Dissipative continuous Euler flows

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Abstract We show the existence of continuous periodic solutions of the 3D incompressible Euler equations which dissipate the total kinetic energy.

1 Introduction

In what follows $\mathbb{T}^3$ denotes the 3-dimensional torus, i.e. $\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$. In this note we prove the following theorem.

Theorem 1.1 *Assume $e : [0, 1] \to \mathbb{R}$ is a positive smooth function. Then there is a continuous vector field $v : \mathbb{T}^3 \times [0, 1] \to \mathbb{R}^3$ and a continuous scalar field $p : \mathbb{T}^3 \times [0, 1] \to \mathbb{R}$ which solve the incompressible Euler equations*

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 \\ \operatorname{div} v = 0 \end{cases} \tag{1}$$

in the sense of distributions and such that

$$e(t) = \int |v|^2(x, t)\, dx \quad \forall t \in [0, 1]. \tag{2}$$

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Obviously, if we choose a strictly decreasing function $e(t)$, Theorem 1.1 yields continuous solutions of the incompressible Euler equations which "dissipate" the total kinetic energy $\frac{1}{2} \int |v|^2(x, t)\, dx$. This is not possible for C^1 solutions: in that case one can multiply the first equation in (1) by v to derive

$$\partial_t \frac{|v|^2}{2} + \operatorname{div}\left(u\left(\frac{|u|^2}{2} + p\right)\right) = 0.$$

Integrating this last identity in x we then conclude

$$\frac{d}{dt} \int_{\mathbb{T}^3} \frac{|v|^2}{2}(x, t)\, dx = 0. \tag{3}$$

Theorem 1.1 shows therefore that this formal computation cannot be justified for distributional solutions, even if they are continuous. The pair (v, p) in Theorem 1.1 solves (1) in the following sense:

$$\int_0^1 \int_{\mathbb{T}^3} (\partial_t \varphi \cdot v + \nabla \varphi : v \otimes v + p \operatorname{div} \varphi)\, dx dt = 0 \tag{4}$$

for all $\varphi \in C_c^\infty(\mathbb{T}^3 \times (0, 1); \mathbb{R}^3)$ and

$$\int_0^1 \int_{\mathbb{T}^3} v \cdot \nabla \psi\, dx dt = 0 \quad \text{for all } \psi \in C_c^\infty\big(\mathbb{T}^3 \times (0, 1)\big).$$

Remark 1 In the usual definition of weak solution, (4) is replaced by the same condition for divergence free test fields: therefore p disappears from the identity. With this alternative definition, for every weak solution v which belongs to L^2 a corresponding pressure field can then be recovered using

$$-\Delta p = \operatorname{div} \operatorname{div}(v \otimes v). \tag{5}$$

p is then determined up to an arbitrary function of t: this arbitrariness can be overcome by imposing, for instance, $\int p(x, t)\, dx = 0$. However, as it is well-known, (5) and the continuity of v does not guarantee the continuity of p.

1.1 Onsager's conjecture

The possibility that weak solutions might dissipate the total kinetic energy has been considered for a rather long time in the fluid dynamics literature: this phenomenon goes under the name of "anomalous dissipation". In fact, to our knowledge, the existence of dissipative solutions was considered for the first time by Lars Onsager in his famous 1949 note about statistical hydrodynamics, see [18]. In that paper Onsager conjectured that

(a) $C^{0,\alpha}$ solutions are energy conservative when $\alpha > \frac{1}{3}$;
(b) There exist dissipative solutions with $C^{0,\alpha}$ regularity for any $\alpha < \frac{1}{3}$ (note that, though Onsager's definition of "weak solution" is, strictly speaking, different from the one given above, it can be easily shown that the two concepts are equivalent).

The first part of the conjecture, i.e. assertion (a), has been shown by Eyink in [11] and by Constantin, E and Titi in [3]. The proof of the last paper amounts to give a rigorous justification of the formal computation sketched above and leading to (3): this is done via a suitable regularization of the equation and some careful commutator estimates. The second part of the conjecture, i.e. statement (b), is still widely open. A first result in that direction was the groundbreaking work of Scheffer [20] which proved the existence of a compactly supported nontrivial weak solution in $\mathbb{R}^2 \times \mathbb{R}$. A different construction of the existence of a compactly supported nontrivial weak solution in $\mathbb{T}^2 \times \mathbb{R}$ was then given by Shnirelman in [21]. In both cases the solutions are only square summable (as a function of both space and time variables): it is therefore not clear whether there are intervals of time in which the total kinetic energy is a monotone function (indeed it is not even clear whether these solutions belong to the energy space $L_t^\infty(L_x^2)$). The first proof of the existence of a solution for which the total kinetic energy is a monotone decreasing function has been given by Shnirelman in [22]. Shnirelman's example is only in the energy space $L^\infty([0, \infty[, L^2(\mathbb{R}^3))$.

1.2 *h*-principle

Our previous work [6, 7] showed the existence of dissipative solutions for which both pressure and velocity are bounded. Besides the obvious improvement (and the discovery of quite severe counterexamples to the uniqueness of admissible solutions, both for incompressible and compressible Euler), in this work we introduced a new point of view in the subject, highlighting connections to other counterintuitive solutions of (mainly geometric) systems of partial differential equations: in geometry these solutions are, according to Gromov, instances of the *h*-principle. In particular Onsager's conjecture bears striking similarities with the rigidity and flexibility properties of isometric embeddings of Riemannian manifolds, pioneered by the celebrated work of Nash [17]. Indeed, results of the same flavor as statements (a) and (b) can be proved in the case of isometric embeddings (see for instance [4] and the references therein): in comparing Onsager's conjecture and these results, the reader should take into account that, in this analogy, the velocity field of the Euler equations corresponds to the differential of the embedding in the isometric embedding problem. All these aspects (and further developments for some PDEs in fluid dynamics inspired by our work) are surveyed in the note [8]. See also [1, 5, 23, 26–28].

1.3 Comments on the proof

As in all applications of convex integration, the solution of (1) is constructed by an iteration scheme: at each stage of this iteration we produce an "almost solution" which solves Euler with an additional error term. We name the resulting "perturbed" system of equations Euler-Reynolds system, since the error term has the typical form of the so-called Reynolds stress in the fluid dynamics literature (see [8] for an informal discussion of this point). This error term converges to 0, while the sequence of almost solutions converge to an exact solution, uniformly in C^0. At each stage the new approximate solution is generated from the previous one by adding fast oscillatory perturbations. Being oscillatory, the linear terms in the perturbation are negligible whereas the nonlinear (quadratic) term is the leading order, which should cancel the previous error.

Although this general underlying philosophy is the same as in our previous papers [6, 7], in the actual proof we need to substantially depart from the previous strategy and introduce several new ideas.

(1) First of all, recall that in [6] we rewrite the Euler equations as a differential inclusion and thus we are able to apply some of the machinery developed in the context of Lipschitz differential inclusions. In particular, in all our results for bounded weak solutions there are two routes: a Baire category approach and a more direct convex integration approach. In fact these approaches are in some sense equivalent, see [25].

For continuous solutions the Baire category approach does not work as there is no way to extract a uniform continuity estimate for approximating sequences. In this paper we are therefore forced to abandon this, essentially "soft" functional analytic technique in favor of a more "hard" PDE approach based on Schauder estimates and oscillatory integrals. This seems to be entirely new in the context of convex integration, even compared to the original proof of Nash [17] for rough isometric embeddings.

(2) In all applications so far of convex integration to fluid mechanics [1, 5, 8, 23, 26–28] a key starting point is a plane-wave analysis, in order to identify compatible plane-waves that will be used in the subsequent construction. Indeed, the use of suitably localized plane-waves for an iteration scheme goes back to the work of Nash [17] and has been widely generalized by Gromov [13] as the technique of convex integration—the simple reason being that one-dimensional oscillations can be "integrated", hence the name convex integration. In the scheme of Nash simple plane-waves suffice to produce a C^1 solution. Nevertheless, it is crucial here that one has an extra dimension (the codimension of the range) to absorb linear errors.

For the Euler equations we have no extra dimension and simple plane-waves do not work for continuous solutions. This leads to two points of departure from existing schemes: we use genuinely multi-dimensional building

blocks, the Beltrami flows, and we retain a linear "transport term" in the iteration, which needs to be controlled with new ideas.

(3) As mentioned above, the main building blocks of our iteration scheme are Beltrami flows, which are truly three-dimensional oscillations. The issue of going beyond one-dimensional oscillations has been raised by Gromov (p. 219 of [13]) as well as Kirchheim-Müller-Šverák (p. 52 of [15]), but as far as we know, there have been no such examples in the literature so far. As a result of using Beltrami flows, the final result of our iteration scheme is the superposition of infinitely many (perturbed) and weakly interacting Beltrami flows. Curiously, the idea that turbulent flows can be understood as a superposition of Beltrami flows has been proposed almost 30 years ago in the fluid dynamics literature: see the work of Constantin and Majda [2]. Indeed, it was Peter Constantin who suggested to us to try Beltrami flows in a convex integration scheme.

