Stochastic Processes 2

Definition of the stochastic process
Daniell-Kolmogorov theorem
Autocovariance and autocorrelation function
Strict and weak stationarity
Properties of autocovariance function

Some important classes of stochastic processes
Markov processes
Independent increments processes
Martingales

October 12, 2015
1 Literature

2 Definitions and basic characteristics
   Definition of the stochastic process
   Daniell-Kolmogorov theorem
   Autocovariance and autocorrelation function
   Strict and weak stationarity
   Properties of autocovariance function

3 Some important classes of stochastic processes
   Markov processes
   Independent increments processes
   Martingales
Basic study text (In Czech):
Z. Prášková: Základy náhodných procesů II, Karolinum 2006

Supplementary texts:

Lecture Notes (website)
Definitions and basic characteristics
Definition:
Let \((\Omega, \mathcal{A}, P)\) be a probability space, \((S, \mathcal{E})\) a measurable space, and \(T \subset \mathbb{R}\). A family of random variables \(\{X_t, t \in T\}\) defined on \((\Omega, \mathcal{A}, P)\) with values in \(S\) is called a stochastic (random) process. If \(S = \mathbb{R}\), \(\{X_t, t \in T\}\) is called a real stochastic process.

Remark:
\(T = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}\) or \(T \subset \mathbb{Z}\) - \(\{X_t, t \in T\}\) is discrete time stochastic process, time series
\(T = [a, b], -\infty \leq a < b \leq \infty\) - \(\{X_t, t \in T\}\) is continuous time process.

Definition:
A pair \((S, \mathcal{E})\), where \(S\) is a set of values of random variables \(X_t\) and \(\mathcal{E}\) is a \(\sigma\)–algebra of subsets of \(S\), is called the state space of the process \(\{X_t, t \in T\}\).
Definition:
A real stochastic process \( \{X_t, t \in T\} \) is said to be measurable, if the mapping \( (\omega, t) \to X_t(\omega) \) is \( A \otimes B_T \)–measurable, where \( B_T \) is a \( \sigma \)–algebra of Borel subsets of \( T \) and \( A \otimes B_T \) is a product \( \sigma \)–algebra.

Finite-dimensional distributions:
\( \forall n \in \mathbb{N}_0 \) and any finite subset \( \{t_1, \ldots, t_n\} \subset T \) there is a system of random variables \( X_{t_1}, \ldots, X_{t_n} \), with the joint distribution function
\[
P \left[ X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n \right] = F_{t_1, \ldots, t_n}(x_1, \ldots, x_n)
\]
for all \( x_1, \ldots, x_n \).

A system of distribution functions is said to be consistent, if
- \( F_{t_1, \ldots, t_n}(x_{i_1}, \ldots, x_{i_n}) = F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \) for any permutation \( (i_1, \ldots, i_n) \) of \( (1, \ldots, n) \) (symmetry)
- \( \lim_{x_n \to \infty} F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = F_{t_1, \ldots, t_n-1}(x_1, \ldots, x_{n-1}) \) (consistency)
A system of characteristic functions
The characteristic function of a random vector $X = (X_1, \ldots, X_n)$ is

$$\varphi(u) := \mathbb{E}e^{iu^\top X} = \mathbb{E}e^{i\sum_{j=1}^{n} u_jX_j}, \quad u = (u_1, \ldots, u_n).$$

Consistent system of characteristic functions:

- $F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) \leftrightarrow \varphi(u_1, \ldots, u_n),$
- symmetry:
  $$\varphi(u_{i_1}, \ldots, u_{i_n}) = \varphi(u_1, \ldots, u_n)$$
  for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n),$
- consistency:
  $$\lim_{u_n \to 0} \varphi_{X_{t_1}, \ldots, X_{t_n}}(u_1, \ldots, u_n) = \varphi_{X_{t_1}, \ldots, X_{t_{n-1}}}(u_1, \ldots, u_{n-1}).$$
Daniell-Kolmogorov theorem

For any stochastic process there exists a consistent system of distribution functions and,

**Theorem 1:**
Let \( \{F_{t_1,\ldots,t_n}(x_1,\ldots,x_n)\} \) be a consistent system of distribution functions. Then there exists a stochastic process \( \{X_t, t \in T\} \) such that for any \( n \in \mathbb{N} \), any \( t_1,\ldots,t_n \in T \) and any real \( x_1,\ldots,x_n \) it holds

\[
P \left[ X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n \right] = F_{t_1,\ldots,t_n}(x_1,\ldots,x_n).
\]

**Definition:**
A complex-valued random variable $X$ is defined by $X = Y + iZ$, where $Y$ and $Z$ are real random variables, $i = \sqrt{-1}$.

