Navier–Stokes Equations

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Contents

1 Introduction **3**
- 1.1 The equations . . . 3
- 1.2 Historical remarks . . . 7
- 1.3 Weaker notion of a solution . . . 8

2 Basic function spaces **10**
- 2.1 Lebesgue and Sobolev spaces . . . 10
- 2.2 Bochner spaces . . . 12
  - 2.2.1 Spaces $L^p(I;X)$ . . . 13
  - 2.2.2 Spaces with time derivative . . . 14
- 2.3 Spaces with zero divergence . . . 21
  - 2.3.1 Temam spaces . . . 21
  - 2.3.2 Sobolev spaces with zero divergence . . . 23
  - 2.3.3 Decomposition of $(L^2(\Omega))^N$. Existence of the pressure. . . 24
- 2.4 Stokes problem . . . 28

3 Weak solution to evolutionary system **32**
- 3.1 Existence of a weak solution . . . 32
- 3.2 Reconstruction of the pressure . . . 42
- 3.3 Regularity ($N = 2$) . . . 49
- 3.4 Uniqueness ($N = 3$) . . . 52
- 3.5 Global-in-time conditional regularity ($N = 3$) . . . 55
- 3.6 Local-in-time regularity ($N = 3$) . . . 57

4 Appendix **60**
- 4.1 Integral operators . . . 60
- 4.2 Bogovskii operator in bounded domains . . . 61
  - 4.2.1 Homogeneous boundary condition . . . 61
  - 4.2.2 Inhomogeneous boundary condition . . . 67
- 4.3 Unbounded domains . . . 70
  - 4.3.1 Whole space . . . 70
  - 4.3.2 Exterior domains . . . 70
  - 4.3.3 Domains with noncompact boundaries . . . 71
  - 4.3.4 Applications . . . 72

Chapter 1

Introduction

We meet the fluids basically everywhere. The water forms 80% of the human body, without drinking we would die in three days. However, how much do we know about the fluids?

We skip all very important facts coming from chemistry, physics and other sciences and we immediately start with the mathematical description. It appears that we know very little about the almost simplest model of a viscous fluid....

1.1 The equations

We assume that the fluid occupies at the time instant t the region $\Omega(t) \subset \mathbf{R}^N$. Typically, we take $N = 2, 3$, as these are the most interesting cases. We will work with the following quantities

- density "$\rho(t,x) = \lim_{r\to 0} \frac{M(t,B_r(x))}{|B_r(x)|}$" with $M(t,B)$ denoting the mass of the fluid contained at the time instant t in the ball B
- velocity field $\mathbf{u}(t,x)$

We assume that we can specify these two quantities at each time instant t at any point $x \in \Omega(t)$. For more details about the assumptions used in continuum mechanics and more detailed explanation, see, e.g., [16].

Let $B \subset \mathbf{R}^N$ be a fixed domain such that $B \subset \Omega(t)$. Then the mass of the fluid remains conserved, i.e., the change of the mass of the fluid contained in B is either due to the changes of the density or due to the convection through the boundary, i.e.,

$$\frac{\mathrm{d}}{\mathrm{dt}} M(t,B) = \frac{\mathrm{d}}{\mathrm{dt}} \int_B \rho(t,\cdot)\,\mathrm{d}x = -\int_{\partial B} \rho(t,\cdot)\mathbf{u}(t,\cdot)\cdot\mathbf{n}(\cdot)\,\mathrm{d}S,$$

where $\mathbf{n}(x)$ denotes the outer normal to B at the point $x \in \partial B$. The Gauss theorem implies the following integral form of the conservation of mass (continuity equation)

$$\int_B \Big(\frac{\partial}{\partial t}\rho(t,\cdot) + \operatorname{div}(\rho\mathbf{u})(t,\cdot)\Big)\,\mathrm{d}x = 0, \tag{1.1}$$

provided the corresponding derivatives exist.

Next, we formulate the Newton law saying that the change of the linear momentum is proportional to the force, in the language of the continuum mechanics. We have

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_B (\rho\mathbf{u})(t,\cdot)\,\mathrm{d}x = -\int_{\partial B} (\rho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\mathbf{n}(\cdot)\,\mathrm{d}S + \mathbf{F}_B,$$

where the boundary term denotes again the flux of the momentum through the boundary and the last term denotes the force applied on the part of the fluid B. In continuum mechanics we consider two kinds of forces, the volume forces (e.g., the gravity force or any other force which acts similarly) and the surface forces (i.e., the tension). Thus

$$\mathbf{F}_B = \int_B (\rho\mathbf{f})(t,\cdot)\,\mathrm{d}x + \int_{\partial B} \mathbf{t}(t,\cdot)\,\mathrm{d}S.$$

It is possible to show that under quite general hypothesis the tension can be written in the form

$$\mathbf{t}(t,x,\mathbf{n}) = \mathbf{T}(t,x)\mathbf{n}(x)\,\Big(=\sum_{j=1}^N T_{ij}(t,x)n_j(x)\Big).$$

Thus

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\int_B (\rho\mathbf{u})(t,\cdot)\,\mathrm{d}x &= -\int_{\partial B} (\rho\mathbf{u}\otimes\mathbf{u})(t,\cdot)\mathbf{n}(\cdot)\,\mathrm{d}S \\ &+\int_B (\rho\mathbf{f})(t,\cdot)\,\mathrm{d}x + \int_{\partial B} \mathbf{T}(t,\cdot)\mathbf{n}(\cdot)\,\mathrm{d}S,\end{aligned}$$

which after the application of the Gauss theorem (again, if the derivatives exist) leads to the integral form of the balance of linear momentum

$$\int_B \Big(\frac{\partial}{\partial t}(\rho\mathbf{u})(t,\cdot) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})(t,\cdot) - (\rho\mathbf{f})(t,\cdot) - \mathrm{div}\,\mathbf{T}(t,\cdot)\Big)\,\mathrm{d}x = \mathbf{0}. \tag{1.2}$$

Next, we could consider also the balance of the angular momentum and energy. However, the balance of the angular momentum, assuming no internal momenta in the fluid, leads to the fact that $\mathbf{T}$ is a symmetric tensor. Further, we neglect any changes of the internal energy, i.e., the balance of the total energy is just the balance of the kinetic energy which is formally (if all quantities are sufficiently smooth) the consequence of the balance of linear momentum.

We return to system (1.1)–(1.2) and assume that all quantities are smooth enough. Multiplying each equation by $\frac{1}{|B|}$ and letting $|B| \to 0^+$ we get the differential form of the balance equations

$$\begin{aligned}\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{u}) &= 0,\\ \frac{\partial}{\partial t}(\rho\mathbf{u}) + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) &= \rho\mathbf{f} + \mathrm{div}\,\mathbf{T}.\end{aligned} \tag{1.3}$$

From now on, we will assume that the fluid is incompressible, i.e., if we follow any part of the fluid, its volume remains unchanged. It means

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{V(t)} \mathrm{dX(t)} = 0;$$

here we integrate at time instant t over the volume which is occupied by the fluid particles which occupied initially (at $t = t_0$) the fixed volume B_{t_0}. It follows by the transport theorem (which is basically the change of variables) that

$$\operatorname{div} \mathbf{u} = 0. \tag{1.4}$$

Moreover, we assume that the density is initially spatially constant, i.e.,

$$\rho(0, x) = \rho_0 = const.$$

Then it follows by $(1.3)_1$ and (1.4) that $\rho = const$ and thus $(1.3)_1$ is reduced to (1.4).

Finally, we have to specify the stress tensor. It is the moment, when modelling starts to play an important role, especially for more complex fluids. First, we write the stress tensor in the form

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S},$$

where the scalar function $p = -\frac{1}{3} \operatorname{tr} \mathbf{T}$ (i.e., $\operatorname{tr} \mathbf{S} = 0$). The so-called viscous part of the stress tensor, the quantity $\mathbf{S}$, must be modelled.

Assuming that the fluid has no memory and the response on the shear is instantaneous (first order), we get

$$\mathbf{S} = \mathbf{S}(\nabla \mathbf{u}).$$

Moreover, using the material frame indifference (i.e., the Galilean invariance), it can be shown that

$$\mathbf{S} = \mathbf{S}(\mathbf{D}(\mathbf{u}))$$

with $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$, the symmetric part of the velocity gradient. Further, as the fluid has the full group of symmetry, the tensor function $\mathbf{S}$ fulfils

$$\mathbf{Q}\mathbf{S}(\mathbf{D}(\mathbf{u}))\mathbf{Q}^T = \mathbf{S}(\mathbf{Q}\mathbf{D}(\mathbf{u})\mathbf{Q}^T)$$

for any orthogonal matrix $\mathbf{Q}$ with $\det \mathbf{Q} = 1$. This leads in three space dimensions to the representation

$$\mathbf{S}(\mathbf{D}(\mathbf{u})) = \alpha_0 \mathbf{I} + \alpha_1 \mathbf{D}(\mathbf{u}) + \alpha_2 \mathbf{D}(\mathbf{u})^2,$$

where the scalars α_0, α_1 and α_2 depend on the invariants of $\mathbf{D}(\mathbf{u})$. Note that in three space dimensions, the invariants of a matrix $\mathbf{A}$ are:

- $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^3 A_{ii}$
- $\operatorname{tr}(\mathbf{A}\mathbf{A}^T) = |\mathbf{A}|^2$
- $\det \mathbf{A}$

For more details see, e.g., [22].

