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The Mandelbrot Set[1]

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Dedicated to Professor Ilja Černý on the occasion of his 70^{th} birthday.

Abstract: We give basic facts about the Mandelbrot set.

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Introduction

One of the most interesting features of the Mandelbrot set $\mathcal{M}$ comes from the fact that we can use computers to generate fairly good pictures of it. These pictures then lead us to conject various properties of $\mathcal{M}$, and this again poses a challenge for mathematicians: to give rigorous proofs. Our paper intends to provide basic mathematical analysis behind the Mandelbrot set.

Firstly, we show basic topological properties of $\mathcal{M}$, namely that it is a non-empty, bounded and closed (hence compact) subset of $\mathbb{C}$, which is symmetric with respect to real axis. We also show that the complement of $\mathcal{M}$ is connected. We then try to give more detailed description of $\mathcal{M}$ - we find an intersection of $\mathcal{M}$ with the real axis and we also explicitely describe the two biggest components of $\operatorname{int}(\mathcal{M})$, namely the "cardioid" and the largest "bulb" adjacent to it. Finally we discuss the implications of our analysis to algorithms for generating pictures of $\mathcal{M}$.

We would like to stress that this paper is by no means any original work about the Mandelbrot set. All facts included here can be found in [B] or [CG].

Notation and terminology

By $\mathbb{C}$ we denote the complex plane, by $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere. When not stated otherwise, we work in topology of $\mathbb{S}$.

If $f : \mathbb{S} \mapsto \mathbb{S}$, by f^n we mean the n-th iteration of f. The sequence $\{f^n(z_0)\}$ is called *orbit* of the point z_0. A point z_0 is called a *fixed point* of f if $f(z_0) = z_0$. If $z_0 \in \mathbb{C}$, we say that z_0 is an *attracting fixed point* provided $|f'(z_0)| < 1$.

Further, we say that the sequence of points $\{z_0, f(z_0), f^2(z_0), \ldots, f^{p-1}(z_0)\}$ forms a *p-cycle* provided p is smallest number such that $f^p(z_0) = z_0$. A p-cycle is called *attracting* if z_0 (or equivalently, any other point of the p-cycle) is an attracting fixed point for f^p.

Finally, if z_0 is a fixed point of f, we call the number $f'(z_0)$ a *multiplier* of z_0. Analogously, we define the multiplier of a p-cycle as $[f^p]'(z_0)$.

Basic properties of $\mathcal{M}$

For any $c \in \mathbb{C}$, we define

$$P_c(z) := z^2 + c.$$

The Mandelbrot set $\mathcal{M}$ is then defined as

$$\mathcal{M} := \{c \in \mathbb{C};\ \{P_c^n(0)\} \text{ is bounded for } n \to \infty\}.$$

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In fact, we are studying the orbit of 0 under the iteration of $P_c(z)$. It is obvious that the only way the orbit becomes unbounded is that it converges to ∞. Hence, we can alternatively say that

$$\mathcal{M} = \{c \in \mathbb{C};\ P_c^n(0) \not\to \infty \text{ for } n \to \infty\}.$$

Some notation is useful here. We denote

$$Q_n(c) := P_c^n(0);$$

note that $Q_1(c) = c$ and $Q_{n+1}(c) = [Q_n(c)]^2 + c$, hence each Q_n is in c a polynomial with real coefficients. This in view of the previous formula implies

$$P_{\bar{c}}^n(0) = \overline{P_c^n(0)}$$

and consequently $c \in \mathcal{M} \iff \bar{c} \in \mathcal{M}$, i.e., $\mathcal{M}$ **is symmetric with respect to the real axis**.

For a given c the orbit of 0 looks like

$$0,\ c,\ c^2 + c,\ (c^2 + c)^2 + c, \dots$$

and naturally we would expect that for c large enough, this tends to infinity, in other words, $\mathcal{M}$ is bounded. In fact, we will prove a little more, using the following lemma.