The mean value of a complex-valued random variable $X = Y + iZ$ is defined by

$$EX = EY + iEZ$$

provided the mean values $EY$ and $EZ$ exist.

The variance of a complex-valued random variable $X = Y + iZ$ is defined by

$$\text{var} X := E[(X - EX)(\overline{X} - \overline{EX})] = E|X - EX|^2 \geq 0$$

provided the second moments of random variables $Y$ and $Z$ exist.

**Definition:**
A complex-valued stochastic process is a family of complex-valued random variables on $(\Omega, \mathcal{A}, P)$. 
Definition:
Let \( \{X_t, t \in T\} \) be a stochastic process such that \( \mathbb{E}X_t \) exists for all \( t \in T \). Then the function \( \mu_t = \mathbb{E}X_t \) defined on \( T \) is called the **mean value** of the process \( \{X_t, t \in T\} \). We say that the process is **centered** if its mean value is zero for all \( t \in T \).

Definition:
Let \( \{X_t, t \in T\} \) be a process with finite second moments, i.e., \( \mathbb{E}|X_t|^2 < \infty, \forall t \in T \). Then a (complex-valued) function defined on \( T \times T \) by

\[
R(s, t) = \mathbb{E} \left[ (X_s - \mu_s)(X_t - \mu_t) \right]
\]

is called the **autocovariance function** of the process \( \{X_t, t \in T\} \). The value \( R(t, t) \) is the **variance** of the process at time \( t \).
Definition:

Autocorrelation function of the process \( \{X_t, t \in T\} \) with positive variances is defined by

\[
r(s, t) = \frac{R(s, t)}{\sqrt{R(s, s)} \sqrt{R(t, t)}}, \quad s, t \in T.
\]

Definition:

Stochastic process \( \{X_t, t \in T\} \) is called **Gaussian**, if for any \( n \in \mathbb{N} \) and \( t_1, \ldots, t_n \in T \), the vector \( (X_{t_1}, \ldots, X_{t_n})^\top \) is normally distributed \( \mathcal{N}_n(\mathbf{m}_t, \mathbf{V}_t) \), where \( \mathbf{m}_t = (\mathbb{E}X_{t_1}, \ldots, \mathbb{E}X_{t_n})^\top \) and

\[
\mathbf{V}_t = \begin{pmatrix}
\text{var}X_{t_1} & \text{cov}(X_{t_1}, X_{t_2}) & \cdots & \text{cov}(X_{t_1}, X_{t_n}) \\
\text{cov}(X_{t_2}, X_{t_1}) & \text{var}X_{t_2} & \cdots & \text{cov}(X_{t_2}, X_{t_n}) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}(X_{t_n}, X_{t_1}) & \text{cov}(X_{t_n}, X_{t_2}) & \cdots & \text{var}X_{t_n}
\end{pmatrix}.
\]
**Definition:**
Stochastic process \(\{X_t, \ t \in T\}\) is said to be **strictly stationary**, if for any \(n \in \mathbb{N}\), for any \(x_1, \ldots, x_n\) real and for any \(t_1, \ldots, t_n\) and \(h\) such that \(t_k \in T, t_k + h \in T, 1 \leq k \leq n\),

\[
F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = F_{t_1+h,\ldots,t_n+h}(x_1, \ldots, x_n).
\]

**Definition:**
Stochastic process \(\{X_t, \ t \in T\}\) with finite second moments is said to be **weakly stationary** or **second order stationary**, if its mean value is constant, \(\mu_t = \mu, \ \forall t \in T\) and if its autocovariance function \(R(s, t)\) is a function of \(s - t\), only. If only the latter condition is satisfied, the process is called **covariance stationary**.
Autocovariance function of weakly stationary process:

\[ R(t) := R(t, 0), \ t \in T, \]

(function of one variable).

Autocorrelation function in such case:

\[ r(t) = \frac{R(t)}{R(0)}. \]

**Theorem 2:**

Strictly stationary stochastic process \( \{X_t, \ t \in T\} \) with finite second moments is also weakly stationary.

