = 0$ or $u_2 = 0$, so that, topologically, $S \cap \Omega_e^0$ is the boundary of S and is not a retract of S. On the other hand, $S \cap \Omega_e^0$ is a retract of Ω_e^0 for a retraction $\pi: \Omega_e^0 \to S \cap \Omega_e^0$ is given by $\pi(t, x, y, z) = (t_0, x_0, y^0, z\tau(t_0) \|y^0\|/\tau(t) \|y\|)$, where $y^0 = y^0(t, y)$ is chosen so that $y^0 - y_\infty = \alpha(y - y_\infty)$ and

$$\alpha = \int_{t_0}^{\infty} \psi \, ds \Big/ \int_{t}^{\infty} \psi \, ds.$$

(That $S \cap \Omega_e^0$ is a retract of Ω_e^0 is geometrically easy to see, because the projection $(t, x, y, z) \to (t, x_0, y, z)$ of the (t, x, y, z)-space into the (t, y, z)-space carries Ω_e^0 into a set which is topologically the boundary of a "cylinder" with $S \cap \Omega_e^0$ corresponding to a section $t = t_0$; cf. Example 2 of § 2.)

Thus by Corollary 3.1, there is a point $(t_0, x_0, y_0, z_0) \in S \cap \Omega^0$ such that a solution arc $(t, x(t), y(t), z(t))$ belonging to (4.11)–(4.12) remains in Ω^0 on its maximal right interval of existence $[t_0, \omega)$. As in the argument at the beginning of the proof of this lemma, $\omega = \infty$ if $\delta = \infty$ or if $\|y_\infty\|$ is sufficiently small; in this case, (4.20), (4.29) hold.

8. Asymptotic Integrations. Logarithmic Scale

Consider again a system of the form

$$\xi' = E\xi + F(t, \xi) \tag{8.1}$$

in which

$$\|F(t, \xi)\| \leqq \psi(t)\,\|\xi\| \tag{8.2}$$

holds. In this section it will be supposed that $\xi = (y, z)$, $F = (F_1, F_2)$, and $E = \operatorname{diag}[P, Q]$, so that initial value problems associated with (8.1) take the form

$$y' = Py + F_1(t, y, z), \qquad z' = Qz + F_2(t, y, z), \tag{8.3}$$

$$y(t_0) = y_0, \qquad z(t_0) = z_0. \tag{8.4}$$

The eigenvalues $p_1, p_2, \ldots$ and $q_1, q_2, \ldots$ of P and Q will be assumed to satisfy

$$\operatorname{Re} p_j \leqq \mu, \qquad \operatorname{Re} q_k > \mu \tag{8.5}$$

for some number μ.

Theorem 8.1. *Let* (8.1) *be equivalent to* (8.3) *where the eigenvalues of* P, Q *satisfy* (8.5); $F(t, \xi)$ *is continuous and satisfies* (8.2) *for* $t \geqq 0$ *and* $\|y\|$, $\|z\| < \delta$ $(\leqq \infty)$; *and* $\psi(t) > 0$ *is continuous for* $t \geqq 0$ *and satisfies*

$$\sup_{s \geqq t} (1 + s - t)^{-1} \int_t^s \psi(r)\, dr \to 0 \qquad \text{as} \quad t \to \infty.$$

When $\mu \geqq 0$, *assume that* $\delta = \infty$. *Then there exist* $T \geqq 0$ *and* $\delta_1 > 0$ *such that for every* $t_0 \geqq T$ *and* y_0 *satisfying* $\|y_0\| < \delta_1$, *there is a* z_0 *with the property that the initial value problem* (8.3)–(8.4) *has a solution for* $t \geqq t_0$ *satisfying either* $(y(t), z(t)) \equiv 0$ *or* $y(t) \not\equiv 0$ *for* $t \geqq t_0$ *and*

$$\|z(t)\| = o(\|y(t)\|) \qquad \text{as} \quad t \to \infty, \tag{8.6}$$

$$\limsup_{t \to \infty} t^{-1} \log \|\xi(t)\| \leqq \mu. \tag{8.7}$$

If μ in (8.7) is replaced by $\mu + \epsilon > \mu$, this follows at once from Lemma 4.1 [with $\delta_1(t_0) = \infty$ if $\delta = \infty$]. Since a linear transformation of the y-variables with constant coefficients does not affect (8.6) but permits an arbitrary choice of $\epsilon > 0$, Theorem 8.1 follows. Assertions (8.6), (8.7) will be improved in § 11 below.

Remark 1. This proof of Theorem 8.1 shows that if the y-variables and z-variables are each subjected to a linear transformation with constant coefficients and $\psi(t)$ is replaced by const. $\psi(t)$ for a suitable constant, then it can be supposed that (4.5) and (4.8) hold. With these choices of coordinates and ψ, the inequalities (4.16)–(4.17) in Lemma 4.1 hold for any solution $(y(t), z(t)) \not\equiv 0$ of (8.3) satisfying (8.6)–(8.7).

Theorem 8.2. *In addition to the conditions of Theorem* 8.1, *assume that* F *satisfies the Lipschitz condition*

$$(8.8)\qquad \|F(t, \xi_1) - F(t, \xi_2)\| \leqq \psi(t)\, \|\xi_1 - \xi_2\|,$$

that t_0 *is sufficiently large, and that* $\|y_0\|$ *is sufficiently small. Then* z_0 *and* $(y(t), z(t))$ *are unique and* $z_0 = g(t_0, y_0)$ *is a continuous function (in fact, uniformly Lipschitz continuous on compact subsets of its domain of definition).*

If, in addition, F is assumed to be smooth (say, of class C^m, $m \geqq 1$, or analytic), then $z_0 = g(t_0, y_0)$ is of the same smoothness. Here, a function of a vector with complex-valued components is said to be of class C^m if it has continuous, mth order partial derivatives with respect to the real and imaginary parts of its variables. In this terminology, the result for $F \in C^1$ is

Theorem 8.3. *Let the conditions of Theorems* 8.1, 8.2 *hold and let* $F(t, \xi)$ *have continuous, first order partial derivatives with respect to the real and imaginary parts of the components of* ξ. *Suppose also that* $\mu < 0$. *Then* $z_0 = g(t_0, y_0)$ *is of class* C^1. *If, in addition, the partial derivatives of* F *with respect to the real and imaginary parts of the components of* ξ *vanish at* $\xi = 0$ *for all* t, *then the partial derivatives of* g *with respect to the real and imaginary parts of the components of* y_0 *vanish at* $y_0 = 0$ *for all* t_0.

The proofs in §§ 9 and 10 will show that Theorems 8.2 and 8.3 are corollaries of Theorem 8.1, which is, in turn, an immediate consequence of Lemma 4.1. For applications, note that the proofs of Theorems 8.1–8.3 imply the following remark.

Remark 2. Let $\epsilon > 0$ be fixed so small that $\mu + \epsilon < 0$ if $\mu < 0$ and that $\operatorname{Re} q_k > \mu + \epsilon$ in (8.5). Then there exists a number $\rho_\epsilon > 0$ with the property that if the condition on $\psi(t)$ is relaxed to

$$(8.9)\qquad (1 + s - t)^{-1} \int_t^s \psi(r)\, dr \leqq \rho_\epsilon \quad \text{for} \quad \text{large } t \text{ and } s \geqq t,$$

then Theorems 8.1–8.3 remain valid if (8.6), (8.7) are replaced by the single condition

$$(8.10)\qquad \limsup_{t \to \infty} t^{-1} \log \|\xi(t)\| \leqq \mu + \epsilon.$$

Notice that the "smallness condition" (8.2) does not seem appropriate if (8.1) is considered only for small ξ, e.g., if $F(t, \xi)$ does not depend on t. In this case, more natural conditions are

$$(8.11)\qquad \frac{\|F(t, \xi)\|}{\|\xi\|} \to 0 \qquad \text{as} \quad (t, \xi) \to (\infty, 0)$$

and, of course, $\mu < 0$.

Corollary 8.1 *Let the assumptions of Theorem* 8.1 *hold except that* (8.11) *replaces* (8.2); *also assume that* $\delta < \infty$ *and* $\mu < 0$. *Then the conclusions of Theorem* 8.1 *remain valid. If, in addition,*

$$(8.12) \qquad \frac{\|F(t, \xi_1) - F(t, \xi_2)\|}{\|\xi_1 - \xi_2\|} \to 0 \quad \text{as} \quad (t, \xi_1, \xi_2) \to (\infty, 0, 0)$$

when $\xi_1 \neq \xi_2$, *then the conclusions of Theorem* 8.2 *hold in the following sense: there exists a small* $\delta_0 > 0$ *with the property that if* t_0 *is sufficiently large and* $\|y_0\|$ *is sufficiently small, then there exists a unique* $z_0 = g(t_0, y_0)$ *such that the solution* $\xi(t) = (y(t), z(t))$ *of* (8.3)-(8.4) *exists and satisfies* $\|\xi(t)\| \leqq \delta_0$ *for* $t \geqq t_0$ *and the conclusions of Theorem* 8.1; *furthermore,* $g(t_0, y_0)$ *is uniformly Lipschitz continuous. Also, if F satisfies the smoothness assumptions of Theorem* 8.3, *then the conclusions of Theorem* 8.3 *are valid.*

This generalizes the last part of Theorem IX 6.1 on the existence of invariant manifolds. The other part will be generalized later in § 11.

Corollary 8.1 follows from the Remark 2 by virtue of the fact that (8.11) implies that, for every $\rho > 0$, there exist $T \geqq 0$ and $\delta_0 > 0$ such that

$$(8.13) \qquad \|F(t, \xi)\| \leqq \rho \, \|\xi\| \quad \text{for} \quad t \geqq T \text{ and } \|\xi\| \leqq \delta_0$$

and correspondingly, (8.12) gives

$$\|F(t, \xi_1) - F(t, \xi_2)\| \leqq \rho \, \|\xi_1 - \xi_2\| \quad \text{for} \quad t \geqq T \text{ and } \|\xi_1\|, \|\xi_2\| \leqq \delta_0;$$

furthermore, if $\epsilon > 0$ is sufficiently small, then $\mu + \epsilon < 0$ and (8.10), (8.11) imply (8.6), (8.7).