We further linearize, i.e., we assume that the dependence of $\mathbf{S}$ on $\mathbf{D}(\mathbf{u})$ is linear, which, together with the incompressibility condition (1.4) leads to

$$\mathbf{S}(\mathbf{D}(\mathbf{u})) = \alpha_1 \mathbf{D}(\mathbf{u}),$$

where α_1 is a constant. Thus we arrive at

$$\begin{gathered}\operatorname{div}\mathbf{u} = 0,\\ \rho_0\frac{\partial\mathbf{u}}{\partial t} + \rho_0\operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla p - \mu\Delta\mathbf{u} = \rho_0\mathbf{f},\end{gathered} \tag{1.5}$$

where we used the fact that

$$\operatorname{div}(\alpha_1\mathbf{D}(\mathbf{u})) = \frac{1}{2}\alpha_1\operatorname{div}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) = \frac{1}{2}\alpha_1(\Delta\mathbf{u} + \nabla\operatorname{div}\mathbf{u}) = \mu\Delta\mathbf{u},$$

and we denoted by $\mu = \frac{1}{2}\alpha_1$ the viscosity. In order to formulate correctly our problem, we must specify the boundary and the initial conditions.

The initial condition can be formulated only for the velocity field $\mathbf{u}$, i.e.,

$$\mathbf{u}(t_0, x) = \mathbf{u}_0(x). \tag{1.6}$$

In what follows, we will take $t_0 = 0$.

Concerning the boundary conditions, the situation is more complex. One can consider problem (1.5)–(1.6) in the full space, i.e., it is enough to specify the behaviour of the solution at infinity. Another possibility is to consider the problem in the space periodic cells and assume that all functions are space periodic. Both these conditions simplify the study considerably, as they avoid any troubles near the boundary. However, we will consider more realistic situation when $\Omega(t) = \Omega$ is a fixed bounded domain. We will mainly consider the case of the homogeneous Dirichlet boundary condition, i.e.,

$$\mathbf{u}(t, x) = \mathbf{0}, \quad t > 0,\ x \text{ on } \partial\Omega. \tag{1.7}$$

Note that condition (1.7) means that the fluid does not penetrate through the boundary (i.e., the normal part of the velocity is zero) and that the fluid adheres at the boundary (i.e., the tangential part is also zero). We could also study more general conditions. Keeping the no-penetration condition

$$\mathbf{u}(t, x)\cdot\mathbf{n}(x) = 0, \quad t > 0,\ x \text{ at } \partial\Omega,$$

we can assume

$$(\mathbf{T}(t, x)\mathbf{n}(x))\cdot\boldsymbol{\tau}(x) + \beta\mathbf{u}(t, x)\cdot\boldsymbol{\tau}(x) = 0$$

for $t > 0$, x at $\partial\Omega$ and $\boldsymbol{\tau}$ any tangent vector. Here, $\beta \geq 0$. Note that the case $\beta = 0$ corresponds to the full slip at the boundary while letting β to infinity we recover (1.7). Indeed, we can also study situations with prescribed flux at the boundary, i.e., the normal velocity component non-zero.

However, we restrict ourselves only on the homogeneous Dirichlet condition, i.e., on problem (1.5)–(1.7). The classical formulation of the problem reads as follows:
For a given $T \in (0, \infty]$, $\mathbf{u}_0$ and $\mathbf{f}$ find[1]

$$\begin{gathered}\mathbf{u} \in \left(C^{1,2}((0,T)\times\Omega)\right)^N \bigcap \left(C([0,T)\times\overline{\Omega})\right)^N, \quad \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega,\\ p \in C^{0,1}((0,T)\times\Omega) \qquad \text{such that in } (0,T)\times\Omega,\\ \operatorname{div}\mathbf{u} = 0,\\ \frac{\partial\mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f}.\end{gathered} \tag{1.8}$$

Note that we denoted $\nu = \frac{\mu}{\rho_0}$ and we redefined the pressure p.

[1]By $C^{\alpha,\beta}((0,T)\times\Omega)$ we mean the set of all functions which are α-times continuously differentiable in the time- and β-times continuously differentiable in the space variables.

1.2 Historical remarks

Equations (1.5) were firstly derived by the French engineer C.M.L.H. Navier in 1822. However, his assumptions, under which he deduced the system from the molecular physics, appeared to be unrealistic. Surprisingly, G. Stokes obtained in 1845 exactly the same system using more or less similar approach as presented above. However, between these years the same model has been proposed by S.D. Poisson (1829) and A.J.C.B. de Saint-Venant (1843). Unfortunately, these names did not enter into the name of this system of equations.

Modern mathematical attempts to study this system go back to twenties of the last century. Swedish mathematician and physicist C.W. Oseen [32] studied mostly the linearized version of our system, but he was also the first one who proposed a weaker version of the formulation to our problem. In the next decade, French mathematician J. Leray followed Oseen's ideas in his doctoral thesis and proved existence and uniqueness of a classical solution in the case when $\Omega = \mathbf{R}^2$, see [25]. However, he failed in the case $\Omega = \mathbf{R}^3$ and therefore he proposed another approach, which is nowadays known as weak formulation. In [26], he proved existence of such solutions for $\Omega = \mathbf{R}^3$. However, he was not able to decide, whether these solutions are unique and whether they are smooth if data are so. He called these solutions turbulent as he believed that the turbulence is responsible for possible irregularities in the flow.

After the second world war J. Leray abandoned the field of mathematical fluid mechanics. However, a new generation represented by E. Hopf [17], O.A. Ladyzhenskaya [21] or J.-L. Lions [27] studied carefully our problem and extended the previous results to many other boundary value problems, with similar results as for the Cauchy problem: in two space dimensions regularity and uniqueness, in three space dimensions only weak solutions with partial results in the direction of regularity and uniqueness.

An interesting idea how to localize the problem of the regularity goes back to early eighties of the last century. Based on the attempts of W. Scheffer, in their seminal paper [4], L. Caffarelli, R. Kohn and L. Nirenberg proposed the suitable weak solution, whose existence they were able to show, together with its partial regularity. We will not treat this problem here, note only that the question whether any weak solution is necessarily a suitable weak solution is still open.

In 2000, inspired by the hundred years old talk of D. Hilbert, the Clay Mathematical Institute [6] offered 1 million US $ for solution of seven open problems in mathematics. And the question, whether weak solutions for the Navier–Stokes equations in three space dimensions are necessarily smooth provided the data are so, was among them. This offer attracted several mathematicians to the problems of mathematical fluid mechanics. It lead to several interesting partial results by P.-L. Lions [28], J. Nečas et al. [30], V. Šverák, G. Seregin [8], [9], [34] ... but the millennium problem remains still open. More complete list of new results can be found in the recent monographs [23], [24], [33] or [40].

Let us finally mention several very recent results. For so-called very weak solution (a solution for which even the first spatial derivatives may not exist) T. Buckmaster a V. Vicol in [2] showed that this definition of a solution is too weak and there may exist many such solutions; even uncountably many. This result is closely connected with similar results for Euler equations of C. De Lellis and L. Székelyhidi (the original paper is [7]).

Moreover, V. Šverák and H. Jia pointed out the possibility [18] that non-smooth initial conditions may generate more than one weak solution of the Navier–Stokes equations. The proof of this property is based on a certain condition. The fact that this condition may be fulfilled was shown "numerically" in [15]. It is not yet an analytical proof, but it shows that this scenario is possible.