(4) The "transport term" is perhaps the most important new aspect of our scheme in comparison with Nash. This arises, roughly speaking, as the linearization of the first equation in (1): this term is typical of an evolution equation, whereas, instead, the equations for isometric embeddings are "static". Moreover, the linear terms in the Nash scheme are set to zero by choosing suitable coordinates and working with the extra dimension. At a first glance this transport term makes it impossible to use a scheme like the one of Nash to prove Theorem 1.1. To overcome this obstruction we need to introduce a phase-function that acts as a kind of discrete Galilean transformation of the (stationary) Beltrami flows, and to introduce an "intermediate" scale along each iteration step on which this transformation acts.

(5) A further difference to the approach in [6, 7] is that we cannot use simple potentials to generate space-time localized versions of the Beltrami flows. One reason is that, in a sense, we cannot simply "integrate" Beltrami flows. More importantly, a space-time cutoff would result in error terms which cannot be controlled in C^0. In order to overcome this issue we introduce a "corrector term" to the main perturbation. This corrector term is not "explicit": it is determined by solving some appropriate elliptic equations.

(6) In comparing Theorem 1.1 to the main result in [7] an important difference is that, whereas in the bounded case e is a function of (x, t), here it is a function of t only. In other words, whereas for bounded solutions we can prescribe $(x, t) \mapsto |v(x, t)|$, for continuous solutions we are only able to prescribe $t \mapsto \int |v(x, t)|^2 \, dx$. As a consequence, our solutions in this paper cannot deal with the local energy inequality (cf. with the discussion in Sect. 3.1 of [8]). We do not know whether this is just a technical issue or a new feature of continuous solutions.

As a minor comment we remark that obviously the smoothness of e in Theorem 1.1 can be relaxed, but we do not pursue this issue here. Moreover, the same theorem can be proved if we replace [0, 1] with [0, ∞[: in this case

we require in addition that e and its derivatives are uniformly bounded and that there is a positive constant c_0 with $e \geq c_0$.

(7) In Theorem 1.1 our aim was to construct continuous weak solutions. In particular we did not address issues concerning the initial-value problem. Thus, setting aside the difference between bounded and continuous solutions, our Theorem 1.1 is weaker than the results proved in [7]. In particular, in this paper we do not prove any non-uniqueness statement.

However, this is a typical way to proceed with problems involving the h-principle. In general one may distinguish two aspects: a local and a global one. In geometric situations the local one is typically a differential constraint whereas the global one is topological (cf. [10]). The flexibility (in other words the lack of uniqueness) that one observes in instances of the h-principle is tied to the specifics of the local aspect. Thus, our Theorem 1.1 deals exclusively with the "local" aspect for the Euler equations, whereas a possible analogue of the global aspect would be the imposition of an initial data, possibly together with an admissibility condition (as in [7]). In subsequent papers we plan to address such "global" issues (e.g. initial data, compactly supported ancient solutions, etc.).

2 Setup and plan of the paper

The proof of Theorem 1.1 will be achieved through an iteration procedure. Along the iteration the maps will be "almost solutions" of the Euler equations. To measure "how far" a solenoidal field is from being a solution of incompressible Euler we introduce a system of differential equations which we call Euler-Reynolds system. The name is justified by the fact that the matrix-field $\mathring{R}$ is a well known object in the theory of turbulence, called "Reynolds stress" (cf. with [8] and the references therein). In what follows $\mathcal{S}_0^{3 \times 3}$ denotes the vector space of symmetric trace-free 3×3 matrices. Moreover we will, as it is usual, use the word smooth for C^∞ maps.

Definition 2.1 Assume $v, p, \mathring{R}$ are smooth functions on $\mathbb{T}^3 \times [0, 1]$ taking values, respectively, in $\mathbb{R}^3, \mathbb{R}, \mathcal{S}_0^{3 \times 3}$. We say that they solve the Euler-Reynolds system if

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}, \\ \operatorname{div} v = 0. \end{cases} \tag{6}$$

We are now ready to state the main proposition of this paper, of which Theorem 1.1 is a simple corollary.

Proposition 2.2 *Let e be as in Theorem 1.1. Then there are positive constants η and M with the following property.*

Let $\delta \leq 1$ be any positive number and $(v, p, \mathring{R})$ a smooth solution of the Euler-Reynolds system (6) *such that*

$$\frac{3\delta}{4}e(t) \leq e(t) - \int |v|^2(x, t)\, dx \leq \frac{5\delta}{4}e(t) \quad \forall t \in [0, 1] \tag{7}$$

and

$$\sup_{x,t}\left|\mathring{R}(x, t)\right| \leq \eta\delta. \tag{8}$$

Then there is a second smooth triple $(v_1, p_1, \mathring{R}_1)$ *which solves as well the Euler-Reynolds system and satisfies the following estimates*:

$$\frac{3\delta}{8}e(t) \leq e(t) - \int |v_1|^2(x, t)\, dx \leq \frac{5\delta}{8}e(t) \quad \forall t \in [0, 1], \tag{9}$$

$$\sup_{x,t}\left|\mathring{R}_1(x, t)\right| \leq \frac{1}{2}\eta\delta, \tag{10}$$

$$\sup_{x,t}\left|v_1(x, t) - v(x, t)\right| \leq M\sqrt{\delta} \tag{11}$$

and

$$\sup_{x,t}\left|p_1(x, t) - p(x, t)\right| \leq M\delta. \tag{12}$$

As already mentioned, Theorem 1.1 follows immediately from Proposition 2.2.

Proof of Theorem 1.1 We start by setting $v_0 = 0$, $p_0 = 0$, $\mathring{R}_0 = 0$ and $\delta := 1$. We then apply Proposition 2.2 iteratively to reach a sequence $(v_n, p_n, \mathring{R}_n)$ which solves (6) and such that

$$\frac{3}{4}\frac{e(t)}{2^n} \leq e(t) - \int |v_n|^2(x, t)\, dx \leq \frac{5}{4}\frac{e(t)}{2^n} \quad \text{for all } t \in [0, 1] \tag{13}$$

$$\sup_{x,t}\left|\mathring{R}_n(x, t)\right| \leq \frac{\eta}{2^n} \tag{14}$$

$$\sup_{x,t}\left|v_{n+1}(x, t) - v_n(x, t)\right| \leq M\sqrt{\frac{1}{2^n}} \tag{15}$$

$$\sup_{x,t}\left|p_{n+1}(x, t) - p_n(x, t)\right| \leq \frac{M}{2^n}. \tag{16}$$

Then $\{v_n\}$ and $\{p_n\}$ are both Cauchy sequences in $C(\mathbb{T}^3 \times [0, 1])$ and converge uniformly to two continuous functions v and p. Similarly $\mathring{R}_n$ converges

uniformly to 0. Moreover, by (13)

$$\int_{\mathbb{T}^3} |v|^2(x,t)\,dx = e(t) \quad \forall t \in [0,1].$$

Passing into the limit in (6) we therefore conclude that (v, p) solves (1). □

2.1 Construction of v_1

The rest of the paper will be dedicated to prove Proposition 2.2. The construction of the map v_1 consists of adding two perturbations to v:

$$v_1 = v + w_o + w_c =: v + w. \tag{17}$$

To specify the form of the perturbation w_o, which is a highly oscillatory function and for which we give a rather explicit formula, we need several ingredients. The vectorfield $v + w_o$ is not in general divergence free. Therefore we add the correction w_c to restore this condition. Having added the correction, the main focus will then be on finding maps $\mathring{R}_1$ and p_1 with the desired estimate and such that

$$\partial_t v_1 + \mathrm{div}_x(v_1 \otimes v_1) + \nabla p_1 = \mathrm{div}_x\,\mathring{R}_1.$$

The perturbation w_o will depend on two parameters, μ and λ, which will satisfy the following conditions

$$\lambda, \mu, \frac{\lambda}{\mu} \in \mathbb{N}. \tag{18}$$

In order to achieve the estimates, λ and μ will be chosen quite large, depending on appropriate norms of v. As already mentioned, the building blocks for the perturbation w_o are Beltrami flows. In order to give the formula leading to the definition of w_o we must, therefore, study closer the particular "geometry" of these flows. This will be done in the next section. We will then be ready to define the perturbations w_o and w_c: this task will be accomplished in Sect. 4 where we also prescribe the constants η and M of the estimates in Proposition 2.2. After recalling some classical Schauder theory in Sect. 5, in the Sects. 6 and 7 we will prove the relevant estimates of the various terms involved in the construction, in terms of the parameters λ and μ. The choice of these parameters will be finally specified in Sect. 8, where we conclude the proof of Proposition 2.2.