For another deduction of the first part of Corollary 8.1 from Theorem 8.1, make the change of variables

$$(8.14) \qquad \xi = e^{-\alpha t}\zeta,$$

where $0 < \alpha < -\mu$. Then (8.1) becomes

$$\zeta' = (E + \alpha I)\zeta + e^{\alpha t}F(t, e^{-\alpha t}\zeta)$$

and μ is replaced by $\mu + \alpha < 0$. For the applicability of Theorem 8.1, it is sufficient to verify the existence of a $\psi(t)$ such that $\psi(t) \to 0$ as $t \to \infty$ and

$$\|e^{\alpha t}F(t, e^{-\alpha t}\zeta)\| \leqq \psi(t) \, \|\zeta\| \qquad \text{for} \quad \|\zeta\| < \tfrac{1}{2}\delta.$$

Note that $\alpha > 0$ and (8.11) imply that such a $\psi(t)$ is given by

$$\psi(t) = \sup_{\|\zeta\| \leqq \frac{1}{2}\delta} \frac{\|F(t, e^{-\alpha t}\zeta)\|}{\|e^{-\alpha t}\zeta\|}.$$

Exercise 8.1. This exercise involves a proof of the conclusions of Theorems 8.1 and 8.2 by the method of successive approximations rather than by the use of Corollary 3.1 (via Lemma 4.1). In view of the change of variables (8.14) with a suitable α, there is no loss of generality in assuming that $\mu < 0$. If $\xi = (y(t), z(t))$ is a solution of (8.3) satisfying (8.7), then it is easy to see that

$$(8.15)\qquad \begin{aligned} y(t) &= e^{P(t-t_0)}y_0 + \int_{t_0}^{t} e^{P(t-s)}F_1(s, y(s), z(s))\, ds, \\ z(t) &= -\int_{t}^{\infty} e^{Q(t-s)}F_2(s, y(s), z(s))\, ds, \end{aligned}$$

where it can be supposed that P, Q are such that

$$(8.16)\qquad \|e^{Pt}\| \leqq e^{(\mu+\epsilon)t}, \qquad \|e^{-Qt}\| \leqq e^{-(\mu+\epsilon)t} \quad \text{for} \quad t \geqq 0.$$

Conversely, if $\xi = (y(t), z(t))$ is a solution of (8.15) satisfying (8.7), then it is a solution of (8.3). Show, by the method of successive approximations, that under the assumptions (8.2), (8.8), where $\psi(t) \to 0$ as $t \to \infty$, (8.15) has a solution (for sufficiently large t_0, small $\|y_0\|$ if $\delta < \infty$) satisfying $y(t_0) = y_0$ and (8.6)–(8.7). Let the 0th approximations be $y_0(t) = e^{P(t-t_0)}y_0$, $z_0(t) = 0$, and the nth approximation be obtained by writing $(y(s), z(s)) = (y_{n-1}(s), z_{n-1}(s))$ on the right of (8.15) and $(y(t), z(t) = (y_n(t), z_n(t))$ on the left. See Coddington and Levinson [2, Chapter 13]. This gives the existence Theorem 8.1 under the additional condition (8.8) and the condition $\psi(t) \to 0$. Theorems 8.2 and 8.3 can also be proved by the considerations of the successive approximations, but note that Theorems 8.2, 8.3 are deduced in §§ 9 and 10 essentially from Theorem 8.1. (Despite the disadvantages of the method of successive approximations in the present situation, this method has important applications in related problems.)

9. Proof of Theorem 8.2

It can be assumed that (4.4), (4.5), and (4.8) hold; cf. Remark 1 following Theorem 8.1. In terms of the function $\sigma(t)$ in (4.18), define

$$(9.1)\qquad u(t, y, z) = 25\sigma^2(t)\,\|z\|^2 - \|y\|^2.$$

It is readily verified (cf. § 5) that if (4.8) holds, $T \geqq 0$ is sufficiently large, and $t \geqq T$, then

$$(9.2)\qquad \dot{u} > 0 \quad \text{when} \quad u = 0.$$

Uniqueness. Suppose that (8.1) has two solutions $\xi_j(t) = (y_j(t), z_j(t))$, where $j = 1, 2$, satisfying $y_j(t_0) = y_0$ and (8.7), but $z_1(t_0) \neq z_2(t_0)$. Put $\xi(t) = \xi_2(t) - \xi_1(t) = (y(t), z(t))$. Then (8.8) implies (4.8) hence, by (9.2),

$du(t, y(t), z(t))/dt > 0$ if $u(t, y(t), z(t)) = 0$. Since $y(t_0) = 0$, $u(t_0, y(t_0), z(t_0)) > 0$ and, consequently, $u(t, y(t), z(t))$ cannot vanish for $t \geqq t_0$. Thus

(9.3) $$\|y(t)\| < 5\sigma(t)\,\|z(t)\| \quad \text{for} \quad t \geqq t_0.$$

It follows from (4.8) and $\sigma(t) \to 0$ as $t \to \infty$ (cf. Exercise 4.1) that

(9.4) $$\liminf_{t\to\infty} t^{-1} \log \|z(t)\| \geqq \mu + c.$$

But this contradicts $\|z(t)\| \leqq \|\xi(t)\| \leqq \|\xi_2(t)\| + \|\xi_1(t)\|$, since both $\xi = \xi_1, \xi_2$ satisfy (8.7).

Continuity of $z_0 = g(t_0, y_0)$. Let $t_0 \geqq T_0$, $\|y_1\| < \delta_1(t_0)$, $z_1 = g(t_0, y_1)$ and $\xi_1(t)$ be the corresponding solution of (8.1). Introduce new variables into (8.1) defined by

(9.5) $$\zeta = \xi - \xi_1(t),$$

so that (8.1) becomes

(9.6) $$\zeta' = E\zeta + F(t, \zeta + \xi_1(t)) - F(t, \xi_1(t)),$$

and, by (8.8),

(9.7) $$\|F(t, \zeta_1 + \xi_1(t)) - F(t, \zeta_2 + \xi_1(t))\| \leqq \psi(t)\,\|\zeta_1 - \zeta_2\|.$$

It follows from the part of Theorem 8.2 already proved that if $\|y_2 - y_1\|$ is sufficiently small, then (9.6) has a unique solution $\zeta(t)$ which satisfies $\zeta(t) \equiv 0$ or $\limsup t^{-1} \log \|\zeta(t)\| \leqq \mu + \epsilon$ as $t \to \infty$, $\zeta(t_0) = (y_2 - y_1, \ldots)$, and

(9.8) $$\|z_2(t) - z_1(t)\| \leqq 5\tau(t)\,\|y_2(t) - y_1(t)\|$$

for $t \geqq t_0$ if $\xi_2(t) = \zeta(t) + \xi_1(t) = (y_2(t), z_2(t))$. (The inequality (9.8) is the analogue of (4.16) is Lemma 4.1.) It follows that $\xi = \xi_2(t)$ is a solution of (8.1) and that $z_2 = g(t_0, y_2)$. Thus $t = t_0$ in (9.8) gives

(9.9) $$\|g(t_0, y_2) - g(t_0, y_1)\| \leqq 5\tau(t_0)\,\|y_2 - y_1\|.$$

Let $\xi = \xi(t, t_0, y_0) = (y(t, t_0, y_0), z(t, t_0, y_0))$ be the unique solution of (8.1) supplied by Theorem 8.1 and the first part of Theorem 8.2. Thus

(9.10) $$\xi(t_0, t_0, y_0) = (y_0, g(t_0, y_0)).$$

The uniqueness of this solution implies that for $t_1 \geqq t_0$,

(9.11) $$\xi(t, t_0, y_0) = \xi(t, t_1, y(t_1, t_0, y_0)).$$

In order to examine the continuity of $g(t_0, y_0)$ with respect to t_0, consider $\xi(t, t_0, y_0) - \xi(t, t_1, y_0)$ for $t_1 \geqq t_0$ and small $\|y_0\|$. In view of (9.11), this difference can be written as $\xi(t, t_1, y(t_1, t_0, y_0)) - \xi(t, t_1, y_0)$. The analogue of (9.8) holds and at $t = t_1$ gives

$$\|z(t_1, t_1, y(t_1, t_0, y_0)) - g(t_1, y_0)\| \leqq 5\tau(t_1)\,\|y(t_1, t_0, y_0) - y_0\|.$$

Since $\xi = \xi(t, t_0, y_0)$ remains in a compact ξ-set for $t_0 \leqq t \leqq t_0 + 1$, and $\|y_0\|$ small, it follows from (8.1) that $\|\xi'(t, t_0, y_0)\| \leqq M$, if M is a bound for F on this set. Hence $\|\xi(t_1, t_0, y_0) - \xi(t_0, t_0, y_0\| \leqq M(t_1 - t_0)$, so that

$$\|z(t_1, t_1, y(t_1, t_0, y_0)) - g(t_0, y_0)\|, \quad \|y(t_1, t_0, y_0) - y_0\| \leqq M(t_1 - t_0).$$

Hence, for $t_0 \leqq t_1 \leqq t_0 + 1$ and $M = M(t_0, y_0)$,

$$(9.12) \qquad \|g(t_1, y_0) - g(t_0, y_0)\| \leqq M(t_1 - t_0)[1 + 5\tau(t_1)].$$

The inequalities (9.9), (9.12) complete the proof of Theorem 8.2.

10. Proof of Theorem 8.3

It will be shown that $\xi(t, t_0, y_0)$ is of class C^1; in particular, $\xi(t_0, t_0, y_0) = (y_0, g(t_0, y_0))$ is of class C^1. The proof will be given as if all variables and functions are real-valued. This is justified since a real system is obtained by separating real and imaginary parts of (8.1); cf. the interpretation $\partial u/\partial y^k = \frac{1}{2}(\partial u/\partial \sigma^k - i\, \partial u/\partial \tau^k)$ if $y^k = \sigma^k + i\tau^k$ mentioned after (3.2).

Let e be a unit vector in the y-space, $h \neq 0$ a small real number. By the Lemma V 3.1, the difference

$$(10.1) \qquad \zeta_h = \frac{\xi(t, t_0, y_0 + he) - \xi(t, t_0, y_0)}{h}$$

satisfies a linear differential equation of the form

$$(10.2) \qquad \zeta' = (E + E_1(t, h, y_0))\zeta,$$

where, in view of the continuity of $\xi(t, t_0, y_0)$,

$$(10.3) \qquad E_1(t, h, y_0) \to \partial_\xi F(t, \xi(t, t_0, y_0)) \qquad \text{as} \quad h \to 0$$

uniformly on bounded t-sets, and $\partial_\xi F$ denotes the Jacobian matrix of F with respect to ξ. By the analogue of (9.8), the function (10.1) is bounded by

$$[1 + 5\tau(t)] \frac{\|y(t, t_0, y_0 + he) - y(t, t_0, y_0)\|}{h}.$$

The derivation of (9.8) and the analogue of the inequality (4.17) in Lemma 4.1 show that this is at most

$$[1 + 5\tau(t)] \exp \int_{t_0}^{t} [\mu + \epsilon + 2\psi(s)]\, ds \equiv r(t).$$

Hence, for fixed (t_0, y_0), the family of functions (10.1) is uniformly bounded and equicontinuous in t on bounded t-intervals of $t \geqq t_0$. Thus there exist sequences $h_1, h_2, \ldots$ such that $h_n \to 0$ and the corresponding functions (10.1) tend to a limit $\zeta(t) = \zeta(t, t_0, y_0)$ uniformly for bounded

t ($\geqq t_0$). This limit satisfies the linear system

$$\zeta' = [E + \partial_\xi F(t, \xi(t, t_0, y_0))]\zeta, \tag{10.4}$$

an initial condition of the form $\zeta(t_0) = (e, z^*)$ for some z^*, and $\|\zeta(t)\| \leqq r(t)$. The last inequality implies that $\zeta(t) = 0$ or

$$\limsup_{t\to\infty} t^{-1} \log \|\zeta(t)\| \leqq \mu + \epsilon. \tag{10.5}$$

By (8.8), $\|\partial_\xi F(t, \xi)\| \leqq \psi(t)$. Then Theorems 8.1, 8.2 imply that if t_0 is sufficiently large, there is a unique z^* such that (10.4) has a solution satisfying $\zeta(t_0) = (e, z^*)$ and (10.5). Consequently, the selection of the sequence $h_1, h_2, \ldots$ is unnecessary and

$$\lim_{h\to 0} \frac{\xi(t, t_0, y_0 + he) - \xi(t, t_0, y_0)}{h} = \zeta(t, t_0, y_0) \tag{10.6}$$

exists uniformly on bounded t-intervals and is the unique solution of (10.4) satisfying (10.5) and $\zeta(t_0) = (e, z^*)$ for a unique z^*.

Hence $\xi(t, t_0, y_0)$ has partial derivatives with respect to the components of y_0. The continuity of these derivatives as functions of (t, t_0, y_0) follows from (10.4) and arguments similar to those just used to prove (10.6). The existence and continuity of $\partial\xi(t, t_0, y_0)/\partial t_0$ follows by the arguments in the proof of formula (V 3.4) in Theorem V 3.1.

Note that if $\partial_\xi F(t, 0) = 0$, then (10.4) reduces for $y_0 = 0$ to $\zeta' = E\zeta$. The only solutions $\zeta = (y(t), z(t))$ of this linear system satisfying (10.5) have $z(t) = 0$; cf. § IV 5. Thus $\partial_{y_0} z(t, t_0, 0) = 0$ and, at $t = t_0$, this gives $\partial_{y_0} g(t_0, 0) = 0$ and proves Theorem 8.3.