**Proof:**

\( \{X_t, \ t \in T\} \) strictly stationary \( \Rightarrow \) \( X_t \) are equally distributed for all \( t \in T \) and thus with the mean value

\[ \mathbb{E}X_t = \mathbb{E}X_{t+h}, \ \forall t \in T, \ \forall h : t + h \in T \]

especially, for \( h = -t \): \( \mathbb{E}X_t = \mathbb{E}X_0 = \text{const} \)
Proof of Theorem 2, continued

Similarly, \((X_t, X_s)\) are equally distributed and

\[
E[X_t X_s] = E[X_{t+h} X_{s+h}] \quad \forall s, t \in T, \forall h : s + h \in T, \; t + h \in T
\]

evenly, for \(h = -t\) : \(E[X_t X_s] = E[X_0 X_{s-t}]\)

is a function of \(s - t\).  \qed
Example:

\( \{X_t, \ t \in T\} \) - a sequence of iid random variables with a distribution function \( F \)

\[
F_{t_1, \ldots, t_n}(x_1, \ldots, x_n) = P[X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n] = \prod_{i=1}^{n} P[X_{t_i} \leq x_i] = \prod_{i=1}^{n} F(x_i),
\]

\[
F_{t_1+h, \ldots, t_n+h}(x_1, \ldots, x_n) = P[X_{t_1+h} \leq x_1, \ldots, X_{t_n+h} \leq x_n] = \prod_{i=1}^{n} P[X_{t_i+h} \leq x_i] = \prod_{i=1}^{n} F(x_i),
\]

\( \Rightarrow \{X_t, \ t \in T\} \) is strictly stationary.
Example:

\{X_t, t \in \mathbb{Z}\} - a sequence defined by

\[ X_t = (-1)^t X, \]

where \( X \) is a random variable:

\[ X = \begin{cases} 
-\frac{1}{4} & \text{with probability } \frac{3}{4}, \\
\frac{3}{4} & \text{with probability } \frac{1}{4}.
\end{cases} \]

Then \( \{X_t, t \in \mathbb{Z}\} \) is weakly stationary, since

\[ \mathbb{E}X_t = 0, \]

\[ \text{var } X_t = \sigma^2 = \frac{3}{16}, \]

\[ R(s, t) = \sigma^2(-1)^{s+t} = \sigma^2(-1)^{s-t}, \]

but it is not strictly stationary (variables \( X \) a \( -X \) are not equally distributed).
**Theorem 3:**
A weakly stationary Gaussian process \( \{X_t, \ t \in T\} \) is also strictly stationary.

**Proof:**
Weak stationarity of the process \( \{X_t, \ t \in T\} \) implies
\[
E X_t = \mu, \ \text{cov} (X_t, X_s) = R(t - s) = \text{cov} (X_{t+h}, X_{s+h}), \ t, s \in T,
\]
thus
\[
E(X_{t_1}, \ldots, X_{t_n}) = E(X_{t_1+h}, \ldots, X_{t_n+h}) = (\mu, \ldots, \mu) := \mu
\]

\[
\text{var} (X_{t_1}, \ldots, X_{t_n}) = \text{var} (X_{t_1+h}, \ldots, X_{t_n+h}) := \Sigma
\]
\[ \Sigma = \begin{pmatrix}
R(0) & R(t_2 - t_1) & \ldots & R(t_n - t_1) \\
R(t_2 - t_1) & R(0) & \ldots & R(t_n - t_2) \\
\vdots & \vdots & \ddots & \vdots \\
\ldots & \ldots & R(0) & 
\end{pmatrix}. \]

Since the normal distribution is uniquely defined by the mean value vector and the variance matrix, 
\((X_{t_1}, \ldots, X_{t_n}) \sim \mathcal{N}(\mu, \Sigma), \text{ and} \)
\((X_{t_1+h}, \ldots, X_{t_n+h}) \sim \mathcal{N}(\mu, \Sigma) \Rightarrow \{X_t, \ t \in T\} \text{ is strictly stationary.} \]
Properties of autocovariance function

**Theorem 4:**
Let \( \{X_t, t \in T\} \) be a process with finite second moments. Then its autocovariance function satisfies

\[
R(t, t) \geq 0, \\
|R(s, t)| \leq \sqrt{R(s, s)} \sqrt{R(t, t)}.
\]