1.3 Weaker notion of a solution

Let as briefly explain the main idea how to weaken the formulation of our problem. Take the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f},$$

multiply it scalarly by a smooth function $\boldsymbol{\varphi}$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$, $\boldsymbol{\varphi}(x, T) = \mathbf{0}$ and $\boldsymbol{\varphi} = \mathbf{0}$ at the boundary. Finally, apply the Gauss theorem. We consider each term separately:

$$\int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = -\int_0^T \int_\Omega \mathbf{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}_0(\cdot) \cdot \boldsymbol{\varphi}(0, \cdot) \, \mathrm{d}x$$

$$\int_0^T \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t$$

$$= -\int_0^T \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\partial\Omega} \underbrace{((\mathbf{u} \otimes \mathbf{u})\mathbf{n}) \cdot \boldsymbol{\varphi}}_{=0} \, \mathrm{d}S \, \mathrm{d}t$$

$$\int_0^T \int_\Omega -\Delta \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_{\partial\Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \underbrace{\boldsymbol{\varphi}}_{=\mathbf{0}} \, \mathrm{d}S \, \mathrm{d}t$$

$$\int_0^T \int_\Omega \nabla p \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\partial\Omega} p \underbrace{\boldsymbol{\varphi} \cdot \mathbf{n}}_{=0} \mathrm{d}S \, \mathrm{d}t - \int_0^T \int_\Omega p \underbrace{\operatorname{div} \boldsymbol{\varphi}}_{=0} \, \mathrm{d}x \, \mathrm{d}t$$

and we get

$$\int_0^T \int_\Omega \Big(-\mathbf{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - (\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + \nu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \Big) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x \tag{1.9}$$

for all $\boldsymbol{\varphi} \in \big(C_0^\infty([0,T) \times \Omega)\big)^N$ such that $\operatorname{div} \boldsymbol{\varphi} = 0$ in $(0,T) \times \Omega$.

Later on, we will slightly modify the definition of our weak solution. Let us now emphasize two important things. Firstly, we separated the pressure from the formulation, which simplifies considerably the situation. We will return to the question if we can reconstruct it when we prove the existence of a weak solution to our problem. Secondly, our solution has much less regularity than the classical solution, thus we have better chance to construct it. Note that we need only $\nabla \mathbf{u} \in \big(L^1((0,T) \times \Omega)\big)^{N^2}$ such that $\mathbf{u} \in \big(L^2((0,T) \times \Omega)\big)^N$.

Remark. Note that we started with integral formulation of the balance laws, then switched, assuming that the solution is sufficiently smooth, to the classical formulation and finally relaxed the regularity assumption to get weak formulation. Is this approach correct, in view of the fact that classical solutions may not exist? Fortunately, we can get the weak formulation without necessity to work with classical solutions. The proof is slightly technical, but can be done using standard methods from the measure theory. C.W. Oseen (see [32]) has already observed this fact, however, he worked with Riemann integral rather than with the Lebesgue one. For another approach, see [11].

The plan of the Lecture notes is following:

- we introduce the function spaces which we will use later on, part of the results without the proof, part will also be proved
- we prove existence of a solution to our problem, in bounded two- and three-dimensional domains
- we show that to a given weak solution the pressure exists
- for two space dimensions, we prove uniqueness and regularity of the solution
- in three space dimensions we show uniqueness under additional regularity assumptions, under similar assumptions also higher regularity
- we prove short time regularity as well as global-in-time regularity for small data

Chapter 2

Basic function spaces

2.1 Lebesgue and Sobolev spaces

We use standard notation for:
Sobolev space: $W^{k,p}(\Omega)$, $k \in \mathbf{N}$, $1 \le p \le \infty$
Lebesgue space: $L^q(\Omega)$, $1 \le q \le \infty$

We assume that the theory of these fundamental spaces is known to the reader. It can be found in many textbooks on partial differential equations (see, e.g., [10]) or in special monographs (see, e.g., [1] or [20]).

Let us only mention the following well known interpolation inequalities:

a) Lebesgue:

Lemma 2.1.1. *Let $f \in L^p(\Omega) \cap L^q(\Omega)$, $1 \le p < q \le \infty$, $\Omega \subset \mathbf{R}^N$. Then $f \in L^r(\Omega)$, $p \le r \le q$, and*

$$\|f\|_r \le \|f\|_p^{\alpha} \, \|f\|_q^{1-\alpha}, \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \quad \alpha \in [0,1] \, . \tag{2.1}$$

Proof. It is left as an exercise for the kind reader. ■

b) Lebesgue, Sobolev:

Let $f \in L^q(\Omega) \cap W^{1,s}(\Omega)$, $1 \le q < \infty$. Is it possible to show inequalities of the type

$$\|f\|_r \le C \|f\|_q^{1-\alpha} \, \|f\|_{1,s}^{\alpha}$$

for certain r, q a s? The answer is affirmative.

Theorem 2.1.1. *Let $\Omega \in C^{0,1}$ be a bounded domain in $\mathbf{R}^N$, $f \in W^{1,s}(\Omega) \cap L^q(\Omega)$, $1 \le q < \infty$.*

a) If $s < N$, then $f \in L^r(\Omega)$, $r \le \frac{Ns}{N-s}$ and for $q \le r \le \frac{Ns}{N-s}$ there exists $C = C(\Omega, N, s, q, r)$:

$$\begin{aligned} &\|f\|_r \le C\, \|f\|_{1,s}^{\alpha} \|f\|_q^{1-\alpha},\ \alpha \in [0,1]\,, \\ &\frac{1}{r} = \alpha\Big(\frac{1}{s} - \frac{1}{N}\Big) + (1-\alpha)\,\frac{1}{q}. \end{aligned} \tag{2.2}$$

b) If $s = N$ we can take in (2.2) $q \le r < \infty$ and $r \le \infty$ for $s > N$.

Proof. The idea of the proof is based on the following two steps:

a) we show that (2.2) holds true for $f \in C_0^\infty(\mathbf{R}^N)$, basically using Gagliardo-Nirenberg type inequalities

b) we use the extension theorem (therefore $\Omega \in C^{0,1}$!) and density of smooth functions or properties of the mollifier to transfer these results to bounded domains.

Remark: If $\Omega = \mathbf{R}^N$ or $f \in W_0^{1,s}(\Omega)$, we can take in (2.2) instead of $\|f\|_{1,s}$ only $\|\nabla f\|_s$.

Remark: We show only two special cases of (2.2) which will be important for us:

$N = 2,\ r = 4,\ s = q = 2:$ $\qquad \dfrac{1}{4} = \alpha\left(\dfrac{1}{2} - \dfrac{1}{2}\right) + (1-\alpha)\,\dfrac{1}{2} \ \Rightarrow\ \alpha = \dfrac{1}{2},$

$N = 3,\ r = 4,\ s = q = 2:$ $\qquad \dfrac{1}{4} = \alpha\left(\dfrac{1}{2} - \dfrac{1}{3}\right) + (1-\alpha)\,\dfrac{1}{2} \ \Rightarrow \alpha = \dfrac{3}{4},$

i.e., $\exists C = C(N) : \forall u \in W^{1,2}(\Omega)$: $\quad \|u\|_4 \le C\, \|u\|_{1,2}^{\frac{1}{2}}\, \|u\|_2^{\frac{1}{2}},\ \Omega \subset \mathbf{R}^2,$
$\quad \|u\|_4 \le C\, \|u\|_{1,2}^{\frac{3}{4}}\, \|u\|_2^{\frac{1}{4}},\ \Omega \subset \mathbf{R}^3.$

Let us prove these inequalities:

a) $N = 2$: Let $u \in C_0^\infty(\mathbf{R}^2)$. Then $\|u\|_4 \le \sqrt{2}\|\nabla u\|_2^{\frac{1}{2}}\, \|u\|_2^{\frac{1}{2}}$.

Proof. The Gagliardo–Nirenberg inequality tells us for $v \in C_0^\infty(\mathbf{R}^2)$

$$\|v\|_2 \le \|\nabla v\|_1.$$

Take $v = |u|^2$. It implies

$$\|u\|_4^2 \le \int_{\mathbf{R}^2} |\nabla(u^2)|\, \mathrm{d}x \le 2 \int_{\mathbf{R}^2} |u||\nabla u|\, \mathrm{d}x \le 2\|u\|_2\|\nabla u\|_2,$$

i.e.,

$$\|u\|_4 \le \sqrt{2}\|u\|_2^{\frac{1}{2}}\, \|\nabla u\|_2^{\frac{1}{2}}.$$

(The constant is not optimal — see R. Temam [38]: $C = 2^{\frac{1}{4}}$.)

b) $N = 3$: Let $u \in C_0^\infty(\mathbf{R}^3)$. Then $\|u\|_4 \le \left(\frac{8}{3}\right)^{\frac{3}{4}} \|\nabla u\|_2^{\frac{3}{4}}\, \|u\|_2^{\frac{1}{4}}$.