11. Logarithmic Scale (Continued)

The object of this section is to obtain improvements of the assertions of Theorem 8.1 without adding additional assumptions on F. To this end, let $\xi = (x, y, z)$, $E = \text{diag}\,[A_1, A_2, A_3]$, and $F(t, x, y, z) = (F_1, F_2, F_3)$, so that initial value problems associated with (8.1) take the form

$$x' = A_1 x + F_1, \qquad y' = A_2 y + F_2, \qquad z' = A_3 z + F_3, \tag{11.1}$$

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \qquad z(t_0) = z_0. \tag{11.2}$$

It will be assumed that the eigenvalues $\alpha_{j1}, \alpha_{j2}, \ldots$ of A_j satisfy

$$\text{Re}\,\alpha_{1k} < \mu, \qquad \text{Re}\,\alpha_{2k} = \mu, \qquad \text{Re}\,\alpha_{3k} > \mu \tag{11.3}$$

for some number μ.

Theorem 11.1. *Let* (8.1) *be equivalent to* (11.1), *where* A_1, A_2, A_3 *are matrices satisfying* (11.3), *and let* $F = (F_1, F_2, F_3)$, $\psi(t)$, δ, *and* μ *be as in Theorem* 8.1. *Then there exist* $\delta_1 > 0$, $T \geqq 0$ *and, for every* $t_0 \geqq T$, *a* *constant* $\sigma(t_0) > 0$ *such that if* $\|x_0\| < 7\sigma(t_0)\,\|y_0\|$ *and* $0 < \|y_0\| < \delta_1$, *then there is a* z_0 *with the properties that* (11.1)–(11.2) *has a solution for* $t \geqq t_0$ *satisfying* $y(t) \neq 0$ *and*

$$\|x(t)\|, \quad \|z(t)\| = o(\|y(t)\|) \quad \text{as} \quad t \to \infty, \tag{11.4}$$

$$\lim_{t\to\infty} t^{-1} \log \|\xi(t)\| = \mu. \tag{11.5}$$

This theorem, which concerns *certain* solutions of (8.1), follows at once from Lemma 4.2 (with $\delta_1 = \infty$ when $\delta = \infty$). Note that if μ is the least [or greatest] real part of the eigenvalues of E (so that there are no x [or z] variables), a corresponding statement holds. In fact, this case is contained in Theorem 11.1 since dummy x or z variables can be added to the system (8.1), with suitable choices of A_1 or A_3 and $F_1 \equiv 0$ or $F_3 \equiv 0$. The next theorem concerns *all* solutions of (8.1).

Theorem 11.2. *Assume the hypotheses of Theorem* 11.1 *on* $F(t, \xi)$. *If* $\delta = \infty$, *let* $\xi_0(t) \not\equiv 0$ *be any solution of* (8.1); *and if* $\delta < \infty$, *let* $\xi_0(t) \not\equiv 0$ *be a solution of* (8.1) *for large* t *satisfying*

$$\limsup_{t\to\infty} t^{-1} \log \|\xi_0(t)\| < 0. \tag{11.6}$$

Then the limit (11.5) *exists and is the real part* μ *of an eigenvalue of* E. *If, in addition, coordinates in the* ξ*-space are chosen so that* (8.1) *is of the form* (11.1), *where* (11.3) *holds, then* $\xi(t) = (x(t), y(t), z(t))$ *satisfies* (11.4).

It is clear that the first part of Corollary 8.1 has a similar improvement:

Corollary 11.1. *Let the assumptions of Theorem* 11.1 [*or Theorem* 11.2] *hold except that* (8.2) *is replaced by* (8.11), *and let* $\delta < \infty$, $\mu < 0$. *Then the conclusions of Theorem* 11.1 [*or Theorem* 11.2] *remain valid.*

Exercise 11.1. (*a*). Consider the case of a linear system of differential equations

$$\xi' = [E + G(t)]\xi, \tag{11.7}$$

where $G(t)$ is a continuous matrix for $t \geqq 0$ such that $\|G(t)\| \leqq \psi(t)$, where $\psi(t)$ is continuous and satisfies (4.24). Let $E = \operatorname{diag}[\lambda_1, \ldots, \lambda_d]$, and let the real parts $\mu_1, \ldots, \mu_d$ of $\lambda_1, \ldots, \lambda_d$ be distinct. Then, for any j, $1 \leqq j \leqq d$, (11.7) has a solution $\xi(t) = (\xi^1(t), \ldots, \xi^d(t))$ such that $\xi^j(t) \neq 0$ for large t, $|\xi^k(t)| = o(|\xi^j(t)|)$ as $t \to \infty$ for $k \neq j$, and $t^{-1} \log |\xi^j(t)| \to \mu_j$ as $t \to \infty$. (*b*) Show that if $\int^\infty \psi(t)\,dt < \infty$, then $\xi^j(t) = [c + o(1)] \exp \lambda_j t$, as $t \to \infty$, for some constant $c \neq 0$.

Exercise 11.2. Let $E = \operatorname{diag}[A_1, A_2, A_3]$, where A_j is a square matrix with eigenvalues $\alpha_{j1}, \alpha_{j2}, \ldots$ satisfying (11.3). Let $G(t)$ be a continuous matrix for $t \geqq 0$ and identify (11.1) with (11.7), where $\xi = (x, y, z)$ and $F(t, \xi) = G(t)\xi$. Suppose that $\|G(t)\| \leqq \psi(t)$, where $\psi(t)$ is continuous and satisfies $\int^{\infty} |\psi(t)|^p \, dt < \infty$ for some p, $1 \leqq p \leqq 2$. Let A_2 be a 1×1 matrix, consisting of the constant λ, $\operatorname{Re} \lambda = \mu$; thus y is 1-dimensional. Let $\xi(t) = (x(t), y(t), z(t))$ be a solution of (11.7) satisfying (11.4), (11.5). Show that there is a constant $c \neq 0$ such that

$$y(t) = [c + o(1)] \exp \int_0^t [\lambda + g(s)] \, ds,$$

where $g(t)$ is the diagonal element of $G(t)$ which is the coefficient of y in the second equation of (11.1). Note that this equation is the form $y' = \lambda y + \Sigma q_j(t)x^j + g(t)y + \Sigma r_k(t)z^k$, where $(q_1, q_2, \ldots, g, r_1, r_2, \ldots)$ is a row of $G(t)$.

Exercise 11.3. Let $f(t, y)$ be continuous and have continuous partial derivatives with respect to the components of y on a (t, y)-domain and be periodic of period p in t, $f(t + p, y) = f(t, y)$. Let

$$y' = f(t, y) \tag{11.8}$$

have a periodic solution $y = \gamma(t)$ of period p. Discuss the behavior of solutions of (11.8) and $y(t_0) = y_0$, where (t_0, y_0) is near the curve $(t, \gamma(t))$, $0 \leqq t \leqq p$, on the basis of the following suggestions: Introduce the new variables

$$\zeta = y - \gamma(t) \tag{11.9}$$

Thus (11.8) becomes

$$\zeta' = f(t, \zeta + \gamma(t)) - f(t, \gamma(t)),$$

which can be written as

$$\zeta' = P(t)\zeta + H(t, \zeta), \tag{11.10}$$

where

$$P(t) = \partial_y f(t, y) = \left(\frac{\partial f^j}{\partial y^k}\right) \quad \text{at} \quad y = \gamma(t),$$

$$H(t, \zeta) = f(t, \zeta + \gamma(t)) - f(t, \gamma(t)) - P(t)\zeta, \tag{11.11}$$

$P(t)$ is a matrix function of period p and $H(t, \zeta)$ is continuous and has continuous partial derivatives with respect to the components of ζ, and $H(t, 0) = 0$, $\partial_\zeta H(t, 0) = 0$. The linear matrix initial value problem

$$J' = P(t)J, \qquad J(0) = I \tag{11.12}$$

has a solution which, by the Floquet theory in § IV 6, is of the form

$$J(t) = K(t)e^{Et}, \tag{11.13}$$

where $K(t + p) = K(t)$ and E is a constant matrix. The change of variables

$$\zeta = K(t)\xi \tag{11.14}$$

transforms (11.10) into

$$\xi' = E\xi + K^{-1}(t)H(t, K(t)\xi). \tag{11.15}$$

Consider the application of the theorems of § 8 and of this section to (11.15) to obtain generalizations of the results of §§ IX 10, 11. (Note that e^E need not have $\lambda = 1$ as an eigenvalue in the situation here.)

12. Proof of Theorem 11.2

It will be shown that it is sufficient to consider the case of linear equations. Note that (8.2) implies that if the solution $\xi = \xi_0(t)$ of (8.1) vanishes at one t-value, then it vanishes for all t. Hence $\xi_0(t) \neq 0$ for large t, say $t \geqq t_0$. Define a matrix $G(t) = (g_{jk}(t))$ as follows: if $F = (F^1, F^2, \ldots)$, put

$$g_{jk}(t) = \frac{F^j(t, \xi_0(t))\bar{\xi}_0^{\,k}(t)}{\|\xi_0(t)\|^2} \tag{12.1}$$

for $t \geqq t_0$. Since $\xi = \xi_0(t)$ is a solution of (8.1), it follows that it is a solution of the linear system

$$\xi' = (E + G(t))\xi. \tag{12.2}$$

Note that (8.2) and (12.1) imply that

$$|g_{jk}(t)| \leqq \frac{\psi(t)\ \|\xi_0(t)\| \cdot |\xi_0^{\,k}(t)|}{\|\xi_0(t)\|^2} \leqq \psi(t).$$

Hence Theorem 11.2 is contained in the following:

Lemma 12.1 *Let $G(t)$ be a continuous matrix for $t \geqq 0$ such that*

$$\|G(t)\| \leqq \psi(t), \tag{12.3}$$

where $\psi(t) > 0$ is a continuous function satisfying (4.24). *Let $\xi = \xi_0(t) \neq 0$ be a solution of* (12.2). *Then the conclusion of Theorem* 11.2 *holds.*

Proof of Lemma 12.1. Let $\mu_1 < \mu_2 < \cdots < \mu_f$ denote the different real parts of the eigenvalues of E. After a change of coordinates, it can be supposed that $E = \text{diag}\,[B_1, B_2, \ldots, B_f]$, where the eigenvalues β_{jk} of B_j satisfy $\text{Re}\,\beta_{jk} = \mu_j$. Correspondingly, let $\xi = (y_1, \ldots, y_f)$, $E\xi = (B_1y_1, \ldots, B_fy_f)$, and let (12.2) be written as

$$y_j' = B_jy_j + \sum_{k=1}^{f} G_{jk}(t)y_k \qquad \text{for}\quad j = 1, \ldots, f, \tag{12.4}$$

where $G_{jk}(t)$ is a rectangular matrix and $\|G_{jk}(t)\| \leqq \psi(t)$.

If $1 \leqq q \leqq f$, t_0 is sufficiently large, and $y_{q0} \neq 0$, then Theorem 11.1 implies that (12.4) has a solution $\xi = (y_1(t), \ldots, y_f(t))$ satisfying

$$(12.5) \qquad y_k(t_0) = 0 \quad \text{if} \quad k < q, \qquad y_q(t_0) = y_{q0},$$

$$(12.6) \qquad \|y_k(t)\| = o(\|y_q(t)\|) \quad \text{as} \quad t \to \infty \quad \text{for} \quad k \neq q,$$

$$(12.7) \qquad t^{-1} \log \|y_q(t)\| \to \mu_q \quad \text{as} \quad t \to \infty.$$

This solution, say $\xi = \xi_q(t, t_0, y_{q0})$, is unique by Theorem 8.2. In fact, it is unique even if (12.6), (12.7) are replaced by

$$(12.8) \qquad \limsup_{t \to \infty} \; t^{-1} \log \|\xi(t)\| \leqq \mu_q$$

(cf. Remark 2 following Theorem 8.3). This uniqueness implies that $\xi_q(t, t_0, y_{q0})$ is linear in y_{q0} (for fixed t, t_0, q).