Lemma 1 *Let $c \in \mathbb{C}$, $\varepsilon > 0$ are given and*

$$W_{c,\varepsilon} := \{z;\ |z| \geq |c| \text{ and } |z| \geq 2 + \varepsilon\}.$$

Then $P_c^n(z_0) \to \infty$ for any $z_0 \in W_{c,\varepsilon}$.

PROOF: Let $z_0 \in W_{c,\varepsilon}$. Then we can write

$$|P_c(z_0)| = |{z_0}^2 + c| \geq |z_0|^2 - |c| \geq |z_0|^2 - |z_0| = |z_0|(|z_0| - 1) \geq |z_0|(1 + \varepsilon).$$

This implies that $P_c(z_0) \in W_{c,\varepsilon}$ and repeating this procedure inductively we obtain $|P_c^n(z_0)| \geq |z_0|(1 + \varepsilon)^n$ and the conclusion follows. $\square$

Note that if $c \in \mathcal{M}$ then $|c| \leq 2$, since otherwise we could apply the previous lemma with $z_0 = c$. Consequently $\mathcal{M}$ **is contained in a closed disk of radius 2**.

Moreover, we obtain third characterization of $\mathcal{M}$, namely

$$\mathcal{M} = \{c \in \mathbb{C};\ |Q_n(c)| \leq 2 \text{ for all } n \in \mathbb{N}\}.$$

Indeed, the set on the right is part of $\mathcal{M}$. On the other hand, if $c \in \mathcal{M}$, then for all n and all $\varepsilon > 0$, $Q_n(c) \notin W_{c,\varepsilon}$ since otherwise the sequence would escape to infinity by Lemma 1. This means that for all $n \in \mathbb{N}$ there must be either $|Q_n(c)| < |c|$ or $|Q_n(c)| \leq 2$ (as ε can be taken arbitrarily small) and we finish by considering the case $n = 1$ which (since $Q_1(c) = c$) implies $|c| \leq 2$.

Now, let us denote $K := \{z \in \mathbb{C};\ |z| \leq 2\}$. The third characterization of $\mathcal{M}$ can be rewritten as

$$\mathcal{M} = \bigcap_{n=1}^{\infty} Q_n^{-1}(K);$$

note that for all n, $Q_n^{-1}(K)$ is closed and bounded set, so $\mathcal{M}$ is expressed as an intersection of family of compact (in $\mathbb{C}$) sets and hence $\mathcal{M}$ **is compact in** $\mathbb{C}$.

Let us turn our attention to the complement $\mathbf{C}(\mathcal{M})$ of $\mathcal{M}$. We have

$$\mathbf{C}(\mathcal{M}) = \bigcup_{n=1}^{\infty} Q_n^{-1}(D),$$

where $D := \{z \in \mathbb{C};\ |z| > 2\}$. In the sequence of two lemmas, we will show that $\mathbf{C}(\mathcal{M})$ is connected.

Lemma 2 *Let P be a non-constant polynomial, let D be an open and connected set, and let U be an arbitrary component of $P^{-1}(D)$. Then $P(U) = D$.*

PROOF: Let $P(U) = E \subsetneq D$. Then (since D is connected) there must exist $w \in \partial E \cap D$. Consequently, there are sequences of points $w_n \in E$ and $z_n \in U$ such that $P(z_n) = w_n$; we also assume $z_n \to z_0$ for some $z_0 \in \overline{U}$. Obviously $P(z_0) = w$ since P is continous and we will show that both $z_0 \in \operatorname{int} U$ and $z_0 \in \partial U$ is impossible.

Indeed, if $z_0 \in \operatorname{int} U$ then $w \in \operatorname{int} E$ as P is an open mapping (see e.g. [C], Theorem 10.4.4, p. 308), which contradicts the choice of w. On the other hand, if $z_0 \in \partial U$ then we can find a neighborhood $V(z_0)$ small enough so that $P(V(z_0)) \subset D$. But then $U \cup V$ is a connected subset of $P^{-1}(D)$ stricly greater than U, a contradiction. $\square$

Lemma 3 *Let P be a non-constant polynomial, and let D be an open and connected set containing infinity. Then also $P^{-1}(D)$ is an open and connected set containing infinity.*

PROOF: Let U be an arbitrary component of $P^{-1}(D)$. By previous lemma, $P(U) = D$. But since $\infty \in D$ and P is a polynomial, it follows $\infty \in U$. We have shown that any component of $P^{-1}(D)$ contains the same point, hence this set is connected. $\square$

As a consequence, $\mathbf{C}(\mathcal{M})$ is an union of open and connected sets containing infinity, hence it has the same property, i.e., $\mathbf{C}(\mathcal{M})$ **is open and connected set containing** ∞.