3 Geometric preliminaries

In this paper we denote by $\mathbb{R}^{n \times n}$, as usual, the space of $n \times n$ matrices, whereas $\mathcal{S}^{n \times n}$ and $\mathcal{S}_0^{n \times n}$ denote, respectively, the corresponding subspaces

of symmetric matrices and of trace-free symmetric matrices. The 3×3 identity matrix will be denoted with Id. For definitiveness we will use the matrix operator norm $|R| := \max_{|v|=1} |Rv|$. Since we will deal with symmetric matrices, we have the identity $|R| = \max_{|v|=1} |Rv \cdot v|$.

3.1 Beltrami flows

We start by recalling a celebrated example of stationary periodic solutions to the 3D Euler equations, the so called Beltrami flows. One important fact which will play a central role in our paper is that the space of Beltrami flows contains linear spaces of fairly large dimension.

Proposition 3.1 (Beltrami flows) *Let $\lambda_0 \geq 1$ and let $A_k \in \mathbb{R}^3$ be such that*

$$A_k \cdot k = 0, \qquad |A_k| = \frac{1}{\sqrt{2}}, \qquad A_{-k} = A_k$$

for $k \in \mathbb{Z}^3$ with $|k| = \lambda_0$. Furthermore, let

$$B_k = A_k + i \frac{k}{|k|} \times A_k \in \mathbb{C}^3.$$

For any choice of $a_k \in \mathbb{C}$ with $\overline{a_k} = a_{-k}$ the vectorfield

$$W(\xi) = \sum_{|k|=\lambda_0} a_k B_k e^{ik \cdot \xi} \tag{19}$$

is divergence-free and satisfies

$$\operatorname{div}(W \otimes W) = \nabla \frac{|W|^2}{2}. \tag{20}$$

Furthermore

$$\langle W \otimes W \rangle = \fint_{\mathbb{T}^3} W \otimes W \, d\xi = \frac{1}{2} \sum_{|k|=\lambda_0} |a_k|^2 \left(\operatorname{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|} \right). \tag{21}$$

In other words $W(\xi)$ defined by (19) is a stationary solution of (1) with pressure $p = -\frac{|W|^2}{2}$. For the rest of this paper we will treat the vectors $A_k \in \mathbb{R}^3$, $B_k \in \mathbb{C}^3$ as fixed (the choice of A_k as prescribed in the proposition is not unique, but this is immaterial for our purposes). The proof of Proposition 3.1 is a classic in the fluid dynamics literature, but we include it for the reader's convenience.

Proof First of all observe that $a_{-k}B_{-k} = \overline{a_k B_k}$. Thus the vector field defined in (19) is real valued. Next notice that

$$\text{div } W(\xi) = \sum_{|k|=\lambda_0} ik \cdot B_k a_k e^{ik\cdot\xi} = 0,$$

because $k \cdot B_k = 0$ for every k.

Observe also that

$$\text{curl } W(\xi) = \sum_{|k|=\lambda_0} ik \times B_k a_k e^{ik\cdot\xi}.$$

On the other hand

$$ik \times B_k = \lambda_0 \left(i\frac{k}{|k|} \times A_k - \frac{k}{|k|} \times \left(\frac{k}{|k|} \times A_k \right) \right)$$

$$= \lambda_0 \left(i\frac{k}{|k|} \times A_k + A_k \right) = \lambda_0 B_k.$$

We therefore infer $\text{curl } W = \lambda_0 W$. Since W is divergence free, $\text{div}(W \otimes W) = (W \cdot \nabla)W$ and we can use the well known vector identity

$$\text{div}(W \otimes W) = (W \cdot \nabla)W = \nabla\frac{|W|^2}{2} - W \times (\text{curl } W).$$

Since we have just seen that $\text{curl } W$ and W are parallel, (20) follows easily.

Finally, we compute

$$W \otimes W = \sum_{k,j} a_k a_j B_k \otimes B_j e^{i(k+j)\cdot\xi} = \sum_{k,j} a_k \overline{a_j} B_k \otimes \overline{B}_j e^{i(k-j)\cdot\xi}.$$

Averaging this identity in ξ we infer

$$\langle W \otimes W \rangle = \sum_{|k|=\lambda_0} |a_k|^2 B_k \otimes \overline{B}_k.$$

However, since $B_k = \overline{B}_{-k}$, we get

$$\langle W \otimes W \rangle = \sum_{|k|=\lambda_0} |a_k|^2 \text{Re}(B_k \otimes \overline{B}_k)$$

$$= \sum_{|k|=\lambda_0} |a_k|^2 \left(A_k \otimes A_k + \left(\frac{k}{|k|} \times A_k \right) \otimes \left(\frac{k}{|k|} \times A_k \right) \right).$$

On the other hand, observe that the triple $\sqrt{2}A_k, \sqrt{2}\frac{k}{|k|} \times A_k, \frac{k}{|k|}$ forms an orthonormal basis of $\mathbb{R}^3$. Thus,

$$2A_k \otimes A_k + 2\left(\frac{k}{|k|} \times A_k\right) \otimes \left(\frac{k}{|k|} \times A_k\right) + \frac{k}{|k|} \otimes \frac{k}{|k|} = \text{Id}.$$

This shows (21) and hence completes the proof. $\square$

3.2 The geometric lemma

One key point of our construction is that the abundance of Beltrami flows allows to find several such flows v with the property that

$$\langle v \otimes v \rangle(t) := \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} v \otimes v(x, t)\, dx$$

equals a prescribed symmetric matrix R. Indeed we will need to select these flows so as to depend smoothly on the matrix R, at least when R belongs to a neighborhood of the identity matrix. In view of (21), such selection is made possible by the following lemma.

Lemma 3.2 (Geometric Lemma) *For every $N \in \mathbb{N}$ we can choose $r_0 > 0$ and $\lambda_0 > 1$ with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\} \quad j \in \{1, \ldots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty\big(B_{r_0}(\text{Id})\big) \quad j \in \{1, \ldots, N\}, k \in \Lambda_j$$

such that

(a) *$k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;*
(b) *For each $R \in B_{r_0}(\text{Id})$ we have the identity*

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} \big(\gamma_k^{(j)}(R)\big)^2 \left(\text{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right) \quad \forall R \in B_{r_0}(\text{Id}). \tag{22}$$

Remark 2 Though it will not be used in the sequel, the cardinality of each set Λ_j constructed in the proof of the lemma is indeed bounded *a priori* independently of all the other parameters. A close inspection of the proof shows that it gives sets with cardinality at most 98. This seems however far from optimal: one should be able to find sets Λ_j with cardinality 14.

The proof of the Geometric Lemma is based on the following well-known fact.

Proposition 3.3 *The set $\mathbb{Q}^3 \cap \mathbb{S}^2$ is dense in $\mathbb{S}^2$.*

Proof Let $\mathbf{s} : \mathbb{R}^2 \to \mathbb{S}^2$ be the inverse of the stereographic projection:

$$\mathbf{s}(u, v) := \left(\frac{2v}{u^2 + v^2 + 1}, \frac{2u}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

It is obvious that $\mathbf{s}(\mathbb{Q}^2) \subset \mathbb{Q}^3$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$ and $\mathbf{s}$ is a diffeomorphism onto $\mathbb{S}^2 \setminus (0, 0, 1)$, the proposition follows trivially. $\qquad\square$

Indeed, much more can be proved: $\frac{1}{n}\mathbb{Z}^3 \cap \mathbb{S}^2$, distributes uniformly on the sphere for $n \in \mathbb{N}$ large whenever $n \equiv 1, 2, 3, 4, 5, 6 \pmod 8$. This problem was raised by Linnik (see [16]), who proved a first result in its direction, and solved thanks to a breakthrough of Iwaniec [14] in the theory of modular forms of half-integral weight (see, for instance, [9] and [19]).

Proof of Lemma 3.2 For each vector $v \in \mathbb{R}^3 \setminus \{0\}$, we denote by M_v the 3×3 symmetric matrix given by

$$M_v = \mathrm{Id} - \frac{v}{|v|} \otimes \frac{v}{|v|}.$$

With this notation the identity (22) reads as

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(R) \right)^2 M_k. \tag{23}$$

Step 1 Fix a $\lambda_0 > 1$ and for each set $F \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\}$ we consider the set $c(F)$ which is the interior of the convex hull, in $\mathcal{S}^{3 \times 3}$, of $\{M_k : k \in F\}$. We claim in this step that it suffices to find a λ_0 and N disjoint subsets $F_j \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\}$ such that

(d) $-F_j = F_j$;
(e) $c(F_j)$ contains a positive multiple of the identity.

Indeed, we will show below that, if F_j satisfies (d) and (e), then we can find a $r_0 > 0$, a subset $\Gamma_j \subset F_j$ and positive smooth functions $\lambda_k^{(j)} \in C^\infty(B_{2r_0}(\mathrm{Id}))$ such that

$$R = \sum_{k \in \Gamma_j} \lambda_k^{(j)}(R) M_k.$$

We then find Λ_j and the functions $\gamma_k^{(j)}$ by

- defining $\Lambda_j := \Gamma_j \cup -\Gamma_j$;
- setting $\lambda_k^{(j)} = 0$ if $k \in \Lambda_j \setminus \Gamma_j$;
- defining

$$\gamma_k^{(j)} := \sqrt{\lambda_k^{(j)} + \lambda_{-k}^{(j)}}$$

for every $k \in \Lambda_j$.