With the understanding that $\xi_q(t, t_0, 0) \equiv 0$, it follows that there exist unique $y_{10}, \ldots, y_{f0}$ such that the given solution $\xi_0(t)$ is of the form

$$(12.9) \qquad \xi_0(t) = \sum_{j=1}^{f} \xi_j(t, t_0, y_{j_0}).$$

In fact, $y_{10}, \ldots, y_{f0}$ are defined recursively as follows: if $\xi_0(t) = (y_1(t), \ldots, y_f(t))$, let $y_{10} = y_1(t_0)$; then let $y_{20} = y_2(t_0) - y_{12}(t_0)$, where $\xi_1(t, t_0, y_{10}) = (y_{11}(t), y_{12}(t), \ldots, y_{1f}(t))$; etc.

Let q be the largest j-value such that $y_{j0} \neq 0$ in (12.9). It is clear that $\xi_0(t) = (y_1(t), \ldots, y_f(t))$ satisfies (12.6), (12.7). This proves the lemma.

13. Asymptotic Integration

The object of this section is to study the asymptotic behavior of solutions $\xi(t)$ of a perturbed linear systems

$$(13.1) \qquad \xi' = E\xi + F(t, \xi),$$

rather than the behavior of $\|\xi(t)\|$ as in § 11.

Suppose that E is in a Jordan normal form $E = \operatorname{diag}[J(1), \ldots, J(g)]$, where $J(j)$ is an $h(j) \times h(j)$ matrix [as in (IV 5.15)–(IV 5.16)]. Thus $J(j) = \lambda(j)I_{h(j)} + K_{h(j)}$, where I_h is the unit $h \times h$ matrix and K_h is 0 if $h = 1$ or is the $h \times h$ matrix with ones on the subdiagonal and other elements zero if $h > 1$. According as $h = 1$ or $h > 1$,

$$(13.2) \quad J(j)y_j = \lambda y_j \quad \text{or} \quad J(j)y_j = (\lambda y_j^{\,1}, \lambda y_j^{\,2} + y_j^{\,1}, \ldots, \lambda y_j^{\,h} + y_j^{h-1}),$$

where $\lambda = \lambda(j)$, $y_j = (y_j^{\,1}, \ldots, y_j^{\,h})$, $h = h(j)$.

Correspondingly, it is supposed that $\xi = (y_1, \ldots, y_g)$,

$$E\xi = (J(1)y_1, \ldots, J(g)y_g), \; F = (F_1, \ldots, F_g),$$

and (13.1) is of the form

(13.3) $$y_j' = J(j)y_j + F_j(t, \xi) \qquad \text{for } j = 1, \ldots, g.$$

Let μ denote one of the numbers $\text{Re}\,\lambda(1), \ldots, \text{Re}\,\lambda(g)$. An index j will be denoted by p, q, or r according as $\text{Re}\,\lambda(j) < \mu$, $\text{Re}\,\lambda(j) = \mu$ or $\text{Re}\,\lambda(j) > \mu$. Put

(13.4) $$h_* = \max_q h(q).$$

Let j_0 be an integer and β a number satisfying

(13.5) $$j_0 \leqq h_* - 1 \qquad \text{and} \qquad \beta \geqq 1,$$

and $l(q)$, $k(q)$ integers, if any, such that

(13.6) $$1 \leqq l(q) \leqq k(q) \leqq \min(h(q), \beta) \qquad \text{and} \quad h(q) - l(q) \leqq j_0.$$

The next theorem concerns sufficient conditions for (13.3) to have a solution with the following asymptotic properties as $t \to \infty$,

(13.7) $$y_q^k = e^{\lambda(q)t}\left\{\sum\nolimits_{\text{I}} c_q^i \frac{t^{k-i}}{(k-i)!} + o(t^{k-\beta})\right\}, \qquad y_j^k = o(e^{\mu t}t^{1-\beta}) \quad \text{if } j \neq q,$$

where c_q^i are constants,

(13.8) $$\sum\nolimits_{\text{I}} = \sum_{i=l(q)}^{\min(k,k(q))},$$

and $\sum_{\text{I}} = 0$ if $l(q)$, $k(q)$ do not exist.

Note that if the o-terms are replaced by 0, then, since $1 \leqq i \leqq k$ in $\sum_{\text{I}}$, (13.7) becomes a solution of the linear system

(13.9) $$\xi' = E\xi; \qquad \text{i.e.,} \quad y_j' = J(j)y_j \qquad \text{for } j = 1, \ldots, g.$$

The choice of the range of summation $l(q) \leqq i \leqq \min(k, k(q))$ is dictated by several considerations. On the one hand, results permitting $i > k$ can easily (but will not) be obtained as a consequence of Theorem 13.1; also the first term in the first line of (13.7) is not significant unless $i \leqq \beta$, hence the choice $i \leqq \min(k, k(q)) \leqq \min(k, \beta)$ since $k \leqq h(q)$. On the other hand, the condition $i \geqq l(q)$ means that the degree of the polynomial $\sum_{\text{I}} c_q^i t^{k-i}/(k-i)!$ does not exceed the given j_0.

Theorem 13.1. *In the system* (13.3), *let $J(j)$ be a Jordan block; cf.* (13.2). *Let $\mu = \text{Re}\,\lambda(j)$ for some j. Let an index $j = 1, \ldots, g$ be denoted by p, q or r according as* $\text{Re}\,\lambda(j) < \mu$, $\text{Re}\,\lambda(j) = \mu$, or $\text{Re}\,\lambda(j) > \mu$, *and* *define* h_* *by* (13.4). *Let* j_0 *be an integer and* β *a number satisfying*

$$(13.10)\qquad 0 \leqq j_0 \leqq h_* - 1, \quad \beta + j_0 \geqq h_*; \qquad \text{thus} \quad \beta \geqq 1.$$

Let $l(q), k(q)$ *be integers (if any) satisfying* (13.6). *Let* $F(t, \xi) = (F_1, \ldots, F_g)$ *be continuous for* $t \geqq 0$ *and all* ξ, *and satisfy*

$$(13.11)\qquad \|F(t, \xi) \leqq \psi_1(t) \|\xi\|,$$

where $\psi_1(t) > 0$ *is a continuous function such that*

$$(13.12)\qquad \int^\infty t^{\beta+j_0-1}\psi_1(t)\,dt < \infty.$$

Let $m = \sum_p h(p) + \sum_q [h(q) - k(q)]$. *For any set of constants* c_q^k, $l(q) \leqq k \leqq k(q)$, *not all* 0, *there exists an m parameter family of solutions* $\xi(t)$ *of* (13.3) *defined for large t and satisfying the asymptotic relations* (13.7) *as* $t \to \infty$.

The part of the assertion concerning "m parameter family of solutions" means essentially that it is possible to specify a partial set of m "initial conditions," as well as the asymptotic behavior (13.7) for $\xi(t)$; cf. the statement following (14.15) in the proof of Theorem 13.1.

Remark 1. Consider a system of differential equations

$$(13.13)\qquad \eta' = E^0\eta + F^0(t, \eta),$$

where E^0 is a constant matrix and $F^0(t, \eta)$ is continuous for $t \geqq 0$ and all η. Let L be a nonsingular constant matrix such that $L^{-1}E^0L$ is a matrix $E = \text{diag}\,[J(1), \ldots, J(g)]$ in a Jordan normal form. Then the change of variables $\eta = L\xi$ reduces (13.13) to (13.1) [i.e., to (13.3)], where $F(t, \xi) = L^{-1}F^0(t, L\xi)$. The applicability of Theorem 13.1, or at least the condition (13.11), can sometimes be verified without the knowledge of L or the explicit reduction of (13.13) to (13.1). For it is clear that $\|F^0(t, \eta)\| \leqq \psi_1(t) \|\eta\|$ implies that $\|F(t, \xi)\| \leqq c\psi_1(t) \|\xi\|$ if, e.g., $c = \|L^{-1}\| \cdot \|L\|$.

Remark 2. The derivation of Theorem 13.1 from Lemma 4.3 will show that the theorem remains valid if $F(t, \xi)$ is defined only for $t \geqq 0$, $\|\xi\| < \delta < \infty$ if $\mu < 0$ (or $\mu = 0$, $h_* = 1$, and the constants $|c_q^k|$ are sufficiently small).

Theorem 13.1 has a partial "converse" dealing with *all* (rather than *certain*) solutions $\xi(t)$ of (13.1) satisfying

$$(13.14)\qquad t^{-1} \log \|\xi(t)\| \to \mu \qquad \text{as} \quad t \to \infty$$

(cf. Theorem 11.2):

Theorem 13.2. *Let* $E = \text{diag}\,[J(1), \ldots, J(g)]$ *and* $F(t, \xi)$ *be as in*

Theorem 13.1 *except that* (13.12) *is replaced by*

$$(13.15) \qquad \int^{\infty} t^{h_0-1}\psi_1(t)\,dt < \infty, \qquad \text{where} \quad h_0 \geqq h_*$$

(*and* h_0 *is not necessarily an integer*). *Let* $\xi(t) \neq 0$ *be a solution of* (13.3) *satisfying* (13.14). *Then there exists constants* c_q^k, $k = 1, \ldots, h(q)$, *not all* 0, *such that if* j_0 *is defined by*

$$(13.16) \qquad j_0 = \max\,[k(q) - k] \qquad \text{for} \quad c_q^k \neq 0,$$

$\beta = h_0 - j_0$, *and* $l(q)$, $k(q)$ *are the least, greatest integers* (*if any*) *satisfying* (13.6), *then* $\xi(t)$ *satisfies the asymptotic relations* (13.7) *as* $t \to \infty$.

Consequences and refinements of Theorem 13.1, 13.2 will be given in § 16; see also § XII 9.

14. Proof of Theorem 13.1

Change of Variables. In order to apply Lemma 4.3, make the linear change of variables

$$(14.1) \qquad \xi = Q(t)\zeta$$

given in terms of $\xi = (y_1, \ldots, y_g)$ and $\zeta = (z_1, \ldots, z_g)$ by the formulae

$$(14.2) \qquad \begin{aligned} y_q^k &= e^{\lambda(q)t}\left\{\sum\nolimits_{\mathrm{I}} \frac{t^{k-i}}{(k-i)!} z_q^i + t^{k-\beta} \sum\nolimits_{\mathrm{II}} \frac{1}{(k-i)!} z_q^i\right\}, \\ y_j^k &= \epsilon^{-k} e^{\mu t} t^{1-\beta} z_j^k \qquad \text{if} \quad j \neq q, \end{aligned}$$

where $0 < \epsilon < 1$, $\sum_{\mathrm{I}}$ is the sum over the i-range $l(q) \leqq i \leqq \min(k, k(q))$ as in (13.8), and $\sum_{\mathrm{II}}$ is the sum over the other indices i on the range $1 \leqq i \leqq k$, so that

$$\sum\nolimits_{\mathrm{I}} + \sum\nolimits_{\mathrm{II}} = \sum_{i=1}^{k}.$$

A solution $\xi(t)$ of (13.3) satisfies (13.7) if the corresponding vector $\zeta(t)$, defined by (14.1), satisfies

$$(14.3) \qquad \begin{aligned} z_q^i &= c_q^i + o(t^{i-\beta}) \qquad &&\text{for} \quad l(q) \leqq i \leqq h(q), \\ z_j^k &= o(1) &&\text{otherwise.} \end{aligned}$$

To clarify the meaning of (14.1) and to calculate the resulting differential equation for ζ, the map (14.1) will be given a decomposition of the form

$$(14.4) \qquad \xi = Q(t)\zeta = D(t)Q_0(t)\zeta,$$

to be described. This factorization is suggested by the fact that if $t^{k-\beta}$ in the first formula of (14.2) is replaced by t^{k-i} (and written behind the sign $\sum_{\mathrm{II}}$), then this formula becomes $y_q = e^{J(q)t}z_q$.