It remains to settle the question whether the set $\mathcal{M}$ itself is connected. The answer is yes, but the proof of this, as found in [B], is quite complicated and is beyond the scope of this work.

Real part of $\mathcal{M}$

We are going to show that intersection of $\mathcal{M}$ with the real axis is the interval $[-2, 1/4]$.

Note first that $-2 \in \mathcal{M}$, since then the orbit of origin looks like $0, -2, 2, 2, \cdots$. On the other hand, no real number lesser then -2 belongs to $\mathcal{M}$ as $\mathcal{M}$ is contained in the disc or radius 2.

Now, consider the function $f(x) := x^2 + 1/4$. Clearly if $x \in [0, 1/2]$, then $f(x) \in [0, 1/2]$, hence the orbit of zero remains bounded and $1/4 \in \mathcal{M}$. On the other hand, if $c = 1/4 + \varepsilon$, $\varepsilon > 0$, then $f(x) - x = (x - 1/2)^2 + \varepsilon \geq \varepsilon$, hence $f^n(0) - 0 \geq n\varepsilon \to \infty$ as $n \to \infty$.

We have shown that $-2,\ 1/4 \in \mathcal{M}$ and $\mathbb{R} \cap \mathcal{M} \subset [-2, 1/4]$. We could use analogous arguments to show that all numbers between -2 and $1/4$ are in $\mathcal{M}$ too, but instead we are going to employ a kind of topological argument, based on the facts proved the previous section.

Assume that there is a number $a \in (-2, 1/4)$ such that $a \in \mathbf{C}(\mathcal{M})$. We will show that this is impossible as long as both $\mathcal{M}$ and $\mathbf{C}(\mathcal{M})$ are connected and symmetric with respect to the real axis.

Since the number $3i$ belongs in $\mathbf{C}(\mathcal{M})$ and $\mathbf{C}(\mathcal{M})$ is open and connected, there must exist a piece-wise linear curve ϕ connecting a to $3i$. It is clear that this curve can be constructed in such a way that it never (except the last point) intersects the circle $\{|z| = 3\}$.

Similarly, we can make the curve lie only (except the point a) in the upper half of $\mathbb{C}$ $\{\mathrm{Im} z > 0\}$. Should any part of the curve lie below the real axis, we can simply replace it by its complex conjugate - it will remain inside $\mathbf{C}(\mathcal{M})$ as this is symmetric with respect to real axis. There can still be points where ϕ touches the real axis, but then such a point lies in $\mathbf{C}(\mathcal{M})$ with a small neighbourhood of it and hence ϕ can circumvent it without touching $\mathbb{R}$. Finally, we can also assume that ϕ is simple, i.e., it never intersects itself.

Consider now the curve ψ, which consists of the curve ϕ, the counter clock-wise arc or radius 3, connecting the points $3i$ and $-3i$, and the curve $\overline{\phi}$, which is a complex conjugate of ϕ.

Now, we can use e.g. Mařík's Theorem (cf. [C], Theorem 3.3.6, p. 93) to observe that the index of -2 w. r. to ψ is 1, while the index of $1/4$ is 0. This is impossible because these points lie in the same connected set $\mathcal{M}$.

<u>Cardioid</u>

There is an important general principle for the complex dynamics of rational functions (see [B]): if a polynomial f of degree at least two has an attracting cycle, then there must exist a critical point of f whose orbit converges to this cycle. But the mapping $P_c(z) = z^2 + c$ has only two critical points: 0 and ∞, and infinity is a fixed point.