Observe that the functions and the sets satisfy both (a) and (b). Moreover, since at least one of the $\lambda_{\pm k}^{(j)}$ is positive on $B_{2r_0}(\mathrm{Id})$, $\gamma_k^{(j)}$ is smooth in $B_{r_0}(\mathrm{Id})$.

We now come to the existence of the set Γ_j. For simplicity we drop the subscripts. The open set $c(F)$ contains an element $\alpha\,\mathrm{Id}$ with $\alpha > 0$. Then there are seven matrices $A_1, \ldots, A_7$ in $c(F)$ such that $\alpha\,\mathrm{Id}$ belongs to the interior of their convex hull, which is an open convex simplex S. We choose ϑ so that the ball $\tilde{U}$ of center $\alpha\,\mathrm{Id}$ and radius ϑ is contained in S. Then each point $R \in \tilde{U}$ can be written in a unique way as a convex combination of the elements A_i:

$$R = \sum_{i=1}^{7} \beta_i(R) A_i$$

and the functions β_i are positive and smooth on $\tilde{U}$.

By Caratheodory's Theorem, each A_i is the convex combination $\sum \lambda_{i,n} M_{v_{i,n}}$ of at most 7 $M_{v_{i,n}}$ with $v_{i,n} \in F$, where we require that each $\lambda_{i,n}$ is positive (observe that Caratheodory's Theorem guarantees the existence of 7 points $M_{v_{i,n}}$ such that A_i belongs to the *closed* convex hull of them; if we insist on the property that the corresponding coefficients are all positive, then we might be obliged to choose a number smaller than 7).

Set $r_0 := \frac{\vartheta}{2\alpha}$. Then,

$$R = \sum_{i,n} \frac{1}{\alpha} \beta_i(\alpha R) \lambda_{i,n} M_{v_{i,n}} \quad \forall R \in B_{2r_0}(\mathrm{Id})$$

and each coefficient

$$\frac{1}{\alpha} \beta_i(\alpha R) \lambda_{i,n}$$

is positive for every $R \in B_{2r_0}(\mathrm{Id})$.

The set Γ_j is then given by $\{v_{i,n}\}$. Note that we might have $v_{i,n} = v_{l,m}$ for two distinct pairs (i,n) and (l,m). Therefore, for $k \in \Gamma_j$, the function λ_k will be defined as

$$\lambda_k(R) = \sum_{(i,n):k=v_{i,n}} \frac{1}{\alpha} \beta_i(\alpha R) \lambda_{i,n}.$$

Step 2 By Step 1, in order to prove the lemma, it suffices to find a number λ_0 and N disjoint families $F_1, \ldots, F_N \subset \lambda_0 \mathbb{S}^2 \cap \mathbb{Z}^3$ such that the sets $c(F_i)$ contain all a positive multiple of the identity. By Proposition 3.3 there is a sequence $\lambda_k \uparrow \infty$ such that the sets $\mathbb{S}^2 \cap \frac{1}{\lambda_k} \mathbb{Z}^3$ converge, in the Hausdorff sense, to the entire sphere $\mathbb{S}^2$.

Given this sequence $\{\lambda_k\}$, we can easily partition each $\lambda_k \mathbb{S}^2 \cap \mathbb{Z}^3$ into N disjoint symmetric families $\{F_j^k\}_{j=1,\ldots,N}$ in such a way that, for each fixed j, the corresponding sequence of sets $\{\frac{1}{\lambda_k} F_j^k\}_k$ converges in the Hausdorff sense to $\mathbb{S}^2$. Hence, any point of $c(\mathbb{S}^2)$ is contained in $c(\frac{1}{\lambda_k} F_j^k)$ provided k is large enough. On the other hand it is easy to see that $c(\mathbb{S}^2)$ contains a multiple of the identity $\alpha \, \mathrm{Id}$ (for instance one can adapt the argument of Lemma 4.2 in [6]). By Step 1, this concludes the proof. $\qquad \square$

4 The maps v_1, $\mathring{R}_1$ and p_1

We have now all the tools to define the maps v_1, $\mathring{R}_1$ and p_1 of Proposition 2.2. Recalling (17) and the discussion in Sect. 2.1, $w := v_1 - v$ is the sum of two maps, w_o and w_c. w_o is a highly oscillatory function based on "patching Beltrami flows" and it will be defined first, in Sect. 4.1. w_c will then be added so as to ensure that v_1 is divergence free: in order to achieve this we will use the classical Leray projector, see Sect. 4.3 for the precise definition. p_1 is related to w_o by a simple formula, given in Sect. 4.4. Finally, in Sect. 4.5 we will define $\mathring{R}_1$. Essentially, this last matrix field can also be thought of as a "corrector term", analogous to w_c. In fact, if we consider the point of view of [6], the Euler-Reynolds system can be stated equivalently as the fact that the 4×4 matrix

$$U := \begin{pmatrix} v_1 \otimes v_1 + p_1 \, \mathrm{Id} - \mathring{R}_1 & v_1 \\ v_1 & 0 \end{pmatrix}$$

is a divergence-free in space-time. $\mathring{R}_1$ has therefore the same flavor as w_c and is also defined through a suitable (elliptic) operator, cf. with Definition 4.2.

4.1 The perturbation w_o

We start by defining a partition of unity on the space of velocities, i.e. the state space. We choose two constants c_1 and c_2 such that $\frac{\sqrt{3}}{2} < c_1 < c_2 < 1$ and we fix a function $\varphi \in C_c^\infty(B_{c_2}(0))$ which is nonnegative and identically 1 on the ball $B_{c_1}(0)$. We next consider the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ and its quotient by $(2\mathbb{Z})^3$, i.e. we define the equivalence relation

$$(k_1, k_2, k_3) \sim (\ell_1, \ell_2, \ell_3) \quad \Longleftrightarrow \quad k_i - \ell_i \text{ is even } \forall i.$$

We then denote by $\mathcal{C}_j$, $j = 1, \ldots, 8$ the 8 equivalence classes of $\mathbb{Z}^3 / \sim$. For each $k \in \mathbb{Z}^3$ denote by φ_k the function

$$\varphi_k(x) := \varphi(x - k).$$

Observe that, if $k \neq \ell \in \mathcal{C}_i$, then $|k - \ell| \geq 2 > 2c_2$. Hence φ_k and φ_ℓ have disjoint supports. On the other hand, the function

$$\psi := \sum_{k \in \mathbb{Z}^3} \varphi_k^2$$

is smooth, bounded and bounded away from zero. We then define

$$\alpha_k(v) := \frac{\varphi_k(v)}{\sqrt{\psi(v)}}$$

and

$$\phi_k^{(j)}(v, \tau) := \sum_{l \in \mathcal{C}_j} \alpha_l(\mu v) e^{-i(k \cdot \frac{l}{\mu})\tau}.$$

Since α_l and $\alpha_{\tilde{l}}$ have disjoint supports for $l \neq \tilde{l} \in \mathcal{C}_j$, it follows that for all v, τ, j

$$\left|\phi_k^{(j)}(v, \tau)\right|^2 = \sum_{l \in \mathcal{C}_j} \alpha_l(\mu v)^2, \tag{24}$$

and in particular $\sum_{j=1}^{8} |\phi_k^{(j)}(v, \tau)|^2 = 1$. Furthermore, for the same reason there exist for any $m = 0, 1, 2, \ldots$ constants $C = C(m)$ such that

$$\sup_{v, \tau} \left| D_v^m \phi_k^{(j)}(v, \tau) \right| \leq C(m) \mu^m. \tag{25}$$

Fix next any (v, τ) and j. Observe that there is at most one $l \in \mathcal{C}_j$ with the property that $\alpha_l(\mu v) \neq 0$ and this l has the property that $|\mu v - l| < 1$. Thus, in a neighborhood of (v, τ) we will have

$$\partial_\tau \phi_k^{(j)} + i(k \cdot v)\phi_k^{(j)} = ik \cdot \left(v - \frac{l}{\mu}\right)\phi_k^{(j)}. \tag{26}$$

Combining (25) and (26), for any $m = 0, 1, 2, \ldots$ we find constants $C = C(m, |k|)$ such that

$$\sup_{v, \tau} \left| D_v^m \left(\partial_\tau \phi_k^{(j)} + i(k \cdot v)\phi_k^{(j)}\right) \right| \leq C\left(m, |k|\right) \mu^{m-1}. \tag{27}$$

We apply Lemma 3.2 with $N = 8$ to obtain $\lambda_0 > 1$, $r_0 > 0$ and pairwise disjoint families Λ_j together with corresponding functions $\gamma_k^{(j)} \in C^\infty(B_{r_0}(\text{Id}))$.