**Proof:**
The first assertion follows from the definition of the variance. The second one follows from the Schwarz inequality, since

\[
|R(s, t)| = |E(X_s - EX_s)(\bar{X}_t - E\bar{X}_t)| \leq E|X_s - EX_s|(|\bar{X}_t - E\bar{X}_t|) \\
\leq (E|X_s - EX_s|^2)^{1/2}(E|X_t - EX_t|^2)^{1/2} = \sqrt{R(s, s)} \sqrt{R(t, t)}.
\]

Thus, for weakly stationary process \( R(0) \geq 0 \) a \( |R(t)| \leq R(0) \).
Definition:
Let \( f(s, t) \) be a complex-valued function defined on \( T \times T \), \( T \subset \mathbb{R} \). We say that \( f \) is positive semidefinite, if \( \forall n \in \mathbb{N} \), any complex numbers \( c_1, \ldots, c_n \) and any \( t_1, \ldots, t_n \in T \) it holds

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} f(t_j, t_k) \geq 0.
\]

We say that a complex-valued function \( g \) on \( T \) is positive semidefinite, if \( \forall n \in \mathbb{N} \), any complex numbers \( c_1, \ldots, c_n \) and any \( t_1, \ldots, t_n \in T \), such that \( t_j - t_k \in T \), it holds

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} g(t_j - t_k) \geq 0.
\]
**Definition:**
We say that a complex-valued function $f$ on $T \times T$ is **Hermitian**, if $f(s, t) = \overline{f(t, s)} \ \forall s, t \in T$. A complex-valued function $g$ of one variable is called **Hermitian**, if $g(-t) = \overline{g(t)} \ \forall t \in T$.

**Theorem 5:**
Any positive semidefinite function is also Hermitian.

Proof:
Use the definition of positive semidefiniteness and for $n = 1$ choose $c_1 = 1$; for $n = 2$ choose $c_1 = 1, c_2 = 1$ and $c_1 = 1, c_2 = i(= \sqrt{-1})$. □

**Remark:**
A positive semidefinite real-valued function $f$ on $T \times T$, is symmetric, i.e., $f(s, t) = f(t, s)$ for all $s, t \in T$. A positive semidefinite real-valued function $g$ on $T$ is symmetric, i.e, $g(t) = g(-t)$ for all $t \in T$. 
Theorem 6:
Let \( \{X_t, t \in T\} \) be a process with finite second moments. Then its autocovariance function is positive semidefinite on \( T \times T \).

Proof:
Suppose wlog that the process is centered. Then for any \( n \in \mathbb{N} \), complex constants \( c_1, \ldots, c_n \) and \( t_1, \ldots, t_n \in T \)

\[
0 \leq \mathbb{E} \left| \sum_{j=1}^{n} c_j X_{t_j} \right|^2 = \mathbb{E} \left[ \sum_{j=1}^{n} c_j X_{t_j} \sum_{k=1}^{n} c_k X_{t_k} \right]
\]

\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} \mathbb{E}(X_{t_j} X_{t_k}) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} R(t_j, t_k).
\]
**Theorem 7:**

To any positive semidefinite function $R$ on $T \times T$ there exists a stochastic process $\{X_t, t \in T\}$ with finite second moments such that its autocovariance function is $R$.

**Proof:**

The proof will be given for real-valued function $R$, only. For the proof in a complex case see, e.g., Loève (1955), Chap. X, Par. 34.

Since $R$ is positive semidefinite, then for any $n \in \mathbb{N}$ and any real $t_1, \ldots, t_n \in T$ the matrix

$$V_t = \begin{pmatrix}
R(t_1, t_1) & R(t_1, t_2) & \cdots & R(t_1, t_n) \\
R(t_2, t_1) & R(t_2, t_2) & \cdots & R(t_2, t_n) \\
\vdots & \vdots & \ddots & \vdots \\
R(t_n, t_1) & R(t_n, t_2) & \cdots & R(t_n, t_n)
\end{pmatrix}$$

is positive semidefinite.
Proof of Theorem 7, continued

Function

\[ \varphi(u) = \exp \left\{ -\frac{1}{2} u^\top V_t u \right\}, \quad u \in \mathbb{R}^n \]

is the characteristic function of the normal distribution \( \mathcal{N}_n(0, V_t) \). In this way, \( \forall n \in \mathbb{N} \) and any real \( t_1, \ldots, t_n \in T \) we get a consistent system of characteristic functions. The corresponding system of the distribution functions is also consistent. Thus according to Daniell-Kolmogorov theorem (Theorem 1) there exists a Gaussian stochastic process, covariances of which are given the values of the function \( R(s, t) \); hence, function \( R \) is the autocovariance function of this process.
Example:
Decide whether function $\cos t, \ t \in T = (-\infty, \infty)$ is an autocovariance function of a stochastic process.