The change of variables $\xi = Q_0(t)w$, $w = (w_1, \ldots, w_q)$, is given by

$$\xi_j = e^{J(j)t}w_j \quad \text{if} \quad j = q, \qquad \xi_j = e^{\mu t}w_j \quad \text{if} \quad j \neq q. \tag{14.5}$$

Thus (13.3) becomes

$$\begin{aligned} w_q' &= e^{-J(q)t}F_q(t, Q_0 w), \\ w_j' &= [J(j) - \mu I_h]w_j + e^{-\mu t}F_j(t, Q_0 w) \quad \text{if} \quad j \neq q, \end{aligned} \tag{14.6}$$

where $h = h(j)$. Finally, let $D(t)$ be the diagonal matrix such that $w = D(t)\zeta$ is given by

$$\begin{aligned} w_q^k &= z_q^k \quad \text{if} \quad l(q) \leqq k \leqq k(q), \\ w_q^k &= t^{k-\beta}z_q^k \quad \text{if} \quad k < l(q) \quad \text{or} \quad k > k(q), \\ w_j^k &= \epsilon^{1-k}t^{1-\beta}z_j^k \quad \text{if} \quad j \neq q. \end{aligned} \tag{14.7}$$

If the resulting differential equation for ζ is written as

$$\zeta' = E_0\zeta + Q^{-1}(t)F(t, Q\zeta), \tag{14.8}$$

then the linear part $\zeta' = E_0\zeta$ is given by

$$\begin{aligned} z_q^{k\prime} &= 0 \quad \text{if} \quad l(q) \leqq k \leqq k(q), \\ z_q^{k\prime} &= (\beta - k)t^{-1}z_q^k \quad \text{if} \quad k < l(q) \quad \text{or} \quad k > k(q), \\ z_j' &= [J_\epsilon(j) - \mu I_h - (\beta - 1)t^{-1}I_h]z_j \quad \text{if} \quad j \neq q, \end{aligned} \tag{14.9}$$

where $J_\epsilon(j)$ is the matrix obtained by replacing the ones on the subdiagonal of $J(j)$ by ϵ; cf. § IV 9. The last part of (14.9) is easy to see if the transformation $w_j \to z_j$ is made in two steps $w_j^k \to z_j^k/\epsilon^{k-1} \to t^{1-\beta}z_j^k/\epsilon^{k-1}$.

Finally, replace the independent variable t by s, where

$$t = e^s, \qquad \text{so that} \quad \frac{d\zeta}{ds} = t\zeta'. \tag{14.10}$$

Thus (14.8) becomes

$$\frac{d\zeta}{ds} = E_0 t\zeta + tQ^{-1}F(t, Q\zeta), \tag{14.11}$$

where the linear part of this equation is

$$\begin{aligned} \frac{dz_q^k}{ds} &= 0 \quad \text{if} \quad l(q) \leqq k \leqq k(q), \\ \frac{dz_q^k}{ds} &= (\beta - k)z_q^k \quad \text{if} \quad k < l(q) \quad \text{or} \quad k > k(q), \\ \frac{dz_j}{ds} &= [(J_\epsilon(j) - \mu I_h)t + (\beta - 1)I_h]z_j \quad \text{if} \quad j \neq q. \end{aligned} \tag{14.12}$$

Preliminary Existence Result. Suppose that there is a continuous function $\psi_0(t)$ for large t such that

$$\|Q^{-1}(t)F(t, Q\zeta)\| \leqq \psi_0(t)\,\|\zeta\| \qquad \text{for} \quad \|\zeta\| < \delta \leqq \infty, \tag{14.13}$$

$$\int^{\infty} \psi_0(t)\,dt < \infty. \tag{14.14}$$

The last condition is equivalent to $\int^{\infty} t\psi_0(t)\,ds < \infty$ since $ds = dt/t$. Then if $\epsilon > 0$ is sufficiently small, Lemma 4.3 is applicable to (14.11) if x is a vector with components z_p^k, and z_q^k, $k > k(q)$; y is the vector with components z_q^k, $l(q) \leqq k \leqq k(q)$; and z is the vector with components z_r^k and z_q^k, $k < l(q)$. Note that (14.12) shows that there is a constant $c > 0$ such that $\operatorname{Re}(z_q^k\, d\bar{z}_q^k/ds) \leqq -c\,|z_q^k|^2$ or $\geqq c\,|z_q^k|^2$ according as $k > k(q) \geqq \beta$ or $k < l(q) \leqq \beta$; also $\operatorname{Re}(z_j \cdot d\bar{z}^j/ds) \leqq -ct\,\|z_j\|^2$ or $\geqq ct\,\|z_j\|^2$ according as $j = p$ (i.e., $\operatorname{Re}\lambda(j) < \mu$) or $j = r$ (i.e., $\operatorname{Re}\lambda(j) > \mu$) if $\epsilon > 0$ is small and $t > 0$ is large.

Thus, by Lemma 4.3, (14.13)–(14.14) imply that if c_q^k, $l(q) \leqq k \leqq k(q)$, are given constants, not all 0, then there exists a solution $\zeta(t)$ of (14.11) such that, as $t \to \infty$,

$$\begin{aligned} z_q^k(t) &= c_q^k + o(1) \qquad \text{for} \quad l(q) \leqq k \leqq k(q), \\ z_j^k &= o(1) \qquad \text{otherwise.} \end{aligned} \tag{14.15}$$

In fact, we can also specify a set of m initial conditions for ζ: $z_p^k(T) = z_{p0}^k$ and $z_q^k(T) = z_{q0}^k$ for $k(q) < k \leqq h(q)$ if T is sufficiently large and $|z_{p0}^k|$, $|z_{q0}^k|$ are sufficiently small numbers.

The Norms $\|Q\|$, $\|Q^{-1}\|$. In order to complete the proof, it remains to show that the assumptions (13.11)–(13.12) imply (14.13)–(14.14) and that a solution $\zeta(t)$ of (14.8) satisfying (14.15) also satisfies (14.3). To this end, it will first be verified that there exist positive constants c, c' such that for large t,

$$c' \leqq \|Q(t)\|\, e^{-\mu t} t^{-j_0} \leqq c, \qquad c' \leqq \|Q^{-1}(t)\|\, e^{\mu t} t^{1-\beta} \leqq c. \tag{14.16}$$

From (14.2), the norm of $Q(t)$ is easily seen to be $O(e^{\mu t} t^{\gamma})$, where $\gamma = \max[h_* - \beta, h(q) - l(q)]$ and the max refers to the set q. From (13.6), $h(q) - l(q) \leqq j_0$ and, from (13.10), $h_* - \beta \leqq j_0$; hence $\|Q(t)\| = O(e^{\mu t} t^{j_0})$ as $t \to \infty$. It is similarly seen that $e^{\mu t} t^{j_0} = O(\|Q(t)\|)$ as $t \to \infty$. This gives the first part of (14.16).

The factorization $Q = DQ_0$ of Q into nonsingular matrices for $t > 0$ shows that Q^{-1} exists and is $Q^{-1} = Q_0^{-1} D^{-1}$. The inverse map

$$\zeta = Q^{-1}\xi \tag{14.17}$$

is easily seen, from $\xi = Q_0 w$, $w = D\zeta$ in (14.5), (14.7), to be

$$z_q^{\,k} = e^{-\lambda(q)t} \sum_{i=1}^{k} \frac{(-1)^i t^{k-i}}{(k-i)!} y_q^{\,i} \qquad \text{if} \quad l(q) \leqq k \leqq k(q),$$

$$(14.18) \quad z_q^{\,k} = e^{-\lambda(q)t} \sum_{i=1}^{k} \frac{(-1)^i t^{\beta-i}}{(k-i)!} y_q^{\,i} \qquad \text{if} \quad k < l(q) \qquad \text{or} \quad k > k(q),$$

$$z_j^{\,k} = \epsilon^{k-1} e^{-\mu t} t^{\beta-1} y_j^{\,k} \qquad \text{if} \quad j \neq q.$$

Thus, for large t, $\|Q^{-1}(t)\|$ is bounded from above and below by a positive constant times $e^{-\mu t} t^{\gamma}$, where $\gamma = \max\,[\beta - 1, k(q) - 1]$. Since $k(q) - 1 \leqq \beta - 1$, by (13.6), the last part of (14.16) follows.

Completion of the Proof. In view of (13.11),

$$\|Q^{-1}F(t, Q\zeta)\| \leqq \|Q^{-1}\|\ \psi_1(t)\ \|Q\| \cdot \|\zeta\|.$$

Hence, by (14.16),

$$(14.19) \qquad \|Q^{-1}F(t, Q\zeta)\| \leqq c^2 t^{\beta + j_0 - 1} \psi_1(t).$$

Thus (13.12) implies that (14.13), (14.14) hold if $\psi_0(t) = c^2 t^{\beta+j_0-1}\,\psi_1(t)$, and so (14.8) has a solution $\zeta(t)$ satisfying (14.15).

In view of the first part (14.9), the corresponding equations in (14.8) are

$$z_q^{k\prime} = (qk)\text{th component of } Q^{-1}F(t, Q\zeta),$$

so that, by (14.18),

$$z_q^{k\prime} = e^{-\lambda(q)t} \sum_{i=1}^{k} \frac{(-1)^i t^{k-i}}{(k-i)!} F_q^{\,i}(t, Q\zeta) \qquad \text{if} \quad l(q) \leqq k \leqq k(q),$$

where $F_q^{\,i}$ is the (qi)th component of F. Hence,

$$z_q^{k\prime} = O(e^{-\mu t} t^{k-1} \psi_1(t)\ \|Q\zeta\|) \qquad \text{as} \quad t \to \infty,$$

by (13.11). In view of (14.16) and the boundedness of $\zeta(t)$ as $t \to \infty$,

$$z_q^{k\prime} = O(t^{k+j_0-1}\psi_1(t)) \qquad \text{as} \quad t \to \infty.$$

Consequently, $k \leqq \beta$ shows that

$$(14.20) \qquad z_q^{\,k}(t) - c_q^{\,k} = t^{k-\beta} O\left(\int_t^\infty s^{\beta+j_0-1}\psi_1(s)\, ds\right).$$

This gives the first part of (14.3) and completes the proof of Theorem 13.1.

15. Proof of Theorem 13.2

This theorem can be reduced to the case of linear equations by the device used at the beginning of § 12. Hence we can be suppose that $F(t, \xi) = G(t)\xi$, where $G(t)$ is a matrix satisfying $\|G(t)\| \leqq \psi_1(t)$ and (13.1) is replaced by

$$(15.1) \qquad \xi' = E\xi + G(t)\xi.$$

Let q_0 denote a fixed value of q and k_0 an integer on the range $1 \leqq k_0 \leqq h(q_0)$. Then the equation (15.1) has a solution $\xi_{k_0 q_0}(t)$ satisfying, as $t \to \infty$,

$$
\begin{aligned}
y_q^k(t) &= \frac{e^{\lambda(q)t} t^{k-k_0}}{(k-k_0)!} + o(e^{\mu t} t^{k-\gamma}) \quad && \text{if} \quad q = q_0, \quad && k_0 \leqq k \leqq h(q_0), \\
y_q^k(t) &= o(e^{\mu t} t^{k-\gamma}) && \text{if} \quad q = q_0, && 1 \leqq k < k_0, \\
y_\gamma^k(t) &= o(e^{\mu t} t^{k-\gamma}) && \text{if} \quad q \neq q_0, && 1 \leqq k \leqq h(q), \\
y_q^k(t) &= o(e^{\mu t} t^{1-\gamma}) && \text{if} \quad j \neq q,
\end{aligned} \tag{15.2}
$$

where $\gamma = h_0 - h(q_0) + k_0 \geqq 1$. This follows from Theorem 13.1 with j_0, β replaced by $h(q_0) - k_0$, $\gamma = h_0 - [h(q_0) - k_0]$ and the choice $c_q^k = 1$ or $c_q^k = 0$ according as $(qk) = (q_0 k_0)$ or $(qk) \neq (q_0 k_0)$.

