Stochastic processes 2

Part 3

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Contents

Lemma 1:

1. Let μ, ν be finite measures on Borel subsets of the interval $[-\pi, \pi]$. If for every $t \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{it\lambda} d\mu(\lambda) = \int_{-\pi}^{\pi} e^{it\lambda} d\nu(\lambda),$$

then $\mu(B) = \nu(B)$ for every $B \subset (-\pi, \pi)$ and $\mu(\{-\pi\} \cup \{\pi\}) = \nu(\{-\pi\} \cup \{\pi\})$.

2. Let μ, ν be finite measures on $(\mathbb{R}, \mathcal{B})$. If for every $t \in \mathbb{R}$

$$\int_{-\infty}^{\infty} e^{it\lambda} d\mu(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda} d\nu(\lambda),$$

then $\mu(B) = \nu(B)$ for all $B \subset \mathcal{B}$.

Proof: Anděl (1976), III.1, Theorems 5 and 6.

Lemma 2 [Helly theorem]:

Let $\{F_n,\ n \in \mathbb{N}\}$ be a sequence on non-decreasing uniformly bounded functions. Then there exists a subsequence $\{F_{n_k}\}$, that, as $k \to \infty,\ n_k \to \infty$, converges weakly to a non-decreasing right-continuous function F, i.e., on the continuity set of F.

Proof: Rao (1978), Theorem 2c.4, I.

Lemma 3 [Helly-Bray]:

Let $\{F_n,\ n \in \mathbb{N}\}$ be a sequence of non-decreasing uniformly bounded functions that, as $n \to \infty$, converges weakly to a non-decreasing bounded right-continuous function F, and $\lim F_n(-\infty) = F(-\infty), \lim F_n(+\infty) = F(+\infty)$. Let f be a continuous bounded function. Then

$$\int_{-\infty}^{\infty} f(x) dF_n(x) \longrightarrow \int_{-\infty}^{\infty} f(x) dF(x) \text{ as } n \to \infty.$$

Proof: Rao (1978), Theorem 2c.4, II.

Remark

The integral at the Helly-Bray theorem is the Riemann-Stieltjes integral of the function f with respect to the function F. If $[a, b]$ is a bounded interval and F is right-continuous, we will understand that

$$\int_a^b f(x)dF(x) := \int_{(a,b]} f(x)dF(x).$$

Theorem 19:
A complex-valued function $R(t)$, $t \in \mathbb{Z}$, is an autocovariance function of a stationary random sequence if and only if

$$R(t) = \int_{-\pi}^{\pi} e^{it\lambda} dF(\lambda) \text{ for all } t \in \mathbb{Z}, \tag{1}$$

where F is a right-continuous non-decreasing bounded function on $[-\pi, \pi]$, $F(-\pi) = 0$. The function F is determined by formula (1) uniquely.
Formula (1) is called the spectral decomposition (or representation) of the autocovariance function of a stationary random sequence. The function F is called the spectral distribution function of a random sequence.

Proof

1. Suppose that (1) holds for any complex-valued function R on $\mathbb{Z}$. Then R is positive semidefinite since for any $n \in \mathbb{N}$, any constants $c_1, \dots, c_n \in \mathbb{C}$ and all $t_1, \dots, t_n \in \mathbb{Z}$

$$\begin{aligned}\sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c}_k R(t_j - t_k) &= \sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c}_k \int_{-\pi}^{\pi} e^{i(t_j - t_k)\lambda} dF(\lambda) \\ &= \int_{-\pi}^{\pi} \left[\sum_{j=1}^{n}\sum_{k=1}^{n} c_j \overline{c}_k e^{it_j\lambda} e^{-it_k\lambda}\right] dF(\lambda) \\ &= \int_{-\pi}^{\pi} \left|\sum_{j=1}^{n} c_j e^{it_j\lambda}\right|^2 dF(\lambda) \geq 0,\end{aligned}$$

because F is non-decreasing in $[-\pi, \pi]$. It means that R is the autocovariance function of a stationary random sequence.

proof, continued

2. Let R be an autocovariance function of a stationary random sequence; it is positive semidefinite, i.e.,

$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} R(t_j - t_k) \geq 0$ for all $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{C}$ and $t_1, \dots, t_n \in \mathbb{Z}$.