Proof. Analogously as above

$$\|v\|_{\frac{3}{2}} \leq \|\nabla v\|_1.$$

Choose $v = |u|^{\frac{8}{3}}$. Then

$$\|u\|_4^{\frac{8}{3}} \leq \int_{\mathbf{R}^3} |\nabla |u|^{\frac{8}{3}}| \,\mathrm{d}x \leq \frac{8}{3} \int_{\mathbf{R}^3} |\nabla u||u|^{\frac{5}{3}} \,\mathrm{d}x$$
$$= \frac{8}{3} \int_{\mathbf{R}^3} |\nabla u||u|^{\frac{5}{3}\alpha}|u|^{(1-\alpha)\frac{5}{3}} \,\mathrm{d}x \leq \frac{8}{3}\|\nabla u\|_2 \|u\|_4^{\frac{4}{3}} \|u\|_2^{\frac{1}{3}}.$$

(As $\frac{1}{2} + \frac{5\alpha}{12} + \frac{5(1-\alpha)}{6} = 1 \Rightarrow \alpha = \frac{4}{5}$.) Altogether

$$\|u\|_4 \leq \left(\frac{8}{3}\right)^{\frac{3}{4}} \|\nabla u\|_2^{\frac{3}{4}} \|u\|_2^{\frac{1}{4}}.$$

(The constant can be decreased to $C = \sqrt{2}$, see [38].)

Especially, if $u \in W_0^{1,2}(\Omega)$, then the inequality holds with the same constant, it is enough to use the density of smooth functions with compact support. In the general case one applies the extension theorem and instead of the norm of gradient the whole $W^{1,2}$-norm appears. ■

2.2 Bochner spaces

We will be interested in functions $u\colon I \subset \mathbf{R} \to X$, where X is a Banach space. The proofs of the following results can be found, e.g., in [20].

Definition 2.2.1. *a) A function $f\colon I \to X$ is called a simple function, if its range is finite, i.e., there exist $c_1 \dots, c_k \in X$ and $O_1, \dots, O_k \subset I$, $O_i \cap O_j = \emptyset$ $i \neq j$, O_i measurable such that $f(t) = \sum_{i=1}^{k} c_i \chi_{O_i}(t)$.*

b) A function $f\colon I \to X$ is called strongly measurable if there exists a sequence of simple functions f_n such that $\lim_{n\to\infty} \|f_n(t) - f(t)\|_X = 0$ for a.e. $t \in I$.

Lemma 2.2.1. *Let f be strongly measurable. Then $\|f(\cdot)\|_X\colon I \to \mathbf{R}$ is measurable in the Lebesgue sense.*

Definition 2.2.2. *A function $f\colon I \to X$ is Bochner integrable, if there exists a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$ such that*

- $\lim_{n\to\infty} \|f_n(t) - f(t)\|_X = 0$ *for a.a. $t \in I$ (i.e., f is strongly measurable),*
- $\lim_{n\to\infty} \int_I \|f_n(\cdot) - f(\cdot)\|_X \,\mathrm{d}t = 0.$

If $J \subseteq I$ and f is Bochner integrable over I, then

$$\int_J f \,\mathrm{d}t = \lim_{n\to\infty} \int_I \chi_J(t) f_n(t) \,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^{k_n} c_i^n |O_i^n \cap J|,$$

where f_n fulfils the assumptions stated above.

Theorem 2.2.1 (Bochner)**.** *A strongly measurable function $f\colon I \to X$ is Bochner integrable over $I \iff \|f(\cdot)\|_X$ has finite Lebesgue integral over I.*

Corollary 2.2.1. *Let I be a bounded open interval in $\mathbf{R}$. If $f \in C\left(\overline{I}; X\right)$, then it is Bochner integrable $\iff \|f(\cdot)\|_X$ has finite integral over I.*

Lemma 2.2.2. *If f is Bochner integrable over I, then*

a) $\|\int\limits_I f \mathrm{d}t\|_X \le \int\limits_I \|f\|_X \,\mathrm{d}t$,

b) $\lim\limits_{|J|\to 0^+,\ J\subset I} \int\limits_J f \,\mathrm{d}t = \mathbf{0} \in X$ *(null element).*

Remark. It follows from the definition that for $\eta \in X^*$ and φ Bochner integrable over I it holds

$$\left\langle \eta, \int\limits_I \varphi(t)\,\mathrm{d}t \right\rangle_{X^*,\, X} = \int\limits_I \langle \eta, \varphi(t)\rangle_{X^*,X} \,\mathrm{d}t.$$

2.2.1 Spaces $L^p(I; X)$

Definition 2.2.3. *Let X be a Banach space, $1 \le p \le \infty$, $I \subset \mathbf{R}$. We denote by $L^p(I; X)$ the set of all strongly measurable functions $f\colon I \to X$ such that*

a) $1 \le p < \infty$

$$\int_I ||f(t)||_X^p \mathrm{d}t < \infty,$$

b) $p = \infty$

$$\operatorname*{ess\,sup}_I \|f(t)\|_X < \infty.$$

□

Theorem 2.2.2. *The spaces $L^p(I; X)$ are linear spaces. We set $f_1 = f_2$ if $f_1(t) = f_2(t)$ for a.a. $t \in I$ (in the sense of X). Then $L^p(I; X)$ are Banach spaces endowed with the norms*

$$\|f\|_{L^p(I;X)} = \Big(\int\limits_I \|f(t)\|_X^p \,\mathrm{d}t\Big)^{1/p}, \qquad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(I;X)} = \operatorname*{ess\,sup}_I \|f(\cdot)\|_X, \qquad p = \infty.$$

Note that if I is a bounded interval, then

- $L^p(I; X) \hookrightarrow L^q(I; X)$, $1 \le q \le p$,
- $\|\int\limits_I f(t)\,\mathrm{d}t\|_X \le \int\limits_I \|f(t)\|_X \,\mathrm{d}t \le \|f\|_{L^p(I;X)} |I|^{1-\frac{1}{p}}$
  $\big(\, \|f(\cdot)\|_X \in L^1(I) \Rightarrow f$ is Bochner integrable.$\big)$

Theorem 2.2.3. *Let X be a reflexive Banach space, let X^* denote its dual, $1 \le p < \infty$. Then each continuous linear functional on $L^p(I;X)$ can be represented as*

$$\langle \Phi, f\rangle_{(L^p(I;X))^*, L^p(I;X)} = \int_I \langle \varphi(t), f(t)\rangle_{X^*,X}\, \mathrm{d}t,\ f \in L^p(I;X),\ \varphi \in L^{p'}(I;X^*).$$

Moreover, if $1 < p < \infty$, then $L^p(I;X)$ is a reflexive Banach space.

Let $I = (0,T)$, $T < \infty$. Extend $f \in L^p(I;X)$ by the null element of X outside of I. For $\omega(\cdot)$ the standard mollifier denote

$$f_h(t) = \frac{1}{h}\int_{\mathbf{R}} \omega\left(\frac{t-s}{h}\right) f(s)\,\mathrm{d}s.$$

Then

$$f_h \in C^\infty\left([0,T];X\right).$$

If $f \in L^p(I;X)$ for $1 \le p < \infty$,

$$f_h \longrightarrow f \text{ in } L^p(0,T;X),$$

and for any $1 \le p \le \infty$

$$\|f_h\|_{L^p(0,T;X)} \le \|f\|_{L^p(0,T;X)}.$$

As a consequence we have

Theorem 2.2.4. *Let $1 \le p < \infty$, X be a separable Banach space. Then also $L^p(I;X)$ is a separable Banach space.*

Proof. It is similar to the case $X = \mathbf{R}$.

In particular, for $1 \le p < \infty$, the functions from $C_0^\infty\left((0,T);X\right)$ are dense in $L^p(0,T;X)$.

2.2.2 Spaces with time derivative

We now define the weak derivative with respect to the variable t. The situation is similar to the definition of the weak derivative for Sobolev spaces.

Definition 2.2.4. *Let $u \in L^1_{loc}(0,T;X)$, $g \in L^1_{loc}(0,T;X)$. Then $g = u'$ $(= \frac{\partial u}{\partial t})$, if*

$$\int_0^T u(t)\varphi'(t)\,\mathrm{d}t = -\int_0^T g(t)\varphi(t)\,\mathrm{d}t \qquad \forall \varphi \in \mathcal{D}(0,T).$$

□

Lemma 2.2.3. *Let X be a Banach space, X^* its dual. Let $u, g \in L^1(0,T;X)$. Then the following assertions are equivalent:*

$$u(t) = \xi + \int_0^t g(s)\,\mathrm{d}s \quad \textit{for a.a. } t \in [0,T]\,,\ \xi \in X, \tag{2.3}$$

$$\forall \varphi \in \mathcal{D}(0,T)\colon \int_0^T u(t)\varphi'(t)\,\mathrm{d}t = -\int_0^T g(t)\varphi(t)\,\mathrm{d}t, \tag{2.4}$$

$$\forall \eta \in X^*\colon \frac{\mathrm{d}}{\mathrm{d}t}\langle \eta, u\rangle_{X^*,X} = \langle \eta, g\rangle_{X^*,X} \ \textit{ in } \mathcal{D}'(0,T). \tag{2.5}$$

If (2.3)–(2.5) holds true, then $u = \widetilde{u}$ a.e. in $[0,T]$, where $\widetilde{u} \in C([0,T];X)$.