Next, set

$$\rho(t) := \frac{1}{3(2\pi)^3}\left(e(t)\left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{T}^3}|v|^2(x,t)\,dx\right)$$

and

$$R(x,t) := \rho(t)\,\mathrm{Id} - \mathring{R}(x,t),$$

and define

$$w_o(x,t) := \sqrt{\rho(t)}\sum_{j=1}^{8}\sum_{k\in\Lambda_j}\gamma_k^{(j)}\left(\frac{R(x,t)}{\rho(t)}\right)\phi_k^{(j)}\big(v(x,t),\lambda t\big)B_k e^{i\lambda k\cdot x}. \quad (28)$$

4.2 The constants η and M

Note that w_o is well-defined only if $\frac{R}{\rho} \in B_{r_0}(\mathrm{Id})$ where r_0 is given in Lemma 3.2. This is ensured by an appropriate choice of η. Indeed,

$$\rho(t) \geq \frac{1}{3(2\pi)^3}\frac{\delta}{4}e(t) \geq c\delta\min_{t\in[0,1]}e(t) =: c\delta m,$$

where c is a dimensional (positive) constant and $m > 0$ by assumption. Then

$$\left\|\frac{R}{\rho(t)} - \mathrm{Id}\right\| \leq \frac{1}{c\delta m}\|\mathring{R}\| \leq \frac{\eta}{cm}.$$

Thus, it suffices to choose

$$\eta := \frac{1}{2}cmr_0 = \frac{r_0}{24(2\pi)^3}\min_{t\in[0,1]}e(t). \quad (29)$$

Observe that this choice is independent of $\delta > 0$.

Notice next that, by our choice of $\rho(t)$ and by (7), $\rho(t) \leq \delta e(t)$. Thus there exists a constant $M > 1$ depending only on e (in particular independent of δ) so that

$$\sup_{t\in[0,1]}\sup_{x\in\mathbb{T}^3}\big|w_o(x,t)\big| \leq \frac{\sqrt{M\delta}}{2}. \quad (30)$$

This fixes the choice of the constant M in Proposition 2.2.

4.3 The correction w_c

We next define the Leray projector onto divergence-free vectorfields with zero average.

Definition 4.1 Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ be a smooth vector field. Let

$$\mathcal{Q}v := \nabla\phi + \fint_{\mathbb{T}^3} v,$$

where $\phi \in C^\infty(\mathbb{T}^3)$ is the solution of

$$\Delta\phi = \operatorname{div} v \text{ in } \mathbb{T}^3$$

with $\fint_{\mathbb{T}^3}\phi = 0$. Furthermore, let $\mathcal{P} = I - \mathcal{Q}$ be the Leray projection onto divergence-free fields with zero average.

The vector field v_1 is then the sum of v with the Leray projection w of w_o, namely

$$v_1(x, t) := v(x, t) + \mathcal{P}w_o(x, t) =: v(x, t) + w(x, t)$$

and hence

$$w_c(x, t) := -\mathcal{Q}w_o(x, t) = w(x, t) - w_o(x, t).$$

4.4 The pressure p_1

We define

$$p_1 := p - \frac{|w_o|^2}{2}. \tag{31}$$

4.5 The Reynolds stress $\mathring{R}_1$

In order to specify the choice of $\mathring{R}_1$ we introduce a new operator.

Definition 4.2 Let $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ be a smooth vector field. We then define $\mathcal{R}v$ to be the matrix-valued periodic function

$$\mathcal{R}v := \frac{1}{4}\left(\nabla\mathcal{P}u + (\nabla\mathcal{P}u)^T\right) + \frac{3}{4}\left(\nabla u + (\nabla u)^T\right) - \frac{1}{2}(\operatorname{div} u)\operatorname{Id},$$

where $u \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ is the solution of

$$\Delta u = v - \fint_{\mathbb{T}^3} v \text{ in } \mathbb{T}^3$$

with $\fint_{\mathbb{T}^3}u = 0$.

Lemma 4.3 $(\mathcal{R} = \operatorname{div}^{-1})$ *For any $v \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$ we have*

(a) *$\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^3$;*
(b) *$\operatorname{div}\mathcal{R}v = v - \fint_{\mathbb{T}^3} v$.*

Proof It is obvious by inspection that $\mathcal{R}v$ is symmetric. Since $\mathcal{P}v$ is divergence-free, we obtain for the trace

$$\text{tr}(\mathcal{R}v) = \frac{3}{4}(2 \operatorname{div} u) - \frac{3}{2} \operatorname{div} u = 0.$$

Similarly, we have

$$\operatorname{div}(\mathcal{R}v) = \frac{1}{4}\Delta(\mathcal{P}u) + \frac{3}{4}(\nabla \operatorname{div} u + \Delta u) - \frac{1}{2}\nabla \operatorname{div} u. \tag{32}$$

On the other hand recall that $\mathcal{P}u = u - \nabla\phi - \fint u = u - \nabla\phi$, where $\Delta\phi = \operatorname{div} u$. Therefore $\Delta(\mathcal{P}u) = \Delta u - \nabla \operatorname{div} u$. Plugging this identity into (32), we obtain

$$\operatorname{div}(\mathcal{R}v) = \Delta u$$

and since u solves $\Delta u = v - \fint v$, (b) follows readily. □

Then we set

$$\mathring{R}_1 := \mathcal{R}\big(\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1\big).$$

Note that $[\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1]$ has average zero. Indeed:

- the vector field $\operatorname{div}(v_1 \otimes v_1) + \nabla p_1$ has average 0 because it is the divergence of the matrix field $v_1 \otimes v_1 + p_1 \operatorname{Id}$;
- for the same reason, the identity $\partial_t v = -\operatorname{div}(v \otimes v + p \operatorname{Id} - \mathring{R})$ shows that $\partial_t v$ has average 0; on the other hand $w = \mathcal{P}w_o$ has average 0 because of the definition of $\mathcal{P}$; this implies that $\partial_t w$ has also average zero and thus we conclude as well that $\partial_t v_1 = \partial_t v + \partial_t w$ has average zero.

Therefore from Lemma 4.3 it follows that $\mathring{R}_1(x, t)$ is symmetric and trace-free and that the identity

$$\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1 = \operatorname{div} \mathring{R}_1$$

holds.

The rest of this note is devoted to prove that the triple $(v_1, p_1, \mathring{R}_1)$ satisfies the estimates (9), (10), (11) and (12). This will be achieved by an appropriate choice of the parameters μ and λ. In particular, we will show that the estimates hold provided μ is sufficiently large and λ much larger than μ.

5 Schauder estimates

In the following $m = 0, 1, 2, \ldots, \alpha \in (0, 1)$, and β is a multiindex. We introduce the usual (spatial) Hölder norms as follows. First of all, the supremum

norm is denoted by $\|f\|_0 := \sup_{\mathbb{T}^3} |f|$. We define the Hölder seminorms as

$$[f]_m = \max_{|\beta|=m} \|D^\beta f\|_0,$$

$$[f]_{m+\alpha} = \max_{|\beta|=m} \sup_{x \neq y} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\alpha}.$$

The Hölder norms are then given by

$$\|f\|_m = \sum_{j=0}^{m} [f]_j$$

$$\|f\|_{m+\alpha} = \|f\|_m + [f]_{m+\alpha}.$$

Recall the following elementary inequalities:

$$[f]_s \leq C\big(\varepsilon^{r-s}[f]_r + \varepsilon^{-s}\|f\|_0\big) \tag{33}$$

for $r \geq s \geq 0$ and all $\varepsilon > 0$, and

$$[fg]_r \leq C\big([f]_r\|g\|_0 + \|f\|_0[g]_r\big) \tag{34}$$

for any $1 \geq r \geq 0$.

Finally, we recall the classical Schauder estimates for the Laplace operator and the corresponding estimates which we can infer for the various operators involved in our construction.