Put $t_j = j, \; c_j = e^{-ij\lambda}$ for a $\lambda \in [-\pi, \pi]$.
Then for every $n \in \mathbb{N}$, $\lambda \in [-\pi, \pi]$,

$$\varphi_n(\lambda) := \frac{1}{2\pi n} \sum_{j=1}^{n} \sum_{k=1}^{n} e^{-i(j-k)\lambda} R(j-k) \geq 0.$$

proof, continued

From here we get

$$\varphi_n(\lambda) = \frac{1}{2\pi n} \sum_{j=1}^{n} \sum_{k=1}^{n} e^{-i(j-k)\lambda} R(j-k)$$

$$= \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} \sum_{j=max(1,\kappa+1)}^{min(n,\kappa+n)} e^{-i\kappa\lambda} R(\kappa)$$

$$= \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} e^{-i\kappa\lambda} R(\kappa)(n-|\kappa|).$$

proof, continued

For any $n \in \mathbb{N}$ let us define function

$$F_n(x) = \begin{cases} 0, & x \le -\pi, \\ \int_{-\pi}^{x} \varphi_n(\lambda) d\lambda, & x \in [-\pi, \pi], \\ F_n(\pi), & x \ge \pi. \end{cases}$$

Obviously, $F_n(-\pi) = 0$ and $F_n(x)$ is non-decreasing on $[-\pi, \pi]$. Compute $F_n(\pi)$:

proof, continued

$$
\begin{aligned}
F_n(\pi) &= \int_{-\pi}^{\pi} \varphi_n(\lambda) d\lambda \\
&= \frac{1}{2\pi n} \int_{-\pi}^{\pi} \left[\sum_{\kappa=-n+1}^{n-1} e^{-i\kappa\lambda} R(\kappa)(n - |\kappa|) \right] d\lambda \\
&= \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} R(\kappa)(n - |\kappa|) \int_{-\pi}^{\pi} e^{-i\kappa\lambda} d\lambda = R(0),
\end{aligned}
$$

since the last integral is $2\pi\delta(\kappa)$.

proof, continued

$\{F_n, \ n \in \mathbb{N}\}$ is a sequence of non-decreasing functions, $0 \leq F_n(x) \leq R(0) < \infty$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

According to the Helly theorem there exists a subsequence $\{F_{n_k}\} \subset \{F_n\}$, $F_{n_k} \to F$ weakly as $k \to \infty, n_k \to \infty$, where F is a nondecreasing bounded right-continuous function and $F(x) = 0, \ x \leq -\pi, \ F(x) = R(0), \ x > \pi$.

From the Helly - Bray theorem for $f(x) = e^{itx}$, where $t \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} e^{it\lambda} dF_{n_k}(\lambda) \longrightarrow \int_{-\pi}^{\pi} e^{it\lambda} dF(\lambda) \quad \text{as } k \to \infty, \ n_k \to \infty.$$

proof, continued

On the other hand,

$$\int_{-\pi}^{\pi} e^{it\lambda} dF_{n_k}(\lambda) = \int_{-\pi}^{\pi} e^{it\lambda} \varphi_{n_k}(\lambda) d\lambda$$

$$= \int_{-\pi}^{\pi} e^{it\lambda} \left[\frac{1}{2\pi n_k} \sum_{\kappa=-n_k+1}^{n_k-1} e^{-i\kappa\lambda} R(\kappa)(n_k - |\kappa|) \right] d\lambda$$

$$= \frac{1}{2\pi n_k} \sum_{\kappa=-n_k+1}^{n_k-1} R(\kappa)(n_k - |\kappa|) \int_{-\pi}^{\pi} e^{i(t-\kappa)\lambda} d\lambda,$$

thus

$$\int_{-\pi}^{\pi} e^{it\lambda} dF_{n_k}(\lambda) = \begin{cases} R(t)\left(1 - \frac{|t|}{n_k}\right), & |t| < n_k \\ 0 & \text{elsewhere.} \end{cases}$$

proof, continued

We get

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} e^{it\lambda} dF_{n_k}(\lambda) = \lim_{k\to\infty} R(t)\left(1 - \frac{|t|}{n_k}\right)$$
$$= R(t) = \int_{-\pi}^{\pi} e^{it\lambda} dF(\lambda)$$

Uniqueness:

Let $R(t) = \int_{-\pi}^{\pi} e^{it\lambda} dG(\lambda)$, where G is a right-continuous non-decreasing bounded function on $[-\pi, \pi]$ and $G(-\pi) = 0$. Then

$$\int_{-\pi}^{\pi} e^{it\lambda} d\mu_F = \int_{-\pi}^{\pi} e^{it\lambda} d\mu_G,$$

where μ_F a μ_G are finite measures on Borel subsets of the interval $[-\pi, \pi]$ induced by functions F a G, respectively. The rest follows from Lemma 1.