Proof.
First note that the mapping $t \mapsto \int_0^t g(s)\,\mathrm{d}s$ is absolutely continuous on $[0,T]$ with values in X. Thus:
(2.3) $\Rightarrow$ (2.4): multiply (2.3) by $\varphi'(t) \in \mathcal{D}(0,T)$ and (2.4) is a consequence of the integration by parts.
(2.3) $\Rightarrow$ (2.5): first apply $\eta \in X^*$ to (2.3), then proceed as above.
(2.5) $\Rightarrow$ (2.4): we know that $\forall \varphi \in \mathcal{D}(0,T)$

$$\int_0^T \langle \eta, u\rangle_{X^*,X}\,\varphi'\,\mathrm{d}t = -\int_0^T \langle \eta, g\rangle_{X^*,X}\,\varphi\,\mathrm{d}t,$$

$\eta \in X^*$. As η is independent of t, the linearity of the integral implies

$$\Big\langle \eta, \int_0^T u\varphi'\,\mathrm{d}t + \int_0^T g\varphi\,\mathrm{d}t \Big\rangle_{X^*,X} = 0 \ \ \forall \eta \in X^*,$$

which gives (2.4).
(2.4) $\Rightarrow$ (2.3): we may assume, without loss of generality, that $g = 0$. Indeed, we set $u_0(t) = \int_0^t g(s)\,\mathrm{d}s$ and $v = u(t) - u_0(t)$. Clearly, $u_0 \in AC([0,T];X)$, ${u_0}' = g$ a.e. in I. Let

$$\int_0^T v\varphi'\,\mathrm{d}t = 0 \qquad \forall \varphi \in \mathcal{D}(0,T).$$

We show that then $v = \mathit{const} \in X$. Each function $\varphi \in \mathcal{D}(0,T)$ can be written as

$$\varphi = \lambda\varphi_0 + \psi'\,, \qquad \lambda = \int_0^T \varphi(s)\,\mathrm{d}s,$$

where $\varphi_0 \in \mathcal{D}(0,T)$ is a fixed function, for which

$$\int_0^T \varphi_0\,\mathrm{d}s = 1,$$

and $\psi \in \mathcal{D}(0,T)$ is a primitive to $\varphi - \lambda\varphi_0$ such that $\psi(0) = 0$. We have

$$\int_0^T (v(t) - \xi)\,\varphi(t)\,\mathrm{d}t = 0 \ \ \forall \varphi \in \mathcal{D}(0,T),\ \xi = \int_0^T v(s)\varphi_0(s)\,\mathrm{d}s.$$

Now, using standard argument (mollification in time) it follows that $v(t) - \xi = 0$ a.e. in $(0,T)$. ■

Consider two separable Hilbert spaces, V (e.g., $W_0^{1,2}(\Omega)$) and H (e.g., $L^2(\Omega)$). Using the Riesz representation theorem we identify $H = H^*$. Let us consider the Gelfand triple

$$V \underset{\text{densely}}{\hookrightarrow} H = H^* \underset{\text{densely}}{\hookrightarrow} V^* \tag{2.6}$$

(we prove the dense embedding of the dual spaces later on, see Lemma 2.2.6). Consider our spaces $V = W_0^{1,2}(\Omega)$ a $H = L^2(\Omega)$. The embedding of V into H represents the identity operator $I\colon W_0^{1,2}(\Omega) \to L^2(\Omega)$. Let us look at the identification of H and H^*. To any $\Phi \in \big(L^2(\Omega)\big)^*$ $\exists! g \in L^2(\Omega)$: $\langle \Phi_g, \varphi\rangle_{(L^2(\Omega))^*, L^2(\Omega)} = \int_\Omega g\varphi\,\mathrm{d}x$, $\|\Phi_g\|_{(L^2(\Omega))^*} = \|g\|_{L^2(\Omega)}$. This functional belongs to $(W_0^{1,2}(\Omega))^*$ in the sense

$$\langle \Phi_g, \psi\rangle_{(W_0^{1,2}(\Omega))^*, W_0^{1,2}(\Omega)} = \int_\Omega g\psi\,\mathrm{d}x \ \ \forall \psi \in W_0^{1,2}(\Omega).$$

Thus for $g \in W_0^{1,2}(\Omega)$

$$\langle g, \psi\rangle_{(W_0^{1,2}(\Omega))^*, W_0^{1,2}(\Omega)} \overset{\text{def}}{=} \langle \Phi_g, \psi\rangle_{(L^2(\Omega))^*, L^2(\Omega)} = \int_\Omega g\psi\,\mathrm{d}x \ \ \forall \psi \in W_0^{1,2}(\Omega).$$

In the general case, we have for u, $v \in V \hookrightarrow H$

$$(Iu, Iv)_H = \langle \Phi_{Iu}, Iv\rangle_{H^*,H}\,,$$

where I is the identity mapping representing the embedding $V \hookrightarrow H$ and $\Phi_{(\cdot)}$ plays, as above, the role of the Riesz representation theorem. Then

$$\langle u, v\rangle_{V^*,V} \overset{\text{def}}{=} \langle \Phi_{Iu}, Iv\rangle_{H^*,H} = (Iu, Iv)_H \ \forall v \in V.$$

In this sense we also understand the embedding $V \hookrightarrow V^*$. We can proceed analogously for V only a reflexive Banach space.

Remark. For spaces V and H as above we can define the time derivative of a function $u \in L^p(0,T;V)$ lying in $L^q(0,T;V^*)$ as follows: we require that

$$\int_0^T \langle u', v\rangle_{V^*,V}\,\psi\,\mathrm{d}t = -\int_0^T (Iu, Iv)_H\,\psi'\,\mathrm{d}t$$

$\forall v \in V$ and $\forall \psi \in C_0^\infty(0,T)$. If moreover $u, v \in L^p(0,T;V)$, $u', v' \in L^{p'}(0,T;V^*)$ and $\psi \in C_0^\infty(0,T)$, $2 \le p < \infty$, then

$$\int_0^T \Big(\langle u', v\rangle_{V^*,V} + \langle v', u\rangle_{V^*,V}\Big)\psi\,\mathrm{d}t = -\int_0^T (u,v)_H\,\psi'\,\mathrm{d}t.$$

The proof follows the same lines as the proof of the lemma below.

In what follows, we skip writing the identity operator I.

Lemma 2.2.4. *Let V be a reflexive Banach space and H a Hilbert spaces, V^* and H^* be the corresponding dual spaces. Let $V \underset{densely}{\hookrightarrow} H = H^* \underset{densely}{\hookrightarrow} V^*$. Let $u \in L^p(0,T;V)$, $u' \in L^{p'}(0,T;V^*)$, $1 < p < \infty$. Then u is equal a.e. in $(0,T)$ to a continuous function from $[0,T]$ to H. Moreover,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_H^2 = 2 \langle u', u \rangle_{V^*,V} \text{ in } \mathcal{D}'(0,T). \tag{2.7}$$

Proof. The proof will be performed in three steps.

Step 1. Let us show (2.7). From Lemma 2.2.3 we know that $u \in C([0,T];V^*)$. Namely, as $V \hookrightarrow V^*$, the functions $u, u' \in L^1(0,T;V^*)$. Further,

$$\|u\|_H^2 = (u,u)_H = \langle u,u \rangle_{H^*,H} = \langle \underbrace{u}_{\in L^\infty(0,T;V^*)}, \underbrace{u}_{\in L^p(0,T;V)} \rangle_{V^*,V} \in L^1(0,T),$$

i.e., $u \in L^2(0,T;H)$. Now, let u_m be the mollification of $\tilde{u}$ ($\tilde{u} = u$ in $[0,T]$, otherwise $\tilde{u}$ is equal to $\mathbf{0} \in V$), $u_m \in C^\infty([0,T];V)$,

$$\begin{aligned} u_m &\longrightarrow u \text{ in } L^p(0,T;V), \\ u'_m &\longrightarrow u \text{ in } L^{p'}(0,T;V^*), \\ u_m &\longrightarrow u \text{ in } L^2(0,T;H). \end{aligned}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|_H^2 = 2 (u'_m, u_m)_H = 2 \langle u'_m, u_m \rangle_{V^*,V} \qquad \forall m \in \mathbf{N},$$

thus

$$-\int_0^T \|u_m\|_H^2 \varphi' \,\mathrm{d}t = 2 \int_0^T \underbrace{\langle u'_m, u_m \rangle_{V^*,V}}_{\in L^1(0,T)} \varphi \,\mathrm{d}t \quad \forall \varphi \in \mathcal{D}(0,T).$$

The limit passage $m \to \infty$ gives

$$-\int_0^T \|u\|_H^2 \varphi' \,\mathrm{d}t = 2 \int_0^T \langle u', u \rangle_{V^*,V} \varphi \,\mathrm{d}t \quad \forall \varphi \in \mathcal{D}(0,T),$$

which is equality (2.7), where we used that the function: $t \mapsto \langle u', u \rangle_{V^*,V}(t) \in L^1(0,T)$. This is a consequence of the fact that $u' \in L^{p'}(0,T;V^*)$ and $u \in L^p(0,T;V)$,

$$\int_0^T \langle u', u \rangle_{V^*,V} \,\mathrm{d}t \le \int_0^T \|u'\|_{V^*} \|u\|_V \,\mathrm{d}t < +\infty$$

and therefore $u \in L^\infty(0,T;H)$. Moreover, $u \in C([0,T];V^*)$ and $\|u\|_H^2 \in C([0,T])$.