The set of solutions $\xi_{qk}(t)$ is a set of $\Sigma h(q)$ linearly independent solutions. Also if $n = \Sigma h(p)$, then Theorem 8.2 implies that there are exactly n linearly independent solutions $\xi_1(t), \ldots, \xi_n(t)$ satisfying

$$\limsup t^{-1} \log \|\xi(t)\| < \mu$$

and $n + \Sigma h(q)$ linearly independent solutions satisfying

$$\limsup t^{-1} \log \|\xi(t)\| \leqq \mu.$$

Hence if $\xi(t) \neq 0$ is a solution of (15.1) satisfying (13.14), then there exist constants $c_1, \ldots, c_n$ and c_q^k such that

$$\xi(t) = \sum_{j=1}^{n} c_j \xi_j(t) + \sum_q \sum_{k=1}^{k(q)} c_q^k \xi_{qk}(t) \tag{15.3}$$

and that not all c_q^k are 0. It will be left to the reader to verify that this implies Theorem 13.2.

16. Corollaries and Refinements

When the matrix E in Theorem 13.1 has simple elementary divisors (e.g., when the eigenvalues of E are distinct) or even if $h_* = 1$, then $h_* = 1, j_0 = 0$, $\beta \geqq 1$, and condition (13.12) reduces to

$$\int^{\infty} t^{\alpha} \psi_1(t)\, dt < \infty \qquad \text{for} \quad \alpha = \beta - 1 \geqq 0;$$

cf. Corollary 4.2. Here, the asymptotic formulae (13.7) reduce to

$$y_q(t) = e^{\lambda(q)t}[c_q + o(t^{-\alpha})], \qquad y_j(t) = e^{\lambda(q)t} o(t^{-\alpha}) \qquad \text{for} \quad j \neq q.$$

For a fixed j_0, the smallest admissible value of β in Theorem 13.1 is $\beta = h_* - j_0$ in which case (13.12) becomes $\int^{\infty} t^{h_*-1} \psi_1(t)\, dt < \infty$. A larger choice of β has the role of possibly increasing the number of significant terms in the asymptotic formulae (13.7) and of improving the error terms. When (13.12) is strengthened to

$$\int^{\infty} t^{2h_*-1}\psi_1(t)\,dt < \infty, \tag{16.1}$$

the maximal number of significant terms is possible. In this case, we have

Corollary 16.1. *Let* $E = \text{diag}\,[J(1), \ldots, J(g)]$, μ, h_* *be as in Theorem* 13.1 *and let* $F(t, \xi)$ *be continuous for* $t \geqq 0$ *and all* ξ *and satisfy* (13.11), *where* $\psi_1(t) > 0$ *is a continuous function satisfying* (16.1). *Let* $\xi = \xi_0(t) \neq 0$ *be a solution of the linear system* $\xi' = E\xi$ *such that* $t^{-1}\log\|\xi(t)\| \to \mu$ *as* $t \to \infty$. *Then* (13.1) *has a solution* $\xi(t)$ *satisfying* $\|\xi(t) - \xi_0(t)\|\, e^{-\mu t} \to 0$, $t \to \infty$.

In this corollary, E is not required to be in a Jordan normal form (cf. Remark 1 following Theorem 13.1). If it is, we can, in addition, assign a partial set of $\Sigma h(p)$ initial conditions, $y_p(t_0) = y_{p0}$ for sufficiently large t_0. Also, $\xi(t)$ satisfies the asymptotic relations (13.7), where $l(q) = 1$, $k(q) = h(q)$, $c_q{}^k$ are suitable constants determined by $\xi_0(t)$, j_0 is defined by (13.16), and $\beta = 2h_* - j_0$. This improves the asymptotic relation claimed in the corollary.

The deduction of Theorem 13.1 from Lemma 4.3 shows that assumptions (13.11), (13.12) can be weakened somewhat.

Corollary 16.2. *Let assumptions* (13.11), (13.12) *of Theorem* 13.1 *be relaxed to*

$$\|Q^{-1}(t)F(t, Q(t)\zeta)\| \leqq \psi_0(t)\,\|\zeta\|, \tag{16.2}$$

or, more generally, to

$$\|Q^{-1}(t)F(t, Q(t)\zeta)\| \leqq \psi(t)\,\|\zeta\| \tag{16.3}$$

$$e^{-\mu t}t^{\beta-i}\,|F_q^{\,i}(t, Q(t)\zeta)| \leqq \psi_0(t)\,\|\zeta\| \qquad \text{for} \quad i = 1, \ldots, k(q), \tag{16.4}$$

where $\xi = Q(t)\zeta$ *is given by* (14.2) *and* $\psi(t)$, $\psi_0(t)$ *are positive continuous functions for* $t > 0$ *such that*

$$\sup_{s \geqq t}\,(1 + s - t)^{-1}\int_s^t \psi(r)\,dr \to 0 \qquad \text{as} \quad t \to \infty, \tag{16.5}$$

$$\int^{\infty} \psi_0(t)\,dt < \infty. \tag{16.6}$$

Then the conclusions of Theorem 13.1 *remain valid.*

Exercise 16.1. By referring to Remark 1 following Lemma 4.3 and to the proof of Theorem 13.1, find sharper estimates for the o-terms in (13.7) under the conditions (16.3)–(16.6) of Corollary 16.2.

Remark 2 following Theorem 13.1 and Corollary 16.2 have important consequences. For example, suppose that F in (13.1) does not depend on

t, so that (13.1) can be written as

$$\xi' = E\xi + F(\xi), \tag{16.7}$$

where $F(\xi)$ is defined for $\|\xi\| < \delta < \infty$ and satisfies

$$\|F(\xi)\| \leqq C_0 \|\xi\|^{1+\theta}, \qquad \theta > 0, \qquad C_0 = \text{const.}, \tag{16.8}$$

or, more generally,

$$\|F(\xi)\| \leqq \frac{C_0 \|\xi\|}{|\log \|\xi\| |^{\nu}}, \qquad \nu > j_0 + \beta, \tag{16.9}$$

or even

$$\|F(\xi)\| \leqq \varphi(\|\xi\|) \|\xi\|, \tag{16.10}$$

where $\varphi(\rho)$ is a nondecreasing function for $0 \leqq \rho < \delta$ such that

$$\int_{+0} \rho^{-1} |\log \rho|^{j_0+\beta-1} \varphi(\rho)\, d\rho < \infty. \tag{16.11}$$

Then (14.16) and (16.10) imply that if $\mu < 0$ and $\|\zeta\| \leqq 1$, then

$$\|Q^{-1}\| \cdot \|F(Q\zeta)\| \leqq ce^{-\mu t} t^{\beta-1} \varphi(ce^{\mu t} t^{j_0}) ce^{\mu t} t^{j_0} \|\zeta\|$$

for large t. Thus the analogue of (16.2) holds with

$$\psi_0(t) = c^2 t^{j_0+\beta-1} \varphi(ce^{\mu t} t^{j_0}).$$

If $\rho = ce^{\mu t} t^{j_0}$ is introduced as a new integration variable in the integral in (16.11) and it is noted that $d\rho/\rho \sim \mu\, dt$ and $\log \rho \sim \mu t$ as $t \to \infty$, then it is seen that (16.6) is a consequence of (16.11).

Corollary 16.3. *In* (16.7), *let* $E = \operatorname{diag}[J(1), \ldots, J(g)]$ *be as in Theorem* 13.1, *let* $F(\xi)$ *be continuous for* $\|\xi\| < \delta$ $(< \infty)$ *and satisfy* (16.10), *where* $\varphi(\rho)$ *is a nondecreasing function of* ρ *satisfying* (16.11). *Let* $\mu < 0$. *Then the conclusions of Theorem* 13.1, *with* (13.3) *replaced by* (16.7), *remain valid.*

Exercise 16.2. By involving the Remark 2 following Lemma 4.3, show that conditions (16.10), (16.11) in Corollary 16.3 can be replaced by

$$\|F(t, \xi)\| \leqq \varphi_0(\|\xi\|) \qquad \text{for} \quad \|\xi\| < \delta, \tag{16.12}$$

where $\varphi_0(\rho)$ is a nondecreasing function of ρ, $0 < \rho < \delta$, such that

$$\int^{\infty} \rho^{-2} |\log \rho|^{\beta+j_0-1} \varphi_0(\rho)\, d\rho < \infty. \tag{16.13}$$

[This is somewhat more general than Corollary 16.3 for (16.12) implies (16.10) with $\varphi_0(\rho) = \varphi(\rho)\rho$. Although (16.10) is a consequence of (16.12) with $\varphi(\rho) = \varphi_0(\rho)/\rho$, the monotony of φ_0 does not imply that of φ.]

Analogously, we obtain the following consequence of the proofs of Theorems 13.1 and 13.2.

Corollary 16.4. *Let E, μ, F, φ be as in Corollary* 16.3 *except that* (16.11) *is replaced by*

(16.14) $$\int^{\infty} \rho^{-1} |\log \rho|^{h_0-1} \varphi(\rho)\, d\rho < \infty, \qquad \text{where} \quad h_0 \geqq h_*$$

(*and h_0 need not be an integer*). *Then the conclusions of Theorem* 13.2, *with* (13.1) *replaced by* (16.7), *are valid.*

17. Linear Higher Order Equations

The results of §§ 4, 11, 13, 16 will be applied in this section to a linear differential equation of order $d > 1$,

(17.1) $$u^{(d)} + [a_1 + p_1(t)]u^{(d-1)} + \cdots + [a_{d-1} + p_{d-1}(t)]u' + [a_d + p_d(t)]\, u = 0,$$

for a real- or complex-valued function u. This will be viewed as a perturbation of the equation

(17.2) $$u^{(d)} + a_1 u^{(d-1)} + \cdots + a_{d-1}u' + a_d u = 0$$

with constant coefficients. The characteristic equation for (17.2) is

(17.3) $$\lambda^d + a_1\lambda^{d-1} + \cdots + a_{d-1}\lambda + a_d = 0.$$

Equation (17.1) can be written as a linear system

(17.4) $$\xi' = [R + G(t)]\xi,$$

for the d-dimensional vector $\xi = (u^{(d-1)}, \ldots, u^{(1)}, u^{(0)})$, where $u = u^{(0)}$ and R, $G(t)$ are the matrices

(17.5) $$R = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{d-1} & -a_d \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

$$G(t) = \begin{pmatrix} -p_1 & -p_2 & -p_3 & \ldots & -p_{d-1} & -p_d \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Note first that if $a_1 = \cdots = a_d = 0$, then R is in the Jordan normal form and consists of one Jordan block with $\lambda = 0$ on its main diagonal. If, the coefficients $p_1(t), \ldots, p_d(t)$ are small, then (17.1) can be considered to be a perturbation of $u^{(d)} = 0$ which has the linearly independent solutions $u = 1, t, \ldots, t^{d-1}$. It will be verified that Corollary 16.2 has the following consequence.