Formula (1) is called the spectral decomposition of the autocovariance function of a stationary random sequence. Function F is called the spectral distribution function of a random sequence.

If there exists a function $f(\lambda) \geq 0$ for $\lambda \in [-\pi, \pi]$ such that $F(\lambda) = \int_{-\pi}^{\lambda} f(x)dx$ (F is absolutely continuous), then f is called spectral density. Obviously $f = F'$.

In case that the spectral density exists, the spectral decomposition of an autocovariance function is of the form

$$R(t) = \int_{-\pi}^{\pi} e^{it\lambda} f(\lambda) d\lambda, \quad t \in \mathbb{Z}. \tag{2}$$

Theorem 20:
A complex-valued function $R(t)$, $t \in \mathbb{R}$, is the autocovariance function of a centered stationary mean square continuous process if and only if

$$R(t) = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda), \quad t \in \mathbb{R}, \tag{3}$$

where F is non-decreasing right-continuous function such that $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = R(0) < \infty$. Function F is determined uniquely.
Function F is called the spectral distribution function of a mean square continuous stochastic process.

Proof.
1. Let R be a complex-valued function on $\mathbb{R}$ that satisfies (3), where F is non-decreasing right-continuous function, $F(-\infty) = 0$, $F(+\infty) = R(0) < \infty$. Then R is positive semidefinite, moreover, it is continuous. According to Theorem 6 there exists a stationary centered process with the autocovariance function R. Since R is continuous (hence, continuous at zero), this process is mean square continuous which follows from Theorem 15.

2. Suppose that R is the autocovariance function of a centered stationary mean square continuous process. Then, it is positive semidefinite and continuous at zero. For the proof that R satisfies (3), see, e.g., Anděl (1976), IV.1, Theorem 2. $\square$

If the spectral distribution function in (3) is absolutely continuous, its derivative f is called spectral density and (3) can be written in the form

$$R(t) = \int_{-\infty}^{\infty} e^{it\lambda} f(\lambda) d\lambda, \quad t \in \mathbb{R}. \tag{4}$$

Remark: Two different stochastic processes may have the same spectral distribution functions and thus the same autocovariance functions.

Theorem 21:
Let K be a complex-valued function of an integer-valued variable, let $\sum_{t=-\infty}^{\infty} |K(t)| < \infty$. Then

$$K(t) = \int_{-\pi}^{\pi} e^{it\lambda} f(\lambda) d\lambda, \quad t \in \mathbb{Z},$$

where

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} e^{-it\lambda} K(t), \quad \lambda \in [-\pi, \pi].$$

Proof:

Let us consider function K, such that $\sum_{t=-\infty}^{\infty} |K(t)| < \infty$. Since the series on the right-hand side of

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} e^{-it\lambda} K(t)$$

is absolutely convergent uniformly for $\lambda \in [-\pi, \pi]$, we can interchange the integration and the summation and for any $t \in \mathbb{Z}$ we get

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{it\lambda} f(\lambda) d\lambda &= \int_{-\pi}^{\pi} e^{it\lambda} \left[\frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\lambda} K(k) \right] d\lambda \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left[K(k) \int_{-\pi}^{\pi} e^{i(t-k)\lambda} d\lambda \right] \\
&= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} K(k) 2\pi \delta(t-k) = K(t).
\end{aligned}$$

Theorem 22:
Let $\{X_t,\ t \in \mathbb{Z}\}$ be a stationary sequence such that its autocovariance function R is absolutely summable, i.e. $\sum_{t=-\infty}^{\infty} |R(t)| < \infty$. Then the spectral density of the sequence $\{X_t,\ t \in \mathbb{Z}\}$ exists and for every $\lambda \in [-\pi, \pi]$

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\lambda} R(k). \tag{5}$$

Proof:
$\sum_{t=-\infty}^{\infty} |R(t)| < \infty \Rightarrow$

$$R(t) = \int_{-\pi}^{\pi} e^{it\lambda} f(\lambda) d\lambda, \quad t \in \mathbb{Z},$$

$$f(\lambda) = \frac{1}{2\pi} \sum_{t=-\infty}^{\infty} e^{-it\lambda} R(t), \quad \lambda \in [-\pi, \pi]$$

(from the previous theorem). Due to the uniqueness of the spectral decomposition (2) it suffices to prove, that $f(\lambda) \geq 0$ for every $\lambda \in [-\pi, \pi]$.
For every $\lambda \in [-\pi, \pi]$,

$$\varphi_n(\lambda) = \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} e^{-i\kappa\lambda} R(\kappa)(n - |\kappa|) \geq 0$$

(it follows from the proof of Theorem 19). We show that $f(\lambda) = \lim_{n\to\infty} \varphi_n(\lambda)$.