Solution:
It suffices to show, that $\cos t$ is a positive semidefinite function. Consider $n \in \mathbb{N}, c_1, \ldots, c_n \in \mathbb{C}$ a $t_1, \ldots, t_n \in \mathbb{R}$. Then we have

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k \cos(t_j - t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c}_k (\cos t_j \cos t_k + \sin t_j \sin t_k)$$

$$= \left| \sum_{j=1}^{n} c_j \cos t_j \right|^2 + \left| \sum_{k=1}^{n} c_k \sin t_k \right|^2 \geq 0.$$

Function $\cos t$ is positive semidefinite, and according to Theorem 6 there exists a (Gaussian) stochastic process $\{X_t, t \in T\}$, autocovariance function of which is $R(s, t) = \cos(s - t)$. 
Theorem 8:
The sum of two positive semidefinite functions is a positive semidefinite function.

Proof:
It follows from the definition of the positive semidefinite function. If $f$ and $g$ are positive semidefinite and $h = f + g$, then for any $n \in \mathbb{N}$, complex $c_1, \ldots, c_n$ and $t_1, \ldots, t_n \in T$

$$
\sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k h(t_j, t_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k [f(t_j, t_k) + g(t_j, t_k)]
$$

$$
= \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k f(t_j, t_k) + \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \bar{c}_k g(t_j, t_k) \geq 0.
$$
**Corollary:**
Sum of two autocovariance functions is an autocovariance function of a stochastic process with finite second moments.

**Proof:**
It follows from Theorems 6 - 8. □

**Theorem 9:**
The real part of an autocovariance function is an autocovariance function. The imaginary part is an autocovariance function if and only if it is zero.

**Proof:**
Wlog, we prove the assertion for centered processes only. If $X_t = Y_t + iZ_t$ is complex with zero mean, then $EY_t = EZ_t = 0$ and $R(s, t) = EX_sX_t = E[(Y_s + iZ_s)(Y_t - iZ_t)] = EY_sY_t + EZ_sZ_t + i(EZ_sY_t - EY_sZ_t)$. The real part is an autocovariance function according to the previous Corollary. Since for $s = t$ the imaginary part is zero, the second assertion holds as well. □
Some important classes of stochastic processes
Markov processes

Definition:
We say that \( \{X_t, t \in T\} \) is a Markov process with state space \((S, \mathcal{E})\), if for any \( t_0, t_1, \ldots, t_n, 0 \leq t_0 < t_1 < \cdots < t_n \), it holds

\[
P(X_{t_n} \leq x | X_{t_{n-1}}, \ldots, X_{t_0}) = P(X_{t_n} \leq x | X_{t_{n-1}}) \text{ a.s.} \quad (1)
\]

for all \( x \in \mathbb{R} \).

The relation (1) is called Markovian property. Simple cases: discrete state Markov processes, i.e., discrete and continuous time Markov chains.
Example:
Markov chain \( \{ X_t, \ t \geq 0 \} \) with state space \( S = \{ 0, 1 \} \), initial distribution \( P(X_0 = 0) = 1, P(X_0 = 1) = 0 \) and the intensity matrix
\[
Q = \begin{pmatrix}
-\alpha & \alpha \\
\beta & -\beta
\end{pmatrix}, \quad \alpha > 0, \ \beta > 0
\]
Let us treat the stationarity of this process.
We know:
\[
p(t)^T = p(0)^T P(t) = (1, 0)^T P(t) = (p_{00}(t), p_{01}(t))^T
\]
\[
P(t) = \frac{1}{\alpha + \beta}
\begin{pmatrix}
\beta + \alpha e^{-(\alpha+\beta)t} & \alpha - \alpha e^{-(\alpha+\beta)t} \\
\beta - \beta e^{-(\alpha+\beta)t} & \alpha + \beta e^{-(\alpha+\beta)t}
\end{pmatrix}.
\]
Example, continued.

Due to the initial distribution we have

$$EX_t = P(X_t = 1) = p_{01}(t) = \frac{1}{\alpha + \beta} \cdot \left(\alpha - \alpha e^{-(\alpha + \beta)t}\right),$$

which depends on $t$, the process is neither strictly nor weakly stationary.