Theorem 17.1 *In* (17.1), *let* $a_1 = \cdots = a_d = 0$ *and let* $p_1(t), \ldots, p_d(t)$ *be continuous complex-valued functions for* $t \geqq 0$ *satisfying*

$$(17.6) \qquad \int^{\infty} t^{k+\alpha-1}\,|p_k(t)|\,dt < \infty \qquad \text{for some} \quad \alpha \geqq 0$$

and $k = 1, \ldots, d$. *Then, for any* j, $0 \leqq j \leqq d-1$, (17.1) *has a solution satisfying* $u(t) = (t^j/j!)(1 + o(t^{-\alpha}))$ *as* $t \to \infty$, *and this relation can be "differentiated"* $d - 1$ *times, i.e.,*

$$(17.7) \qquad \begin{aligned} u^{(k)} &= \frac{t^{j-k}}{(j-k)!}\left(1 + o\left(\frac{1}{t^{\alpha}}\right)\right) \qquad &&\text{for} \quad k = 0, \ldots, j, \\ u^{(k)} &= o(t^{j-k-\alpha}) \qquad &&\text{for} \quad k = j+1, \ldots, d-1. \end{aligned}$$

It will be clear from the proof that, for a *given* j (rather than for *any* j) on the range $0 \leqq j \leqq d-1$, a sufficient condition for the existence of a solution satisfying (17.7) is that

$$\int^{\infty} \left(\sum_{k=1}^{d-j-1} |p_k(t)|\, t^{k-1} + \sum_{k=d-j}^{d} |p_k(t)|\, t^{k+\alpha-1}\right) dt < \infty.$$

Proof. Since R in (17.5) is in a Jordan normal form, (17.4) can be identified with (13.3) if $F(t, \xi) = G(t)\xi$, where $\xi = (\xi^1, \ldots, \xi^d)$ and $\xi^k = u^{(d-k)}$. In order to verify the conditions of Corollary 16.2, note that the sets of p and r are vacuous and that there is only one q. Correspondingly, $\lambda(q) = 0$ and $h(q) = d$. Let $j_0 = j$ be the index j in (17.7), $\beta = d - j + \alpha$, and $l(q) = k(q) = d - j$. Thus $\sum_{\rm I}$ in (13.8) contains no terms if $k < d - j$ or exactly one term $i = d - j$ if $d - j \leqq k \leqq d$. Also, let $c_q{}^i = 1$ or $c_q{}^i = 0$ according as $i = d - j$ or $i \neq d - j$, so that the desired asymptotic relation (17.7) is identical with (the first part of) (13.7).

Consider $F(t, Q(t)\zeta) = G(t)Q(t)\zeta$. Since only the first row of $G(t)$ contains nonzero elements, this can be written as $F(t, Q\zeta) = (F^1, 0, \ldots, 0)$, where, by (14.1)–(14.2) and (17.5),

$$F^1 = -\sum_{k=d-j}^{d} \frac{p_k t^{k-d+j}}{(k-d+j)!} z^{d-j} - \sum_{k=1}^{d} p_k t^{k-\beta} \sum\nolimits_{\rm II} \frac{1}{(k-i)!} z^i$$

$\zeta = (z^1, \ldots, z^d)$, and $\sum_{\rm II}$ is the sum over the set of i-values, $1 \leqq i \leqq k$ and $i \neq d - j$. Consequently, $\|Q^{-1}(t)\| \leqq ct^{\beta-1}$ implies that

$$\|Q^{-1}(t)\| \cdot \|F(t, Q\zeta)\| \leqq c_0\left(\sum_{k=1}^{d-j-1} |p_k(t)|\, t^{k-1} + \sum_{k=d-j}^{d} |p_k(t)|\, t^{k+\alpha-1}\right) \|\zeta\|$$

for a suitable constant c_0 and large t. Since the coefficient of $\|\zeta\|$ is a function $\psi_0(r)$ satisfying $\int^{\infty} \psi_0(t)\,dt < \infty$ by (17.6), Theorem 17.1 follows from Corollary 16.2.

When all the roots of (17.3) are the same, say λ, this can be reduced to the situation of Theorem 17.1 by replacing u by the new dependent variable $v = ue^{-\lambda t}$. In the other extreme case, when λ is a simple root of (17.3), we have

Theorem 17.2 *Let* (17.3) *have a simple root, say* λ, *and suppose that if* λ_0 *is any other root, then* $\operatorname{Re}\lambda \neq \operatorname{Re}\lambda_0$. *Let* $p_1(t), \dots, p_d(t)$ *be continuous functions for* $t \geqq 0$ *satisfying*

$$\int^{\infty} |p_k(t)|\, t^{\alpha}\, dt < \infty \qquad \text{for some} \quad \alpha \geqq 0 \tag{17.8}$$

and $k = 1, \dots, d$. *Then* (17.1) *has a solution* $u(t)$ *satisfying*

$$u^{(k)}(t) = e^{\lambda t}\left[\lambda^k + o\left(\frac{1}{t^{\alpha}}\right)\right] \qquad \text{for} \quad k = 0, \dots, d-1 \tag{17.9}$$

as $t \to \infty$.

Proof. This is the simplest case of Corollary 16.2 when $h(q) = 1$. Let $j_0 = 0$, $\beta = 1 + \alpha$. Let (17.4) be identified with (13.13) in Remark 1 following Theorem 13.1. Then

$$\|F^0(t, \xi)\| = \|G(t)\xi\| \leqq c\,\|\xi\| \sum_{k=1}^{d} |p_k(t)|$$

for some constant c. Thus (13.11) holds with $\psi_1(t) = c\Sigma\,|p_k(t)|$ and the theorem follows from Corollary 16.2.

Consider the general case where (17.3) has a root, say $\lambda = 0$, of multiplicity h, $1 \leqq h \leqq d$.

Theorem 17.3. *Let* $\lambda = 0$ *be a root of* (17.3) *of multiplicity* h, $1 \leqq h \leqq d$; *i.e., let* $a_{d-h+1} = \cdots = a_d = 0$ *and* $a_{d-h} \neq 0$; *and suppose that if* $\lambda_0 \neq 0$ *is any other root, then* $\operatorname{Re}\lambda_0 \neq 0$. *Let* $p_1(t), \dots, p_d(t)$ *be continuous functions for* $t \geqq 0$ *such that*

$$\int^{\infty} t^{k-d+h-\alpha-1} |p_k(t)|\, dt < \infty \qquad \text{for} \quad k = d-h+1, \dots, d. \tag{17.10}$$

$$\int^{\infty} |p_k(t)|\, t^{\alpha}\, dt < \infty \qquad \text{for} \quad k = 1, \dots, d-h, \tag{17.11}$$

for some $\alpha \geqq 0$. *Then, for any* j, $0 \leqq j \leqq h-1$, (17.1) *has a solution* $u(t)$ *satisfying*

$$\begin{aligned} u^{(k)} &= \frac{t^{j-k}}{(j-k)!} + o(t^{1-h+j-\alpha}) \qquad &&\text{for} \quad k = 0, \dots, j, \\ u^{(k)} &= o(t^{1-d+j-\alpha}) \qquad &&\text{for} \quad k = j+1, \dots, d-1, \end{aligned} \tag{17.12}$$

as $t \to \infty$.

Exercise 17.1. Prove Theorem 17.3.

Exercise 17.2. Restate Theorem 17.3 when $\lambda = 0$ is replaced by an arbitrary λ.

Theorems 17.1–17.3 depend on §§ 13, 16; we can also apply the results of § 11:

Theorem 17.4. *Let λ be a simple root of* (17.3) *and suppose that if λ_0 is any other root, then* Re $\lambda_0 \neq$ Re λ. *Let $p_1(t), \ldots, p_d(t)$ be continuous functions for $t \geqq 0$ satisfying*

$$(17.13) \qquad p_k(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{for} \quad k = 1, \ldots, d$$

or, more generally,

$$(17.14) \qquad \sup_{s \geqq t} (1 + s - t)^{-1} \int_t^s |p_k(r)|\, dr \to 0 \quad \text{as} \quad t \to \infty$$

for $k = 1, \ldots, d$. Then (17.1) *possesses a solution $u(t) \neq 0$ for large t such that*

$$(17.15) \qquad u^{(k)}(t) = u(t)[\lambda^k + o(1)] \text{ for } k = 1, \ldots, d - 1 \text{ as } t \to \infty.$$

Proof. It is sufficient to prove this theorem in the case that $\lambda = 0$, otherwise $ue^{-\lambda t}$ is introduced as a new dependent variable in (17.1). Thus $a_d = 0$. Write (17.1) as the system (17.4), (17.5). Let Y be a constant nonsingular matrix such that $Y^{-1}RY = E = \text{diag}\,[J(1), \ldots, J(g)]$ is in a Jordan normal form. The first column of Y can be taken to be $(0, \ldots, 0, 1)$, since this is an eigenvector of R belonging to the simple eigenvalue $\lambda = 0$. Thus $J(1)$ is the 1×1 zero matrix and the diagonal elements $\lambda(j)$ of $J(j)$ are such that Re $\lambda(j) \neq 0$ for $j = 2, \ldots, d$. The change of variables $\xi = Y\eta$ reduces (17.4) to

$$(17.16) \qquad \eta' = E\eta + Y^{-1}G(t)Y\eta.$$

If $\eta = (\eta^1, \ldots, \eta^d)$, it follows from Theorem 11.1 that (17.16) has a solution $\eta(t) \neq 0$ such that $\eta^k(t) = o(|\eta^1(t)|)$ as $t \to \infty$ for $k = 2, \ldots, d$. The corresponding solution $\xi(t) = Y\eta(t)$ of (17.4), where $\xi = (\xi^1, \ldots, \xi^d)$, satisfies $\xi^k(t) = o(|\xi^d(t)|)$ as $t \to \infty$ for $k = 1, \ldots, d - 1$. Since $u(t) = \xi^d(t)$ and $u^{(d-k)} = \xi^k$ for $k = 1, \ldots, d - 1$, the relations (17.15) follow.

It cannot be expected that condition (17.14) in Theorem 17.4 can be improved. This is shown by the following exercise.

Exercise 17.3. (*a*) In the second order equation,

$$(17.17) \qquad u'' - [\lambda^2 + q(t)]u = 0,$$

let $q(t)$ be continuous for $t \geqq 0$ and Re $\lambda \neq 0$. Show that a necessary condition for (17.17) to possess a solution $u(t)$ which does not vanish for large t and satisfies $u'/u \to \lambda$ as $t \to \infty$ is that

$$\text{(17.18)} \qquad \sup_{s \geq t} (1 + s + t)^{-1} \left| \int_t^s q(r)\, dr \right| \to 0 \quad \text{as} \quad t \to \infty.$$

(*b*) Prove that the necessary condition (17.18) in (*a*) is sufficient if λ is a positive number and $q(t)$ is real-valued; see Hartman [5]. For a related result, see Exercise XI 7.5.

Theorem 17.5. *Assume the conditions of Theorem* 17.4 *with* (17.14) *strengthened to*

$$\text{(17.19)} \qquad \int^\infty |p_k(t)|^p\, dt < \infty \qquad \text{for some } p, \quad 1 \leqq p \leqq 2,$$

for $k = 1, \ldots, d$. *Then a solution* $u(t) \not\equiv 0$ *of* (17.1) *satisfying* (17.15) *also satisfies*

$$\text{(17.20)} \qquad u(t) = [c + o(1)] \exp \int^t [\lambda + g(s)]\, ds \qquad \text{as} \quad t \to \infty,$$

where $c \neq 0$ *is a constant,*

$$\text{(17.21)} \qquad g(t) = -\frac{1}{F'(\lambda)} \sum_{k=1}^{d} \lambda^{d-k} p_k(t),$$

$F' = dF/d\lambda$ *and* F *is the polynomial on the left of* (17.3) [*so that* $F'(\lambda) = (\lambda - \lambda_2) \ldots (\lambda - \lambda_d)$ *if* $\lambda_2, \ldots, \lambda_d$ *are the roots of* (17.3) *distinct from* λ].