This has an important consequence: if for some $c \in \mathbb{C}$, $P_c(z)$ has an attracting cycle in $\mathbb{C}$, then zero must converge to this cycle and hence $c \in \mathcal{M}$. We will use this principle in following two sections to describe explicitely two parts of $\mathcal{M}$ - the cardioid and the largest bulb adjacent to it.

First we are going to find all those c such that $P_c(z)$ has an attracting fixed point. The equation for fixed point reads:

$$P_c(z) - z = z^2 - z + c = 0;$$

we denote its roots z_1, z_2 and rewrite the equation as

$$(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2$$

to observe (by comparison to previous equation)

$$z_1 + z_2 = 1, \qquad z_1 z_2 = c.$$

Let us fix z_1. Then $z_2 = 1 - z_1$ and $c = z_1(1 - z_2)$. Note that since $P_c'(z) = 2z$, there is a one-to-one correspondence between the fixed point z_1 and its multiplier $2z_1$. We can thus define a map g

$$g(\lambda) := \frac{\lambda}{2}\left(1 - \frac{\lambda}{2}\right)$$

which to every λ assigns a c such that $P_{g(\lambda)}(z)$ has a fixed point with multiplier λ.

Consequently, the set of all parameters c such that 0 converges to a attracting fixed point is simply the domain $g(U(0,1))$. Its boundary is a cardioid, parametrically written as

$$x = \frac{1}{4}(2\cos t - \cos 2t),\ y = \frac{1}{4}(2\sin t - \sin 2t), \quad t \in [0, 2\pi].$$

The largest bulb

Analogously, we are going to find the set of all c such that $P_c(z)$ possesses an attracting 2-cycle. Denote for simplicity $P_c(z) = f(z)$. The equation for a 2-cycle reads

$$f^2(z) - z = z^4 - 2cz^2 - z + c^2 + c = 0.$$

But if any point is fixed for $f(z)$, it must be fixed for $f^2(z)$ too, hence we can factor out $z^2 - z + c$:

$$f(z) - z = (z^2 - z + c)(z^2 + z + c + 1)$$

and the 2-cycle is formed by the roots of the second polynomial on the right. Denoting these z_1, z_2 we have

$$f(z_1) = z_2, \qquad f(z_2) = z_1, \qquad z_1 z_2 = c + 1,$$

and the multiplier of the 2-cycle is

$$[f^2]'(z_1) = f'(f(z_1))f'(z_1) = f'(z_2)f'(z_1) = 4z_2z_1 = 4(c+1),$$

which has modulus smaller than 1 if and only if $c \in U(-1, 1/4)$. So the largest bulb is simply a disc of radius 1/4, centered at -1.

Computers and the Mandelbrot set

Let us make some remarks on how to generate those beautiful pictures of $\mathcal{M}$. The basic algorithm is very simple: You choose some number N - this is the maximal number of iterations. Then, to assign a colour to a given point of complex plane c, you simply let the computer evaluate the orbit of 0 under P_c:

$$0,\ c,\ c^2 + c,\ (c^2 + c)^2 + c, \ldots$$

However, according to our characterization of $\mathcal{M}$, you can stop whenever the absolute value of your iteration gets over 2 - you are definitely outside $\mathcal{M}$, and you can colour the point simply white or by colour depending on the number of iterations you have done. If, however, you make all N iterations without leaving the disc of radius 2, you will assume that you are in $\mathcal{M}$ and colour the point black. The number of iterations you need to get decent global picture of $\mathcal{M}$ is at least 100, but if making more detailed pictures, you will need more iterations.

Our analysis provides us with several hints how to fasten our computations. First of all, do not forget that $\mathcal{M}$ is symmetric with respect to real axis! This might save you 50% of time. Also, in Section 3 we have identified certain parts of $\mathcal{M}$ explicitely - it would be a waste of time to evaluate orbits of points which come from here. Instead, you can a apriori colour in black points that lie in $U(-1, 1/4)$ or inside the cardioid - this corresponds to the condition

$$[2a - \frac{1}{2} + (2a - \frac{1}{2})^2 + (2b)^2]^2 < (2a - \frac{1}{2})^2 + (2b)^2$$

where $c = a + ib$.

References

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