One the other hand, if the initial distribution is the stationary distribution of the Markov chain, then $\{X_t, t \geq 0\}$ is a strictly stationary process with the mean $EX_t = \frac{\alpha}{\alpha + \beta}$ and the autocovariance function

$$R(s, t) = \frac{\alpha \beta}{(\alpha + \beta)^2} e^{-(\alpha + \beta)|s-t|}.$$  

It means it is also weakly stationary.
Example, continued.

Proof:
stationary distribution - solution of $\pi^T = \pi^T \mathbf{P}(t)$

$\pi_0 = \frac{\beta}{\alpha + \beta}, \pi_1 = \frac{\alpha}{\alpha + \beta}$

stationary distribution implies:
$p(t) = \pi \Rightarrow E X_t = P(X_t = 1) = \frac{\alpha}{\alpha + \beta}$

for $t < s$,

$$E(X_t X_s) = 1 \cdot P(X_t = 1, X_s = 1)$$

$$= P(X_s = 1 | X_t = 1)P(X_t = 1)$$

$$= p_{11}(s - t)P(X_t = 1)$$
Independent increments processes

**Definition:**
A process \( \{X_t, t \in T\} \), where \( T \) is an interval, has **independent increments**, if for any \( t_1, t_2, \ldots, t_n \in T \) such that \( t_1 < t_2 < \cdots < t_n \), the random variables \( X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \) are independent.

If for any \( s, t \in T, s < t \), the distribution of the increments \( X_t - X_s \) depends only on \( t - s \), we say that \( \{X_t, t \in T\} \) has **stationary increments**.

**Example:**
*Poisson process* with intensity \( \lambda \) is a continuous time Markov chain \( \{X_t, t \geq 0\} \) such that \( X_0 = 0 \) a.s. and for \( t > 0 \), \( X_t \) has the Poisson distribution with parameter \( \lambda t \). Increments \( X_t - X_s, s < t \) have the Poisson distribution with the parameter \( \lambda(t - s) \). Poisson process is neither strictly nor weakly stationary.
Wiener process (Brownian motion process) is a Gaussian stochastic process \( \{ W_t, t \geq 0 \} \) with the properties

1. \( W_0 = 0 \) a.s. and \( \{ W_t, t \geq 0 \} \) has continuous trajectories

2. For any \( 0 \leq t_1 < t_2 < \cdots < t_n \),
   \[ W_{t_1}, W_{t_2} - W_{t_1}, W_{t_3} - W_{t_2}, \ldots, W_{t_n} - W_{t_{n-1}} \]
   are independent random variables (independent increments).

3. For any \( 0 \leq t < s \), the increments \( W_s - W_t \) have normal distribution with zero mean and the variance \( \sigma^2(s - t) \), where \( \sigma^2 \) is a positive constant. Especially, for any \( t \geq 0 \),
   \( \mathbb{E}W_t = 0 \) and \( \text{var} W_t = \sigma^2t \).

The Wiener process is Markovian but it is neither strictly nor weakly stationary.
Martingales

Definition:
Let \( \{\Omega, \mathcal{A}, P\} \) be a probability space, \( T \subset \mathbb{R}, T \neq \emptyset \). Let for any \( t \in T \), \( \mathcal{F}_t \subset \mathcal{A} \) be a \( \sigma \)-algebra (\( \sigma \)-field). The system of \( \sigma \)-fields \( \{\mathcal{F}_t, t \in T\} \) such that \( \mathcal{F}_s \subset \mathcal{F}_t \) for any \( s, t \in T, s < t \) is called a filtration.

Definition:
Let \( \{X_t, t \in T\} \) be a stochastic process defined on \( \{\Omega, \mathcal{A}, P\} \), and let \( \{\mathcal{F}_t, t \in T\} \) be a filtration. We say that \( \{X_t, t \in T\} \) is adapted to \( \{\mathcal{F}_t, t \in T\} \) if for any \( t \in T \), \( X_t \) is \( \mathcal{F}_t \) measurable.

Definition:
Let \( \{X_t, t \in T\} \) be adapted to \( \{\mathcal{F}_t, t \in T\} \) and \( E|X_t| < \infty \) for all \( t \in T \). Then \( \{X_t, t \in T\} \) is said to be a martingale if \( EX_t|\mathcal{F}_s = X_s \) a.s. for any \( s < t, s, t, \in T \).