Step 2. It holds:

Lemma 2.2.5. *Let X, Y be Banach spaces. Let X be reflexive and $X \underset{densely}{\hookrightarrow} Y$. Let $\varphi \in L^\infty(0,T;X) \bigcap C([0,T];Y_w)$. Then $\varphi \in C([0,T];X_w)$.*

The proof of Lemma 2.2.5 will be given later on. Just recall (at the endpoints, the limits are one-sided):

$$\begin{aligned}
\varphi &\in C([0,T];Y) \Longleftrightarrow \lim_{t\to t_0} \|\varphi(t)-\varphi(t_0)\|_Y = 0 \quad \forall t_0 \in [0,T],\\
\varphi &\in C([0,T];Y_w) \Longleftrightarrow \lim_{t\to t_0} \langle \eta, \varphi(t)\rangle - \langle \eta, \varphi(t_0)\rangle\\
&= \lim_{t\to t_0} \langle \eta, \varphi(t)-\varphi(t_0)\rangle = 0 \quad \forall \eta \in Y^*,\ \forall t_0 \in [0,T].
\end{aligned}$$

Evidently, $\varphi \in C([0,T];Y) \Rightarrow \varphi \in C([0,T];Y_w)$, the opposite implication holds true only for Y finite dimensional. Therefore we have $u \in C([0,T];V^*)$, which implies $u \in C([0,T];V^*_w)$ and due to Lemma 2.2.5 and identification $H = H^*$ we know that $u \in C([0,T];H_w)$.

Step 3. Let us show that $u \in C([0,T];H)$. Let $t_0 \in I$. Compute

$$\|u(t)-u(t_0)\|_H^2 = \|u(t)\|_H^2 - 2\,(u(t),u(t_0))_H + \|u(t_0)\|_H^2 \ .$$

Thus, due to the fact that $\|u(\cdot)\|_H^2 \in C([0,T])$ and $u(t) \rightharpoonup u(t_0)$ for $t \to t_0$ (at the endpoints the limits are one-sided),

$$\begin{aligned}
&\lim_{t\to t_0} \|u(t)-u(t_0)\|_H^2\\
&= \lim_{t\to t_0} \underbrace{\|u(t)\|_H^2}_{\to \|u(t_0)\|_H^2} - \lim_{t\to t_0} \underbrace{2\,(u(t),u(t_0))_H}_{\to 2(u(t_0),u(t_0))_H \text{ due to Step 2}} + \|u(t_0)\|_H^2\\
&= \|u(t_0)\|_H^2 - 2\,\|u(t_0)\|_H^2 + \|u(t_0)\|_H^2 = 0.
\end{aligned}$$

■

It remains to prove Lemma 2.2.5. First, recall that

Lemma 2.2.6. *Let X be a reflexive Banach space, Y a Banach space and let $X \underset{densely}{\hookrightarrow} Y$. Then $Y^* \underset{densely}{\hookrightarrow} X^*$.*

Proof. Denote

$$i\colon X \longrightarrow Y$$

the mapping defining the embedding $X \hookrightarrow Y$, i.e., a continuous injective mapping from X to Y, defined on the whole X. According to our assumptions we know that $i(X)$ is dense in Y. Define

$$i^*\colon Y^* \longrightarrow X^*$$

as follows:

$$\langle i^*(y^*), x\rangle_{X^*,X} := \langle y^*, i(x)\rangle_{Y^*,Y}.$$

We show that i^* defines the embedding Y^* to X^*, i.e., it is a continuous injective mapping defined on the whole Y^*, such that $i^*(Y^*)$ is dense in X^*.

Let $i^*(y^*) = 0$, i.e., $\langle y^*, i(x)\rangle_{Y^*,Y} = 0$ for all $x \in X$. As $i(X)$ is dense in Y, we get $y^* = 0$. Now, let X be a reflexive Banach space. Suppose that $\overline{Y^*} \neq X^*$. Then $\exists\ x^{**} \in X^{**}$: $\forall\ y^* \in Y^*$ is $\langle x^{**}, i^*(y^*)\rangle_{X^{**},X^*} = 0$, but $x^{**} \neq 0$. Due to the reflexivity of X there exists $x \in X$: $x^{**} = \mathcal{J}(x)$ ($\mathcal{J}(x)$ is the canonical mapping) such that

$$\begin{aligned}
\langle i^*(y^*), x\rangle_{X^*,X} = 0 \ \ \forall y^* \in Y^* &\quad\Longrightarrow\\
\langle y^*, i(x)\rangle_{Y^*,Y} = 0 \ \ \forall y^* \in Y^* &\quad\Longrightarrow\quad i(x) = 0,
\end{aligned}$$

hence, as i is injective, $x = 0$, which contradicts to $\overline{Y^*} \neq X^*$. ■

We can now prove Lemma 2.2.5 which is of independent interest.

Proof (of Lemma 2.2.5)**.** As $X \underset{\text{densely}}{\hookrightarrow} Y$, it is $Y^* \underset{\text{densely}}{\hookrightarrow} X^*$. Due to the assumptions we know that

$$\langle \eta, \varphi(t)\rangle_{Y^*,Y} \underset{t\to t_0}{\longrightarrow} \langle \eta, \varphi(t_0)\rangle_{Y^*,Y} \quad \forall \eta \in Y^*.$$

We aim to show that

$$\langle \mu, \varphi(t)\rangle_{X^*,X} \underset{t\to t_0}{\longrightarrow} \langle \mu, \varphi(t_0)\rangle_{X^*,X} \quad \forall \mu \in X^*.$$

Define $\widetilde{\varphi}(t) \in X$ as follows

$$\langle \mathcal{J}(\widetilde{\varphi}(t)), \mu\rangle_{X^{**},X^*} = \liminf_{\substack{h\to 0\\ t+h\in I}} \frac{1}{h}\int_t^{t+h} \langle \mu, \varphi(s)\rangle_{X^*,X}\, \mathrm{d}s.$$

Evidently, the right-hand side is bounded by $\|\varphi\|_{L^\infty(0,T;X)} \|\mu\|_{X^*}$ and thus $\mathcal{J}(\widetilde{\varphi}(t)) \in X^{**}$. Due to the reflexivity of X, $\widetilde{\varphi}(t) \in X$ is uniquely defined. Moreover,

$$\|\widetilde{\varphi}(t)\|_X = \sup_{\|\mu\|_{X^*}\le 1} \langle \mu, \tilde{\varphi}(t)\rangle \le \sup_{\|\mu\|_{X^*}\le 1} \|\varphi\|_{L^\infty(0,T;X)} \|\mu\|_{X^*} \le \|\varphi\|_{L^\infty(0,T;X)}\,.$$

In particular, for $\mu \in Y^* (\underset{\text{densely}}{\hookrightarrow} X^*)$ we see that $\widetilde{\varphi}(t) = \varphi(t)$ in $[0,T]$. Thus $\|\varphi(t)\|_X \le \|\varphi\|_{L^\infty(0,T;X)} \quad \forall t \in [0,T]$. As Y^* is dense in X^*, $\forall \mu \in X^*$ and $\forall \varepsilon > 0\ \exists \mu_\varepsilon \in Y^*\colon\ \|\mu_\varepsilon - \mu\|_{X^*} < \varepsilon$. Fix $\varepsilon > 0$. Therefore

$$\langle \mu, \varphi(t) - \varphi(t_0)\rangle_{X^*,X} = \langle \mu - \mu_{\tilde{\varepsilon}}, \varphi(t) - \varphi(t_0)\rangle_{X^*,X} + \langle \mu_{\tilde{\varepsilon}}, \varphi(t) - \varphi(t_0)\rangle_{X^*,X}\,.$$