Proof, continued

We have

$$
\begin{aligned}
|f(\lambda) - \varphi_n(\lambda)| &\leq \left| \frac{1}{2\pi} \sum_{|k| \geq n} e^{-ik\lambda} R(k) \right| \\
&+ \left| \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} e^{-i\kappa\lambda} R(\kappa) |\kappa| \right| \\
&\leq \frac{1}{2\pi} \sum_{|k| \geq n} |R(k)| + \frac{1}{2\pi n} \sum_{\kappa=-n+1}^{n-1} |R(\kappa)| |\kappa| \longrightarrow 0
\end{aligned}
$$

(Kronecker lemma: $\sum_{k=1}^{\infty} a_k < \infty \Rightarrow \frac{1}{n} \sum_{k=1}^{n} k\, a_k \to 0$ pro $n \to \infty$). $\square$

Formula (5) is called inverse formula for computing the spectral density of a stationary random sequence.

Theorem 23:
Let $\{X_t,\ t \in \mathbb{R}\}$ be a centered weakly stationary mean square process. If its autocovariance function R satisfies condition $\int_{-\infty}^{\infty} |R(t)| dt < \infty$ then there exists the spectral density of the process and it holds

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-it\lambda} R(t) dt, \quad \lambda \in (-\infty, \infty). \tag{6}$$

The proof is quite analogous to computation of a probability density function by means of a characteristic function (Fourier transformation)

Example (white noise):
Let $\{X_t,\ t \in \mathbb{Z}\}$ be a sequence of uncorrelated random variables with zero mean and a finite positive variance σ^2:
$\mathrm{E}X_t = 0, \mathrm{var}X_t = \sigma^2, \mathrm{cov}(X_s, X_t) = \sigma^2\delta(s-t) = R(s-t)$.

$\sum_{t=-\infty}^{\infty} |R(t)| = \sigma^2 < \infty \Rightarrow$ spectral density exists
According to the inverse formula

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\lambda} R(k) = \frac{1}{2\pi} R(0) = \frac{\sigma^2}{2\pi}, \quad \lambda \in [-\pi, \pi].$$

spectral distribution function of the white noise

$$
\begin{aligned}
F(\lambda) &= 0, && \lambda \le -\pi, \\
&= \frac{\sigma^2}{2\pi}(\lambda + \pi), && \lambda \in [-\pi, \pi], \\
&= \sigma^2, && \lambda \ge \pi.
\end{aligned}
$$

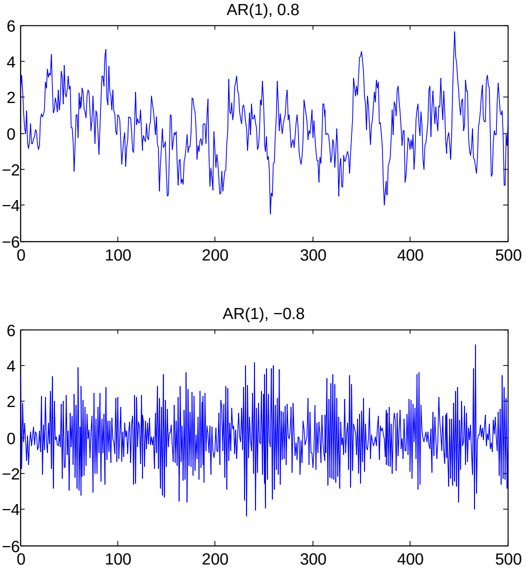
Notation: $WN(0, \sigma^2)$ (white noise)

Example :

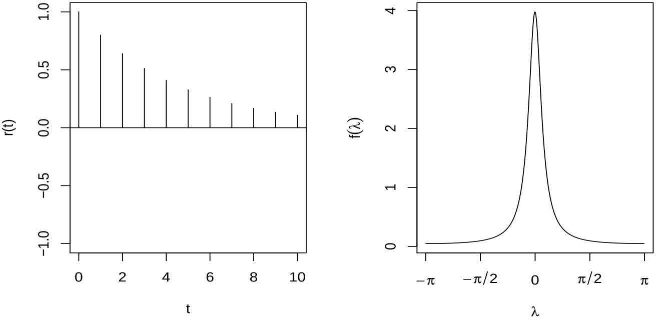
Consider a stationary sequence with the autocovariance function $R(t) = a^{|t|}$, $t \in \mathbb{Z}$, $|a| < 1$.