Proof. Write (17.1) as the system (17.4), (17.5) and make the change of variables $\xi = Y\eta$, where $Y = Y(0)$ is the constant matrix given in Exercise IV 8.2 and having $(\lambda^{d-1}, \ldots, \lambda, 1)$ as its first column. Then (17.4) becomes (17.16), where $E = \text{diag}\,[J(1), \ldots, J(g)]$ and $J(1)$ is the 1×1 matrix λ. Since Y is a constant matrix, (17.19) implies that the pth power of the absolute values of the elements of $Y^{-1}G(t)Y$ are integrable over $0 \leqq t < \infty$. Hence, it follows from Theorem 11.1 that (17.16) has solutions $\eta(t)$ such that if $\eta = (\eta^1, \ldots, \eta^d)$, then $\eta^1(t) \not\equiv 0$ for large t and $\eta^j(t) = o(|\eta^1(t)|)$ as $t \to \infty$ for $j = 2, \ldots, d$. Furthermore, by Exercise 11.2, any such solution satisfies

$$\eta^1(t) = [c + o(1)] \exp \int_0^t [\lambda + g(s)]\, ds \qquad \text{as} \quad t \to \infty,$$

where $g(t)$ is the element in the first row and first column of $Y^{-1}G(t)Y$.

In order to calculate $g(t)$, note that since the first column of Y is $(\lambda^{d-1}, \ldots, \lambda, 1)$, the element in the first row, first column of $G(t)Y$ is $-\Sigma\lambda^{d-k}p_k(t)$. All elements of $G(t)Y$ not in the first row are 0. Hence, the upper left corner element of $Y^{-1}G(t)Y$ is $-\Sigma\lambda^{d-k}p_d(t)$ times the corresponding element of Y^{-1}. This element of Y^{-1} is the cofactor A of the corresponding element of Y divided by det Y. If the distinct roots of (17.3) and their multiplicities are $\lambda, \lambda(2), \ldots, \lambda(g)$ and $1, h(2), \ldots, h(g)$, respectively, then

$$\det Y = \prod_{j=2}^{g} [\lambda - \lambda(j)]^{h(j)} \prod_{2 \leqq i < j} [\lambda(i) - \lambda(j)]^{h(i)h(j)};$$

see Exercise IV 8.2. The determinant which is the cofactor A has the same form as det Y, except that λ does not occur. It follows that A is the second of the two products above. Hence

$$g = -\frac{A}{\det Y} \sum \lambda^{d-k} p_k = -\prod_{j=2}^{g} [\lambda - \lambda(j)]^{-h(j)} \sum \lambda^{d-k} p_k,$$

i.e., (17.21) holds. The relations $\xi = Y\eta$, $\xi^k = u^{(d-k)}$ and the fact that the first column of Y is $(\lambda^{d-1}, \ldots, \lambda, 1)$ completes the proof of Theorem 17.5.

As an illustration of Theorem 17.5, consider the second order equation (17.17) in which Re $\lambda \neq 0$ and $\int^{\infty} |q(t)|^p \, dt < \infty$ for some p, $1 \leqq p \leqq 2$. Then (17.17) has a pair of solutions satisfying

(17.22) $$u' \sim \pm\lambda u, \qquad u \sim \exp \pm \int^{t} \left[\lambda + \frac{1}{2\lambda} q(s)\right] ds \qquad \text{as} \quad t \to \infty.$$

Exercise 17.4. Let $q(t)$ be real-valued and continuous for $t \geqq 0$, $q(t) \to 0$ as $t \to \infty$, and $q(t)$ of bounded variation for $t \geqq 0$ [e.g., let $q(t)$ be monotone or let $q(t)$ have a continuous derivative such that $\int^{\infty} |q'(t)| \, dt < \infty$]. Show that (*a*) $u'' + [1 + q(t)]u = 0$ has solutions $u(t)$ satisfying

$$u' \sim \pm iu \qquad \text{and} \qquad u(t) \sim \exp \pm i \int^{t} [1 + q(s)]^{1/2} \, ds \qquad \text{as} \quad t \to \infty,$$

and that (*b*) $u'' + [-1 + q(t)]u = 0$ has solutions $u(t)$ satisfying

$$u' \sim \pm u \qquad \text{and} \qquad u(t) \sim \exp \pm \int^{t} [1 - q(s)]^{1/2} \, ds \qquad \text{as} \quad t \to \infty.$$

(*c*) State an analogous result for (17.17) where $\lambda \neq 0$ and it is not assumed that λ or $q(t)$ are real-valued; cf. Exercises XI 8.4(b).

Exercise 17.5. In the differential equation

(17.23) $$u'' - f(t)u = 0,$$

let $f(t)$ be a continuously differentiable, complex-valued function for $t \geqq 0$ such that

(17.24) $$\operatorname{Re} f^{1/2}(t) \neq 0 \qquad \text{and} \qquad \int^{\infty} |\operatorname{Re} f^{1/2}(t)| \, dt = \infty.$$

(*a*) Show that if $f'/|f| \cdot |\mathrm{Re}\, f^{1/2}| \to 0$ as $t \to \infty$, then (17.23) has solutions satisfying

(17.25) $$u' \sim \pm f^{1/2}(t)u \quad \text{as} \quad t \to \infty.$$

(*b*) Show that if $\int^\infty |\mathrm{Re} f^{1/2}(t)|^{p-1} |f'(t)/f(t)|^p \, dt < \infty$ for some p, $1 \leqq p \leqq 2$, then (17.23) has solutions satisfying (17.25) and

(17.26) $$u(t) \sim f^{-1/4}(t) \exp \pm \int^t f^{1/2}(r) \, dr \qquad \text{as} \quad t \to \infty,$$

(*c*) Show that if $f'(t)/f^{3/2}(t)$ is of bounded variation, i.e.,

$$\int^\infty |d(f'/f^{3/2})| < \infty,$$

$f'(t)/f^{3/2}(t) \to 0$ as $t \to \infty$, then (17.23) has a pair of solutions satisfying (17.25) and

(17.27) $$u(t) \sim f^{-1/4}(t) \exp \pm \int^t f^{1/2}(1 + f'^2/16f^3)^{1/2} \, dr \qquad \text{as} \quad t \to \infty$$

For other results of this type, see § XI 9. For analogous results when $\mathrm{Re} f^{1/2} \equiv 0$, see Exercise XI 8.5.

Exercise 17.6. As a simple application of the last exercise, consider Weber's equation

(17.28) $$u'' + tu' - 2\lambda u = 0,$$

where λ is a constant. (*a*) By introducing the new independent variable $s = \frac{1}{2}t^2$, deduce from Theorem 17.4 that (17.28) has a pair of solutions $u_0(t)$, $u_1(t)$ which do not vanish for large t and satisfy $u_0' \sim -tu_0$, $u_1' = o(tu_1)$ as $t \to \infty$. (*b*) Show that (17.28) has a pair of solutions u_0, u_1 satisfying $u_0 \sim t^{-1-2\lambda}e^{-t^2/2}$, $u_1 \sim t^{2\lambda}$ as $t \to \infty$. (*c*) Find asymptotic relations for derivatives u' of solutions u of (17.28) by differentiating (17.28) and applying (*b*). (See also Exercise XI 9.7.)

Notes

For references and other treatments of the topics in this chapter, see Cesari [2] and Bellman [4].

SECTION 1. The main results, Theorems 1.1 and 1.2, are due to Wintner [3], [7], [8], who gave the existence assertions essentially in the form stated in Exercise 1.2. Linear cases, where $F(t, \xi) = G(t)\xi$ for a matrix $G(t)$, are much older; see Dunkel [1]. For Exercise 1.1, see Hale and Onuchic [1]; cf. § XII 9. For Exercise 1.4, see Wintner [21].

SECTION 2. Theorem 2.1 was formulated by Ważewski [5] and is a very useful tool in the study of differential equations. Special cases of this theorem and the arguments in its proof had been used earlier; cf. Hartman and Wintner [1] or Nemytzkiĭ and

Stepanov [1, p. 93]. For another type of topological argument, useful for similar purposes, cf. Atkinson [2]. Exercise 2.1 is due to Pliś [1].

SECTION 3. The results of this section are due to Ważewski [5].

SECTIONS 4–7. Lemmas 4.1 and 4.2 are related to results of Ważewski [6], Szmydtówna [1], Lojasiewicz [1], and Hartman and Wintner [17], [19]. The proofs in the text are adapted from those of Ważewski and his students just mentioned; for other proofs, see the papers of Hartman and Wintner. Lemma 4.3 and applications were given in the papers of Hartman and Wintner. Conditions of the type (4.24) were introduced by Hartman [5]. For Exercise 4.6, see Levinson [3] (for the part dealing with boundedness, see Cesari [1]); an analogous result (see Exercise 17.4) on a second order equation was given by Wintner [10]. See Cesari [2, pp. 38–42], for related results and references. For results related to Exercises 4.8, 4.9 and applications, see Coffman [2].

SECTIONS 8–12. Results related to those occurring in § 8 for analytic systems are the oldest in this chapter and go back to Poincaré and to Lyapunov [2]. For particular cases for linear differential equations, see Poincaré [4] and Perron [2]. Cotton [1] and then Perron [9], [10], [12] systematically investigated nonanalytic, nonlinear cases, but under conditions heavier than those in the text. Their results depended on the method of successive approximations. See also Bellman [1], who used fixed point theorems to obtain an analogue of Theorem 8.1, and the references above for §§ 4–7 to Ważewski, Hartman, and Wintner, etc. The relaxation of the condition "$\psi(t) \to 0$ as $t \to \infty$" to (4.24) is due to Hartman [5] and to Hartman and Wintner [19]. A form of Theorem 8.2 involving stronger hypothesis and weaker assertions was given by Petrovsky [1]. The last two parts of Corollary 8.1 are proved in Coddington and Levinson [2, Chap. 13] by the method of successive approximations; cf. Exercise 8.1. For another application of a related method of successive approximations, see Lillo [1]. The comparatively simple proofs in the text for Theorems 8.2, 8.3 and Corollary 8.1 are new. Theorem 11.1 is a slight improvement of a result of Lettenmeyer [2]. Theorem 11.2 is given by Hartman and Wintner [19]. Results of the type in Exercise 11.1 go back to Bôcher [2] and Dunkel [1]; cf. notes on §§ 13–16 below. Exercise 11.2 was first given by Hartman [5] for the case of a second order equation (see Theorem 17.5 with $d = 2$) and generalized to the situation in Exercise 11.2 by Hartman and Wintner [17].

SECTIONS 13–16. Results of the type in Theorem 13.1 were first given by Bôcher [2] for a second order, linear equation. Using successive approximations similar to those of Exercise 8.1, Dunkel [1] generalized Bôcher's result to arbitrary linear systems (13.1), where $F(t, \xi) = G(t)\xi$, but his results are not as sharp as those given here. Theorems 13.1, 13.2 and their corollaries in § 16 are due to Hartman and Wintner [19]. The proofs in the text, which take full advantage of Ważewski's principle of § 2, depend in an essential way on the change of variables (14.1)–(14.2) similar to those introduced by Hartman and Wintner [17] and simplified by Coffman [2]. See also Olech [1].

SECTION 17. When $\alpha = 0$, Theorem 17.1 is due to Bôcher [2] for $d = 2$ and, in a weakened form, it is contained in Dunkel's result [1] for arbitrary d. For $\alpha = 0$, it is given by Faedo [1] and Ghizetti [2]. Theorem 17.2 and a less precise form of Theorem 17.3 with $\alpha = 0$, are also contained in Dunkel [1]; Faedo [1], [2]; and Ghizetti [1]. Theorem 17.4 is a generalization of results of Poincaré [4] and Perron [2] and is contained in Hartman and Wintner [17]. For Exercise 17.3, see Hartman [5]. Theorem 17.5 for $d = 2$ is due to Hartman [5]; the result formulated in the text is new. For a generalization of the case $d = 2$, see Bellman [3]. Exercise 17.4 is due to Wintner [10], [13] and is contained in the more general result of Exercise 4.6; cf. also Exercise XI 8.4(b). Results of the type in Exercise 17.5(a) go back to Wiman [1], [2]; for both parts (a) and (b), see Hartman and Wintner [17].