Now, for $\tilde{\varepsilon}$ chosen appropriately, the first term

$$\begin{aligned} &|\,\langle \mu - \mu_{\tilde{\varepsilon}}, \varphi(t) - \varphi(t_0)\rangle_{X^*,X}\,| \\ &\le \|\mu - \mu_{\tilde{\varepsilon}}\|_{X^*} \|\varphi(t) - \varphi(t_0)\|_X \le 2\,\|\varphi\|_{L^\infty(0,T;X)}\,\tilde{\varepsilon} < \frac{\varepsilon}{2}. \end{aligned}$$

The second term is small for t sufficiently close to t_0, as $\mu_{\tilde{\varepsilon}} \in Y^*$:

$$|\,\langle \mu_{\tilde{\varepsilon}}, \varphi(t) - \varphi(t_0)\rangle_{X^*,X}\,| = |\,\langle \mu_{\tilde{\varepsilon}}, \varphi(t) - \varphi(t_0)\rangle_{Y^*,Y}\,| < \frac{\varepsilon}{2}.$$

Thus, to any $\varepsilon > 0\ \ \exists \delta > 0\ \ \forall t \in U_\delta(t_0)\colon |\,\langle \mu, \varphi(t) - \varphi(t_0)\rangle_{X^*,X}\,| < \varepsilon$. ■

We will need the compact embedding of the space

$$W = W^{\alpha_0,\alpha_1}_{X_0,X_1} = \{v \in L^{\alpha_0}(0,T;X_0);\, v' \in L^{\alpha_1}(0,T;X_1)\}$$

into a suitable space $L^\alpha(0,T;X)$. Set

$$\|v\|_W = \|v\|_{L^{\alpha_0}(0,T;X_0)} + \|v'\|_{L^{\alpha_1}(0,T;X_1)}\,.$$

It holds

Theorem 2.2.5 (Aubin–Lions)**.** *Let X_0, X_1, X be three Banach spaces satisfying $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Let X_0, X_1 be additionally reflexive. Further, let $1 < \alpha_i < \infty$, $i = 0, 1$.*

Then for $0 < T < \infty$, $W \hookrightarrow\hookrightarrow L^{\alpha_0}(0,T;X)$.

Remark. It is possible to take $\alpha_1 = 1$, but the proof is more complicated and we need neither complications nor the strength of this assertion.

We first prove:

Lemma 2.2.7. *Let X_0, X_1, X be Banach spaces such that $X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$. Then $\forall \eta > 0 \ \exists c_\eta$ such that $\forall v \in X_0$*

$$\|v\|_X \le \eta \|v\|_{X_0} + c_\eta \|v\|_{X_1}. \tag{2.8}$$

Proof. We prove the lemma by contradiction. Let (2.8) be not satisfied, i.e., $\exists \eta > 0$: $\forall m \in \mathbf{N}\ \exists w_m \in X_0$ that

$$\|w_m\|_X > \eta \|w_m\|_{X_0} + m \|w_m\|_{X_1}.$$

We set

$$v_m = \frac{w_m}{\|w_m\|_{X_0}},$$

thus

$$\|v_m\|_X > \eta + m \|v_m\|_{X_1}.$$

As $\|v_m\|_{X_0} = 1$, v_m is bounded in X (due to the embedding) and

$$\|v_m\|_{X_1} \longrightarrow 0 \text{ for } m \longrightarrow \infty.$$

Further, there is a subsequence v_{m_k} strongly convergent in X ($X_0 \hookrightarrow\hookrightarrow X$) and thus $v_{m_k} \to 0$ in X. But $\|v_{m_k}\|_X > \eta > 0$, which leads to the contradiction. ∎

Proof (Aubin–Lions)**.** We proceed in four steps.

Step 1. Let u_m be a bounded sequence in W. We aim to show that there is a subsequence u_{m_k}, strongly convergent in $L^{\alpha_0}(0,T;X)$. As X_0, X_1 are reflexive, $1 < \alpha_i < \infty$, W is also reflexive a thus there is a $u \in W$ such that

$$u_{m_k} \rightharpoonup u \text{ in } W,$$

therefore

$$\begin{aligned} u_{m_k} &\rightharpoonup u \text{ in } L^{\alpha_0}(0,T;X_0), \\ u'_{m_k} &\rightharpoonup u' \text{ in } L^{\alpha_1}(0,T;X_1). \end{aligned}$$

We have to show that $v_{m_k} = u_{m_k} - u \to 0$ in $L^{\alpha_0}(0,T;X)$.

Step 2. It is enough to show that $v_{m_k} \to 0$ in $L^{\alpha_0}(0,T;X_1)$. Indeed, in such a case

$$\|v_{m_k}\|_{L^{\alpha_0}(0,T;X)} \le \eta \|v_{m_k}\|_{L^{\alpha_0}(0,T;X_0)} + c_\eta \|v_{m_k}\|_{L^{\alpha_0}(0,T;X_1)},$$

and due to the boundedness of v^{m_k} in W we have

$$\|v_{m_k}\|_{L^{\alpha_0}(0,T;X)} \le C\eta + c_\eta \|v_{m_k}\|_{L^{\alpha_0}(0,T;X_1)}.$$

To any $\varepsilon > 0$ there is $\eta > 0$: $C\eta < \frac{\varepsilon}{2}$ and there is n_0: $\forall m_k > n_0$ we have $c_\eta \|v^{m_k}\|_{L^{\alpha_0}(0,T;X_1)} < \frac{\varepsilon}{2}$. Thus

$$\|v_{m_k}\|_{L^{\alpha_0}(0,T;X)} < \varepsilon$$

and as $\varepsilon > 0$ was arbitrary, the assertion of the theorem is proved.

Step 3. Let us show that $W \hookrightarrow C([0,T];X_1)$. We know that each element from W belongs (after a possible change on a subset of $[0,T]$ of measure zero) to $C([0,T];X_1)$ due to Lemma 2.2.3. The continuity of the embedding is immediate, as Lemma 2.2.3 implies

$$u(t) = u(0) + \int_0^t u'(s)\,\mathrm{d}s$$

and thus

$$\|u(t)\|_{X_1} \le \|u(0)\|_{X_1} + \|u'\|_{L^1(0,T;X_1)}\,.$$

Integrating the equality over $(0,T)$ reads

$$\begin{aligned}&T\|u(0)\|_{X_1} \le \|u\|_{L^1(0,T;X_1)} + T\|u'\|_{L^1(0,T;X_1)}\\&\le C\|u\|_{L^1(0,T;X_0)} + T\|u'\|_{L^1(0,T;X_1)}\\&\Longrightarrow \max_{t\in[0,T]}\|u(t)\|_{X_1} \le C\|u\|_W\,.\end{aligned}$$

Step 4. We know that $\|v_{m_k}(t)\|_{X_1} \le C \quad \forall t \in [0,T]$ and to be able to apply the Lebesgue dominated convergence theorem, it is enough to show that

$$v_{m_k}(t) \longrightarrow 0 \ \text{ strongly in } X_1.$$

Choose, e.g., $t = 0$. Then

$$v_{m_k}(0) = v_{m_k}(t) - \int_0^t v'_{m_k}(\tau)\,\mathrm{d}\tau.$$

Integrate this equality from 0 to s:

$$\begin{aligned}v_{m_k}(0) &= \frac{1}{s}\Big\{\int_0^s v_{m_k}(t)\,\mathrm{d}t - \int_0^s\Big(\int_0^t v'_{m_k}(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}t\Big\}\\&= \frac{1}{s}\int_0^s v_{m_k}(t)\,\mathrm{d}t - \frac{1}{s}\int_0^s (s-\tau)v'_{m_k}(\tau)\,\mathrm{d}\tau := a_{m_k} + b_{m_k}.\end{aligned}$$

Choose $\varepsilon > 0$. We easily see that $\|b_{m_k}\|_{X_1} \le \int_0^s \|v'_{m_k}(\tau)\|_{X_1}\,\mathrm{d}\tau < \frac{\varepsilon}{2}$ for s sufficiently small ($\alpha_1 > 1$!). We know that $v_{m_k} \rightharpoonup 0$ in $L^{\alpha_0}(0,T;X_0)$ and thus $a_{m_k} = \frac{1}{s}\int_0^s v_{m_k}(t)\,\mathrm{d}t \rightharpoonup 0$ in X_0; whence $a_{m_k} \to 0$ in X_1. As s is fixed, for n_0 sufficiently large $\|a_{m_k}\|_{X_1} < \frac{\varepsilon}{2} \ \forall m_k > n_0$. ■

2.3 Spaces with zero divergence

2.3.1 Temam spaces

We define

Definition 2.3.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain. We set for $1 \le p < \infty$*

$$\begin{aligned}&E^p(\Omega) = \{\boldsymbol{g} \in (L^p(\Omega))^N; \operatorname{div}\boldsymbol{g} \in L^p(\Omega)\},\\&\|\boldsymbol{g}\|_{E^p(\Omega)} = \|\boldsymbol{g}\|_p + \|\operatorname{div}\boldsymbol{g}\|_p,\\&E_0^p(\Omega) = \overline{(C_0^\infty(\Omega))^N}^{\|\cdot\|_{E^p(\Omega)}}.\end{aligned}$$

Evidently, both spaces are Banach spaces which are for $1 < p < \infty$ reflexive. We would like to show that the smooth functions up to the boundary are dense in $E^p(\Omega)$. To this aim we need the notion of a star shaped domain.