Proposition 5.1 *For any $\alpha \in (0,1)$ and any $m \in \mathbb{N}$ there exists a constant $C(\alpha, m)$ with the following properties. If $\phi, \psi : \mathbb{T}^3 \to \mathbb{R}$ are the unique solutions of*

$$\begin{cases} \Delta\phi = f, \\ \fint \phi = 0, \end{cases} \qquad \begin{cases} \Delta\psi = \operatorname{div} F, \\ \fint \psi = 0, \end{cases}$$

then

$$\|\phi\|_{m+2+\alpha} \leq C(m,\alpha)\|f\|_{m,\alpha} \quad and \quad \|\psi\|_{m+1+\alpha} \leq C(m,\alpha)\|F\|_{m,\alpha}. \tag{35}$$

Moreover we have the estimates

$$\|\mathcal{Q}v\|_{m+\alpha} \leq C(m,\alpha)\|v\|_{m+\alpha} \tag{36}$$

$$\|\mathcal{P}v\|_{m+\alpha} \leq C(m,\alpha)\|v\|_{m+\alpha} \tag{37}$$

$$\|\mathcal{R}v\|_{m+1+\alpha} \leq C(m,\alpha)\|v\|_{m+\alpha} \tag{38}$$

$$\|\mathcal{R}(\operatorname{div} A)\|_{m+\alpha} \leq C(m,\alpha)\|A\|_{m+\alpha} \tag{39}$$

$$\left\| \mathcal{R}\mathcal{Q}(\operatorname{div} A) \right\|_{m+\alpha} \le C(m, \alpha) \|A\|_{m+\alpha}. \tag{40}$$

Proof The estimates (35) are the usual Schauder estimates, see for instance [12, Chap. 4]. The meticulous reader will notice that the estimates in [12] are stated in $\mathbb{R}^n$ for the potential-theoretic solution of the Laplace operator. The periodic case is however an easy corollary. Take for instance ϕ and f and consider them as periodic functions defined on $\mathbb{R}^3$. Consider $g = f\chi$, where χ is a cut-off function supported in $B_{6\pi}(0)$ and identically 1 on $B_{4\pi}(0)$. Let $\tilde{\phi}$ be the potential-theoretic solution in $\mathbb{R}^3$ of $\Delta\tilde{\phi} = g$. For $\tilde{\phi}$ we can invoke the Schauder estimates as in [12, Chap. 4]. Moreover $\phi - \tilde{\phi}$ is an harmonic function in $B_{4\pi}(0)$. Obviously $\|\phi\|_{L^2(B_{4\pi}(0))}$ can be easily bounded using $\Delta\phi = f$, $\fint\phi = 0$ and the Parseval identity. Thus, standard properties of harmonic functions give $\|\phi - \tilde{\phi}\|_{C^{m,\alpha}([2\pi]^3)} \le C(m, \alpha)\|f\|_0$.

The estimates (36), (37), (38) and (39) are easy consequences of (35) and the definitions of the operators. The estimate (40) requires a little more care. Let $u : \mathbb{T}^3 \to \mathbb{R}^3$ be the unique solution of

$$\Delta\Delta u_i = \partial_i \sum_{j,n} \partial^2_{jn} A_{jn}$$

with $\fint u = 0$. Then

$$\|u\|_{m+1+\alpha} \le C(m, \alpha)\|A\|_{m+\alpha}. \tag{41}$$

First of all, with the argument above, one can reduce this estimate to a corresponding one for the potential-theoretic solution of the biLaplace operator in $\mathbb{R}^3$. For this case we can then invoke general estimates for elliptic k-homogeneous constant coefficients operators (see for instance [24, Theorem 1]) or use the same arguments of [12, Chap. 4] replacing the fundamental solution of the Laplacian with that of the biLaplacian. Finally, (40) follows from the identity

$$\mathcal{R}\mathcal{Q}(\operatorname{div} A) = \frac{1}{4}\big(\nabla\mathcal{P}u + (\nabla\mathcal{P}u)^T\big) + \frac{3}{4}\big(\nabla u + (\nabla u)^T\big) - \frac{1}{2}(\operatorname{div} u)\operatorname{Id}$$

and the estimates (41) and (37). $\qquad\square$

In what follows we will use the convention that Greek subscripts of Hölder norms denote always exponents in the open interval $(0, 1)$.

Proposition 5.2 *Let $k \in \mathbb{Z}^3 \setminus \{0\}$ and $\lambda \ge 1$ be fixed.*
(i) *For any $a \in C^\infty(\mathbb{T}^3)$ and $m \in \mathbb{N}$ we have*

$$\left| \int_{\mathbb{T}^3} a(x)e^{i\lambda k \cdot x}\, dx \right| \le C\frac{[a]_m}{\lambda^m}. \tag{42}$$

(ii) *Let* $\lambda \in \mathbb{N}\setminus\{0\}$ *and* $\phi_\lambda \in C^\infty(\mathbb{T}^3)$ *be the solution of*

$$\Delta\phi_\lambda = f_\lambda \text{ in } \mathbb{T}^3$$

with $\int_{\mathbb{T}^3} \phi_\lambda = 0$, *where*

$$f_\lambda(x) := a(x)e^{i\lambda k \cdot x} - \fint_{\mathbb{T}^3} a(y)e^{i\lambda k \cdot y}\, dy.$$

Then for any $\alpha \in (0, 1)$ *and* $m \in \mathbb{N}$ *we have the estimate*

$$\|\nabla\phi_\lambda\|_\alpha \le \frac{C}{\lambda^{1-\alpha}}\|a\|_0 + \frac{C}{\lambda^{m-\alpha}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\alpha}, \tag{43}$$

where $C = C(\alpha, m)$.

Proof For $j = 0, 1, \ldots$ define

$$A_j(y, \xi) := -i\left[\frac{k}{|k|^2}\left(i\frac{k}{|k|^2} \cdot \nabla\right)^j a(y)\right]e^{ik\cdot\xi},$$

$$F_j(y, \xi) := \left[\left(i\frac{k}{|k|^2} \cdot \nabla\right)^j a(y)\right]e^{ik\cdot\xi}.$$

Direct calculation shows that

$$F_j(x, \lambda x) = \frac{1}{\lambda}\operatorname{div}\left[A_j(x, \lambda x)\right] + \frac{1}{\lambda}F_{j+1}(x, \lambda x).$$

In particular for any $m \in \mathbb{N}$

$$a(x)e^{i\lambda k\cdot x} = F_0(x, \lambda x) = \frac{1}{\lambda}\sum_{j=0}^{m-1}\frac{1}{\lambda^j}\operatorname{div}\left[A_j(x, \lambda x)\right] + \frac{1}{\lambda^m}F_m(x, \lambda x).$$

Integrating this over $\mathbb{T}^3$ and using that $|k| \ge 1$ we obtain (42).

Next, using (33) and (34) we have, for any $j \le m - 1$,

$$\|A_j(\cdot, \lambda\cdot)\|_\alpha \le C\left(\lambda^\alpha[a]_j + [a]_{j+\alpha}\right)$$

$$\le C\lambda^{j+\alpha}\left(\lambda^{-m}[a]_m + \|a\|_0\right)$$

and similarly

$$\|F_m(\cdot, \lambda\cdot)\|_\alpha \le C\left(\lambda^\alpha[a]_m + [a]_{m+\alpha}\right).$$

Moreover, according to the standard estimate (35),

$$\|\nabla\phi\|_\alpha \le C\left(\frac{1}{\lambda}\sum_{j=0}^{m-1}\frac{1}{\lambda^j}\big\|A_j(\cdot,\lambda\cdot)\big\|_\alpha + \frac{1}{\lambda^m}\big\|F_m(\cdot,\lambda\cdot)\big\|_\alpha\right.$$

$$\left. + \left|\fint_{\mathbb{T}^3} F_0(x,\lambda x)\,dx\right|\right),$$

hence, using (42) for the last term,

$$\|\nabla\phi\|_\alpha \le \frac{C}{\lambda^{1-\alpha}}\|a\|_0 + \frac{C}{\lambda^{m-\alpha}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\alpha}$$

as required. $\square$

Corollary 5.3 *Let $k \in \mathbb{Z}^3 \setminus \{0\}$ be fixed. For a smooth vectorfield $a \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ let $F(x) := a(x)e^{i\lambda k\cdot x}$. Then we have*

$$\big\|\mathcal{R}(F)\big\|_\alpha \le \frac{C}{\lambda^{1-\alpha}}\|a\|_0 + \frac{C}{\lambda^{m-\alpha}}[a]_m + \frac{C}{\lambda^m}[a]_{m+\alpha},$$

where $C = C(\alpha, m)$.

Proof This is an immediate consequence of the definition of $\mathcal{R}$, the Schauder estimate (37) for $\mathcal{P}$ and Proposition 5.2 above. $\square$

6 Estimates on the corrector and the energy

In all subsequent estimates, unless otherwise stated, C denotes a generic constant that can vary from line to line, and depends on e, v, $\mathring{R}$ as well as on λ_0, α and δ, but is independent of λ and μ. Smallness of the respective quantities will be achieved by an appropriate choice of λ, μ in Sect. 8. Moreover, all estimates will implicitly assume (18), in particular that $1 \le \mu \le \lambda$.

Our aim is to estimate the space-time sup-norm of $v_1 - v = w = w_o + w_c$ and $\mathring{R}_1$. Since w_c and $\mathring{R}_1$ are defined in terms of the singular integral operators $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, which act in space, instead of obtaining directly estimates of the C^0 norm, we will use Schauder estimates to obtain bounds on spatial Hölder norms. Thus, in the sequel the Hölder norms will denote spatial norms, and are understood to be uniform in time $t \in [0, 1]$. Moreover, if the Hölder exponent is denoted by a Greek letter, then it is a number in the open interval $(0, 1)$.