$$\sum_{t=-\infty}^{\infty} |R(t)| = \sum_{t=-\infty}^{\infty} |a|^{|t|} = 1 + 2\sum_{t=1}^{\infty} |a|^t < \infty,$$

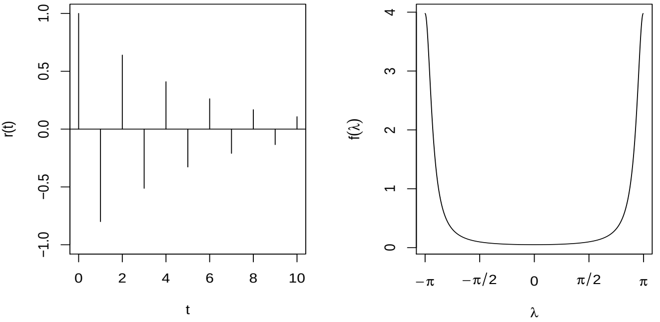
$$\begin{aligned}
f(\lambda) &= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-ik\lambda} a^{|k|} \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} e^{-ik\lambda} a^k + \frac{1}{2\pi} \sum_{k=-\infty}^{-1} e^{-ik\lambda} a^{-k} \\
&= \frac{1}{2\pi} \sum_{k=0}^{\infty} \left(ae^{-i\lambda}\right)^k + \frac{1}{2\pi} \sum_{k=1}^{\infty} \left(ae^{i\lambda}\right)^k \\
&= \frac{1}{2\pi} \frac{1}{1 - ae^{-i\lambda}} + \frac{1}{2\pi} \frac{ae^{i\lambda}}{1 - ae^{i\lambda}} \\
&= \frac{1}{2\pi} \frac{1 - a^2}{|1 - ae^{-i\lambda}|^2} = \frac{1}{2\pi} \frac{1 - a^2}{1 - 2a\cos\lambda + a^2}
\end{aligned}$$



trajectories of a process with the a autocovariance function $R(t) = a^{|t|}$, up: $a = 0,8$, down $a = -0,8$



autocovariance function (left) and spectral density (right), $a = 0,8$



autocovariance function (left) and spectral density (right), $a = -0,8$

Example :

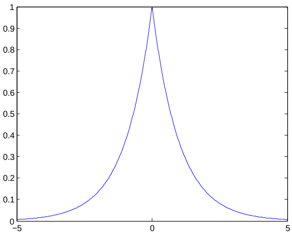
Centered weakly stationary process with the autocovariance function $R(t) = ce^{-\alpha|t|}$, $t \in \mathbb{R}$, $c > 0, \alpha > 0$.
The process is mean square continuous.
It holds

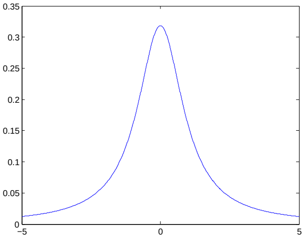
$$\int_{-\infty}^{\infty} |R(t)|dt = \int_{-\infty}^{\infty} ce^{-\alpha|t|}dt < \infty,$$

thus, the spectral density exists and by formula (6)

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\lambda}R(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\lambda}ce^{-\alpha|t|}dt \\ &= \frac{c}{2\pi}\int_{-\infty}^{\infty}(\cos\lambda t - i\sin\lambda t)e^{-\alpha|t|}dt \\ &= \frac{c}{\pi}\int_{0}^{\infty}\cos(\lambda t)e^{-\alpha t}dt = \frac{c\alpha}{\pi}\frac{1}{\alpha^2+\lambda^2} \end{aligned}$$

for every $\lambda \in \mathbb{R}$.





autocovariance function (left) and spectral density (right), $c = 1, \alpha = 1$

Example :

A centered mean square process with the spectral distribution function process

$$\begin{aligned} F(\lambda) &= 0, & \lambda < -1, \\ &= \frac{1}{2}, & -1 \le \lambda < 1, \\ &= 1, & \lambda \ge 1. \end{aligned}$$

Spectral distribution function is not absolutely continuous; the spectral density of the process does not exist. According to (3) the autocovariance function is

$$R(t) = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda) = \frac{1}{2} e^{-it} + \frac{1}{2} e^{it} = \cos t, \quad t \in \mathbb{R}.$$

The process has a discrete spectrum with non-zero value at frequencies $\lambda_1 = -1, \lambda_2 = 1$.

Example :

The process $\{X_t,\ t \in \mathbb{R}\}$ of uncorrelated random variables with zero mean and a finite positive variance does not satisfy decomposition (3), since it is not mean square continuous.