It will be convenient to write w_o as

$$w_o(x, t) = W(x, t, \lambda t, \lambda x),$$

where

$$W(y, s, \tau, \xi) := \sum_{|k|=\lambda_0} a_k(y, s, \tau) B_k e^{ik \cdot \xi}$$

$$= \sqrt{\rho(s)} \sum_{j=1}^{8} \sum_{k \in \Lambda_j} \gamma_k^{(j)} \left(\frac{R(y, s)}{\rho(s)} \right) \phi_k^{(j)}(v(y, s), \tau) B_k e^{ik \cdot \xi}.$$

(44)

We summarize the main properties of the coefficients W:

Proposition 6.1 (i) *Let* $a_k \in C^\infty(\mathbb{T}^3 \times [0, 1] \times \mathbb{R})$ *be given by* (44). *Then for any* $r \geq 0$

$$\left\| a_k(\cdot, s, \tau) \right\|_r \leq C\mu^r,$$

$$\left\| \partial_s a_k(\cdot, s, \tau) \right\|_r \leq C\mu^{r+1},$$

$$\left\| \partial_\tau a_k(\cdot, s, \tau) \right\|_r \leq C\mu^r,$$

$$\left\| (\partial_\tau a_k + i(k \cdot v)a_k)(\cdot, s, \tau) \right\|_r \leq C\mu^{r-1}.$$

(ii) *The matrix-function* $W \otimes W$ *can be written as*

$$(W \otimes W)(y, s, \tau, \xi) = R(y, s) + \sum_{1 \leq |k| \leq 2\lambda_0} U_k(y, s, \tau) e^{ik \cdot \xi}, \qquad (45)$$

where the coefficients $U_k \in C^\infty(\mathbb{T}^3 \times [0, 1] \times \mathbb{R}; \mathcal{S}^{3 \times 3})$ *satisfy*

$$U_k k = \frac{1}{2}(\mathrm{tr}\, U_k)k \qquad (46)$$

and for any $r \geq 0$

$$\left\| U_k^\mu(\cdot, s, \tau) \right\|_r \leq C\mu^r,$$

$$\left\| \partial_s U_k^\mu(\cdot, s, \tau) \right\|_r \leq C\mu^{r+1},$$

$$\left\| \partial_\tau U_k(\cdot, s, \tau) \right\|_r \leq C\mu^r,$$

$$\left\| (\partial_\tau U_k^\mu + i(k \cdot v)U_k^\mu)(\cdot, s, \tau) \right\|_r \leq C\mu^{r-1}.$$

In all these estimates the constant C *depends on* r *and* $e, v, \mathring{R}$ *but is independent of* (s, τ) *and* μ.

Proof The estimates for a_k are a consequence of (25) and (27). Indeed, since the constants in the estimates are allowed to depend on $e, v, \mathring{R}$, one only needs

to keep track of the number of derivatives of $\phi_k^{(j)}$ with respect to v. Then the estimates on a_k, $\partial_s a_k$ and $\partial_\tau a_k + i(k \cdot v)a_k$ immediately follow. From the triangle inequality we can then also conclude the estimate on $\partial_\tau a_k$.

Next, consider the expansion of $\xi \mapsto W \otimes W$ into a Fourier series in ξ, i.e.

$$(W \otimes W)(y, s, \tau, \xi) = U_0(y, s, \tau) + \sum_{1 \leq |k| \leq 2\lambda_0} U_k(y, s, \tau) e^{ik \cdot \xi}.$$

Since each U_k is the sum of finitely many terms of the form $a_{k'} a_{k''}$, the estimates for U_k follow from those for a_k.

Next, since U_0 is given by the average (in ξ), in order to obtain (45) we need to show that

$$\fint_{\mathbb{T}^3} W \otimes W(y, s, \tau, \xi) \, d\xi = R(y, s).$$

To this end we calculate:

$$\fint_{\mathbb{T}^3} W \otimes W \, d\xi \overset{(21)}{=} \frac{\rho}{2} \sum_j \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(\rho^{-1} R)\right)^2 |\phi_k^{(j)}(v, \tau)|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)$$

$$\overset{(24)}{=} \frac{\rho}{2} \sum_j \sum_{k \in \Lambda_j} \sum_{l \in \mathcal{C}_j} \left(\gamma_k^{(j)}(\rho^{-1} R)\right)^2 \left(\alpha_l(\mu v)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)$$

$$\overset{(22)}{=} R \sum_j \sum_{l \in \mathcal{C}_j} \left(\alpha_l(\mu v)\right)^2 = R \sum_{l \in \mathbb{Z}^3} \left(\alpha_l(\mu v)\right)^2 = R.$$

Finally, (46) is a direct consequence of Proposition 3.1, in particular of (20). □

After this preparation we are ready to estimate all the terms in the perturbation scheme. First of all we verify that the corrector term w_c is indeed much smaller than the main perturbation term w_o:

Lemma 6.2 (Estimate on the corrector)

$$\|w_c\|_\alpha \leq C \frac{\mu}{\lambda^{1-\alpha}}. \tag{47}$$

Proof We start with the observation that, since $k \cdot B_k = 0$,

$$w_o(x, t) = \frac{1}{\lambda} \nabla \times \left(\sum_{|k|=\lambda_0} -i a_k(x, t, \lambda t) \frac{k \times B_k}{|k|^2} e^{i\lambda x \cdot k}\right)$$

$$+ \frac{1}{\lambda} \sum_{|k|=\lambda_0} i \nabla a_k(x, t, \lambda t) \times \frac{k \times B_k}{|k|^2} e^{i\lambda x \cdot k}.$$

Hence

$$w_c(x,t) = -\frac{1}{\lambda}\mathcal{Q}u_c(x,t), \tag{48}$$

where

$$u_c(x,t) = \sum_{|k|=\lambda_0} i\nabla a_k(x,t,\lambda t) \times \frac{k \times B_k}{|k|^2}e^{i\lambda x \cdot k}.$$

The estimate (47) then follows from the Schauder estimate (36) for $\mathcal{Q}$ combined with

$$\|u_c\|_\alpha \leq C\mu\lambda^\alpha. \qquad \square$$

Next, we verify the estimate on the energy (9).

Lemma 6.3 (Estimate on the energy)

$$\left| e(t)\left(1 - \frac{1}{2}\delta\right) - \int_{\mathbb{T}^3}|v_1|^2\,dx \right| \leq C\frac{\mu}{\lambda^{1-\alpha}}. \tag{49}$$

Proof Taking the trace of identity (45) in Proposition 6.1 we have

$$\left|W(y,s,\tau,\xi)\right|^2 = \operatorname{tr}R(y,s) + \sum_{1 \leq |k| \leq 2\lambda_0} c_k(y,s,\tau)e^{ik\cdot\xi}$$

for some coefficients $c_k \in C^\infty(\mathbb{T}^3 \times [0,1] \times \mathbb{R})$, which satisfy the estimates

$$\left\|c_k(\cdot,s,\tau)\right\|_r \leq C\mu^r.$$

From part (i) of Proposition 5.2 with $m = 1$ we deduce

$$\left|\int_{\mathbb{T}^3}|w_o|^2 - \operatorname{tr}R\,dx\right| \leq C\frac{\mu}{\lambda}$$

and

$$\left|\int_{\mathbb{T}^3}v \cdot w_o\,dx\right| \leq C\frac{\mu}{\lambda}.$$

Hence, combining with (47) we see that

$$\left|\int_{\mathbb{T}^3}|v_1|^2 - |v|^2 - |w_o|^2\,dx\right| \leq C\frac{\mu}{\lambda^{1-\alpha}}.$$

Recalling that

$$\operatorname{tr}R = 3\rho = \frac{1}{(2\pi)^3}\left(e(t)\left(1 - \frac{1}{2}\delta\right) - \int_{\mathbb{T}^3}|v|^2\,dx\right),$$

we conclude (49). $\qquad \square$

7 Estimates on the Reynolds stress

Rewrite

$$\partial_t v_1 + \mathrm{div}(v_1 \otimes v_1) + \nabla p_1$$

$$= [\partial_t w_o + v \cdot \nabla w_o] + \left[\mathrm{div}\left(w_o \otimes w_o - \frac{1}{2}|w_o|^2 \,\mathrm{Id} + \mathring{R} \right) \right]$$

$$+ \left[\partial_t w_c + \mathrm{div}(v_1 \otimes w_c + w_c \otimes v_1 - w_c \otimes w_c + v \otimes w_o) \right]. \quad (50)$$

In other words we split the Reynolds stress into the three parts on the right hand side. We will refer to them as the *transport part*, the *oscillation part*, and the *error*. In the following we will estimate each term separately.

Lemma 7.1 (The transport part)

$$\left\| \mathcal{R}(\partial_t w_o + v \cdot \nabla w_o) \right\|_\alpha \le C\left(\frac{\lambda^\alpha}{\mu} + \frac{\mu^2}{\lambda^{1-\alpha}} \right). \quad (51)$$

Proof Observe that

$$\mathcal{R}(\partial_t w_o + v \cdot \nabla w_o) = \lambda \mathcal{R}\left(\sum_{|k|=\lambda_0} \left(\partial_\tau a_k + i(k \cdot v)a_k \right)(x, t, \lambda t) B_k e^{i\lambda k \cdot x} \right)$$

$$+ \mathcal{R}\left(\sum_{|k|=\lambda_0} (\partial_s a_k + v \cdot \nabla_y a_k)(x, t, \lambda t) B_k e^{i\lambda k \cdot x} \right).$$

For the first term Corollary 5.3 with $m = 2$ implies the bound

$$\frac{\lambda^\alpha}{\mu} + \frac{\mu}{\lambda^{1-\alpha}} + \frac{\mu^{1+\alpha}}{\lambda},$$

whereas for the second term Corollary 5.3 with $m = 1$ implies the bound

$$\frac{\mu}{\lambda^{1-\alpha}} + \frac{\mu^2}{\lambda^{1-\alpha}} + \frac{\mu^{2+\alpha}}{\lambda}.$$

Since $1 \le \mu \le \lambda$, we obtain (51). $\qquad\square$

Lemma 7.2 (The oscillation part)

$$\left\| \mathcal{R}\left(\mathrm{div}\left(w_o \otimes w_o - \frac{1}{2}|w_o|^2 \,\mathrm{Id} + \mathring{R} \right) \right) \right\|_\alpha \le C \frac{\mu^2}{\lambda^{1-\alpha}}. \quad (52)$$

Proof Recall the formula (45) from Proposition 6.1. Since ρ is a function of t only, we can write the oscillation part in (50) as

$$\text{div}\left(w_o \otimes w_o - \frac{1}{2}\left(|w_o|^2 - \rho\right)\text{Id} + \mathring{R}\right)$$

$$= \text{div}\left(w_o \otimes w_o - R - \frac{1}{2}\left(|w_o|^2 - \text{tr }R\right)\text{Id}\right)$$

$$= \text{div}\left[\sum_{1 \leq |k| \leq 2\lambda_0}\left(U_k - \frac{1}{2}(\text{tr }U_k)\text{Id}\right)(x, t, \lambda t)e^{i\lambda k \cdot x}\right]$$

$$\overset{(46)}{=} \sum_{1 \leq |k| \leq 2\lambda_0}\text{div}_y\left[U_k - \frac{1}{2}(\text{tr }U_k)\text{Id}\right](x, t, \lambda t)e^{i\lambda k \cdot x}.$$

Corollary 5.3 with $m = 1$ then implies (52). $\qquad\square$

Concerning the error, we are going to treat three terms separately, as follows.

Lemma 7.3 (Estimate on the error I)

$$\left\|\mathcal{R}(\partial_t w_c)\right\|_\alpha \leq C\frac{\mu^2}{\lambda^{1-\alpha}}. \tag{53}$$

Proof Recall from (48) that $w_c = -\frac{1}{\lambda}\mathcal{Q}u_c$. Now

$$\partial_t u_c(x, t) = \lambda \sum_{|k|=\lambda_0} i(\nabla\partial_\tau a_k)(x, t, \lambda t) \times \frac{k \times B_k}{|k|^2}e^{i\lambda x \cdot k}$$

$$+ \sum_{|k|=\lambda_0} i(\nabla\partial_s a_k)(x, t, \lambda t) \times \frac{k \times B_k}{|k|^2}e^{i\lambda x \cdot k}.$$

Moreover, for any $c_k \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ we have

$$c_k(x, t, \lambda t)e^{i\lambda x \cdot k} = \frac{1}{i\lambda}\text{div}\left[c_k(x, t, \lambda t) \otimes \frac{k}{|k|^2}e^{i\lambda x \cdot k}\right]$$

$$- \frac{1}{i\lambda}\left[\left(\frac{k}{|k|^2} \cdot \nabla\right)c_k(x, t, \lambda t)\right]e^{i\lambda x \cdot k}.$$

Therefore $\partial_t u_c$ can be written as

$$\partial_t u_c = \text{div }U_c + \tilde{u}_c,$$

where

$$\|U_c\|_\alpha \le C\mu\lambda^\alpha, \qquad \|\tilde u_c\|_\alpha \le C\mu^2\lambda^\alpha.$$

Therefore we have $\mathcal{R}\partial_t w_c = -\frac{1}{\lambda}(\mathcal{R}\mathcal{Q}\operatorname{div}U_c + \mathcal{R}\mathcal{Q}\tilde u_c)$. From the Schauder estimate (40) for the operator $\mathcal{R}\mathcal{Q}\operatorname{div}$, we conclude that

$$\|\mathcal{R}\partial_t w_c\|_\alpha \le \frac{1}{\lambda}\big(\|\mathcal{R}\mathcal{Q}\operatorname{div}U_c\|_\alpha + \|\mathcal{R}\mathcal{Q}\tilde u_c\|_\alpha\big)$$

$$\le \frac{C}{\lambda}\big(\|U_c\|_\alpha + \|\tilde u_c\|_\alpha\big) \le C\frac{\mu^2}{\lambda^{1-\alpha}}. \qquad \square$$

Lemma 7.4 (Estimate on the error II)

$$\big\|\mathcal{R}\big(\operatorname{div}(v_1 \otimes w_c + w_c \otimes v_1 - w_c \otimes w_c)\big)\big\|_\alpha \le C\frac{\mu}{\lambda^{1-2\alpha}}. \qquad (54)$$

Proof We first estimate

$$\|v_1 \otimes w_c + w_c \otimes v_1 - w_c \otimes w_c\|_\alpha$$

$$\le C\big(\|v_1\|_0\|w_c\|_\alpha + \|v_1\|_\alpha\|w_c\|_0 + \|w_c\|_0\|w_c\|_\alpha\big)$$

$$\overset{(47)}{\le} C\frac{\mu}{\lambda^{1-\alpha}}\big(\|v_1\|_\alpha + \|w_c\|_\alpha\big)$$

$$\le C\frac{\mu}{\lambda^{1-\alpha}}\big(\|v\|_\alpha + \|w_c\|_\alpha + \|w_o\|_\alpha\big)$$

$$\le C\frac{\mu}{\lambda^{1-\alpha}}\Big(C + C\frac{\mu}{\lambda^{1-\alpha}} + C\lambda^\alpha\Big).$$

Recall that $1 \le \mu \le \lambda$ and hence $1 \le \frac{\mu}{\lambda^{1-\alpha}} \le \lambda^\alpha$. Thus we conclude

$$\|v_1 \otimes w_c + w_c \otimes v_1 - w_c \otimes w_c\|_\alpha \le C\frac{\mu}{\lambda^{1-2\alpha}}.$$

(54) follows from the latter inequality and the Schauder estimate (39). $\qquad \square$

Lemma 7.5 (Estimate on the error III)

$$\big\|\mathcal{R}\big(\operatorname{div}(v \otimes w_o)\big)\big\|_\alpha \le C\frac{\mu^2}{\lambda^{1-\alpha}}. \qquad (55)$$

Proof Since $B_k \cdot k = 0$, we can write

$$\operatorname{div}(v \otimes w_o) = w_o \cdot \nabla v + (\operatorname{div}w_o)v$$

$$= \sum_{|k|=\lambda_0} \big[a_k(B_k \cdot \nabla)v + v(B_k \cdot \nabla a_k)\big]e^{i\lambda k \cdot x}.$$

The claim follows from Corollary 5.3 with $m = 1$. □

8 Conclusion: proof of Proposition 2.2

In this section we collect the estimates from the preceding sections. For simplicity we set

$$\mu = \lambda^\beta.$$

It should be noted, however, that due to the requirement (18) we can only ensure $\mu \sim \lambda^\beta$ for large λ.

We claim that for an appropriate choice of α and β, the estimates (9), (10), (11) and (12) will be satisfied for sufficiently large λ. First of all recall that, by the choice of M we have

$$\|w_o\|_0 \leq \frac{\sqrt{M\delta}}{2} \tag{56}$$

(cf. with (30)) and $M > 1$. Therefore (11) follows from the estimate $\|w_c\|_\alpha \leq C\mu\lambda^{\alpha-1}$ (cf. with (47)) if, for instance, we can prescribe

$$C\frac{\mu}{\lambda^{1-\alpha}} = C\lambda^{\alpha+\beta-1} \leq \frac{\sqrt{\delta}}{2}.$$

On the other hand (12) follows easily from (56) and the identity $p_1 - p = -\frac{1}{2}|w_o|^2$.

Also, from (49) it follows that (9) is satisfied provided

$$C\lambda^{\alpha+\beta-1} \leq \frac{1}{8}\delta \min_{t\in[0,1]} e(t).$$

Finally, (51), (52), as well as the estimates on the error (53)–(55) imply that

$$\|\mathring{R}_1\|_\alpha \leq C\big(\lambda^{\alpha-\beta} + \lambda^{\alpha+2\beta-1} + \lambda^{2\alpha+\beta-1}\big).$$

Therefore, any choice of α, β such that

$$\alpha < \beta, \quad \alpha + 2\beta < 1 \tag{57}$$

will ensure that (9), (10), (11) and (12) will be valid for sufficiently large λ. This completes the proof. As a side remark observe that (57) requires $\alpha < \frac{1}{3}$.

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