Definition 2.3.2. *A domain $\Omega \subset \mathbf{R}^N$ is called star-shaped with respect to a point $x_0 \in \Omega$, if there is a continuous positive function $h\colon \partial B_1 \to \mathbf{R}$ such that*

$$\Omega = \Big\{x \in \mathbf{R}^N; |x - x_0| < h\Big(\frac{x - x_0}{|x - x_0|}\Big)\Big\}.$$

A domain $\Omega \subset \mathbf{R}^N$ is called star-shaped with respect to a ball $B \subset \Omega$, if it is star-shaped with respect to all points $x \in B$.

Domains with Lipschitz boundary can be decomposed into star-shaped domains. It holds (see [12]):

Lemma 2.3.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with Lipschitz boundary. Then there exists a family of bounded domains*

$$\mathcal{G} = \{G_1, G_2, \dots, G_r, G_{r+1}, \dots, G_{r+m}\}, \quad r, m \in \mathbf{N}$$

such that

(i) $\overline{\Omega} \subset \bigcup_{i=1}^{r+m} G_i$,

(ii) $\partial\Omega \subset \bigcup_{i=1}^{r} G_i$,

(iii) there exists a family of balls

$$\mathcal{B} = \{B_1, B_2, \dots, B_{r+m}\}$$

such that each domain

$$\Omega_i = \Omega \cap G_i, \qquad i = 1, \dots, r + m$$

is star-shaped with respect to the ball B_i.

Further, let $f \in C_0^\infty(\Omega)$ and $\int_\Omega f \,\mathrm{d}x = 0$. Then there is a family of functions

$$\mathcal{F} = \{f_1, \dots, f_r, f_{r+1}, \dots, f_{r+m}\}$$

such that

(i) $f_i \in C_0^\infty(\Omega_i)$, $\int_{\Omega_i} f_i \,\mathrm{d}x = 0$,

(ii) $f(x) = \sum_{i=1}^{r+m} f_i(x)$,

(iii)

$$\|f_i\|_{k,q,\Omega_i} \le C(m, q, \Omega_1, \dots, \Omega_{r+m}, \Omega)\|f\|_{k,q,\Omega},$$

$1 < q < \infty$, $k = 0, 1, \dots$.

It holds

Theorem 2.3.1. *Let $\Omega \in C^{0,1}$, $1 \le p < \infty$.*
Then $E^p(\Omega) = \overline{(C^\infty(\overline{\Omega}))^N}^{\|\cdot\|_{E^p(\Omega)}}$.

Proof. We will only sketch the main ideas of the proof:

a) $\Omega = \mathbf{R}^N$ the result follows by standard mollification

b) $\Omega = C^{0,1}$, bounded domain

we use local description of the boundary and partition of unity

$$\Omega \subset V \cup \bigcup_{i=1}^{m} V_i$$

on V – we use standard mollification

on V_i – by translation and additional partition of unity we can decompose V_i^+ (i.e., $V_i \cap \Omega$) into domains, which are star-shaped with respect to the origin, see Lemma 2.3.1 (here we use that $\Omega \in C^{0,1}$!). On the star-shaped domain we "shift out" the function by

$$\mathbf{u}_\lambda(x) = \mathbf{u}\left(\frac{x}{\lambda}\right), \quad \lambda > 1$$

and we regularize this shifted function. Passing $\lambda \to 1^+$ and $h \to 0^+$ (mollification factor) we show that $\mathbf{u}_n \to \mathbf{u}$ in $E^p(\Omega)$, $\mathbf{u}_n \in (C^\infty(\Omega))^N$, where

$$\mathbf{u}_n(x) = (\mathbf{u}_{\lambda_n})_{h_n}.$$

The precise proof can be found in book [39]. ■

2.3.2 Sobolev spaces with zero divergence

For $1 \le p < \infty$ we will consider spaces of the type

$$W^{1,p}_{0,\mathrm{div}}(\Omega) = \left\{\mathbf{u} \in (W^{1,p}_0(\Omega))^N; \operatorname{div}\mathbf{u} = 0\right\},$$

and

$$\overline{W^{1,p}_{0,\mathrm{div}}(\Omega)} = \overline{\{\mathbf{u} \in (C^\infty_0(\Omega))^N; \operatorname{div}\mathbf{u} = 0\}}^{\|\cdot\|_{1,p}},$$

respectively. We will show that for $\Omega \in C^{0,1}$ the spaces coincide. This is based on the following result

Lemma 2.3.2 (Bogovskii, Solonnikov, Ladyzhenskaya, Borchers, Sohr, and others)**.** *Let $\Omega \in C^{0,1}$ be a bounded domain in $\mathbf{R}^N$. Let $f \in W^{m,q}_0(\Omega)$, $m \ge 0$, $1 < q < \infty$, $\int_\Omega f\,\mathrm{d}x = 0$. Then $\exists \boldsymbol{v} \in (W^{m+1,q}_0(\Omega))^N$, a solution to*

$$\begin{aligned} \operatorname{div}\boldsymbol{v} &= f \quad \text{in } \Omega, \\ \boldsymbol{v}|_{\partial\Omega} &= \boldsymbol{0} \end{aligned}$$

such that

$$\|\nabla \boldsymbol{v}\|_{m,q} \le C\,\|f\|_{m,q},$$

where C is independent of f. In particular, if $f \in C^\infty_0(\Omega)$, then also $\boldsymbol{v} \in (C^\infty_0(\Omega))^N$.

If $f = \operatorname{div}\boldsymbol{g}$, $\boldsymbol{g} \in E^q_0(\Omega)$, then also

$$\|\boldsymbol{v}\|_q \le C\|\boldsymbol{g}\|_q.$$

Moreover, the operator $T\colon \{f \in W^{m,q}_0(\Omega)\colon \int_\Omega f\,\mathrm{d}x = 0\} \to (W^{m+1,q}_0(\Omega))^N$ such that $Tf = \boldsymbol{v}$ is linear (the same holds also in the case when $f = \operatorname{div}\boldsymbol{g}$ with $\boldsymbol{g} \in E^q_0(\Omega)$).

Proof. The proof can be found in the book of Novotný, Straškraba [31] or Galdi [12], or also in Appendix to these Lecture notes. ■

Lemma 2.3.3. *Let $\Omega \in C^{0,1}$, $1 < p < \infty$. Then $W^{1,p}_{0,\mathrm{div}}(\Omega) = \overline{W^{1,p}_{0,\mathrm{div}}(\Omega)}$.*

Proof. Evidently, $\overline{W^{1,p}_{0,\mathrm{div}}(\Omega)} \subseteq W^{1,p}_{0,\mathrm{div}}(\Omega)$. Let us show the opposite inclusion. Let $\mathbf{u} \in W^{1,p}_{0,\mathrm{div}}(\Omega)$. As $W^{1,p}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{1,p}}$, there exists $\{\mathbf{u}_n\}_{n=1}^\infty \in (C^\infty_0(\Omega))^N$ such that $\|\mathbf{u}_n - \mathbf{u}\|_{1,p} \underset{n\to\infty}{\longrightarrow} 0$. However, generally $\operatorname{div}\mathbf{u}_n \neq 0$. On the other hand, $\operatorname{div}\mathbf{u}_n \underset{L^p(\Omega)}{\longrightarrow} \operatorname{div}\mathbf{u} = 0$. It follows from Lemma 2.3.2 that the problem

$$\begin{aligned} \operatorname{div}\mathbf{v}_n &= \operatorname{div}\mathbf{u}_n, \\ \mathbf{v}_n|_{\partial\Omega} &= \mathbf{0}, \\ \|\nabla\mathbf{v}_n\|_p &\leq C\,\|\operatorname{div}\mathbf{u}_n\|_p \end{aligned} \tag{2.9}$$

(and due to the boundary condition also $\|\mathbf{v}_n\|_p \leq C(\Omega)\,\|\nabla\mathbf{v}_n\|_p$) has a solution (the compatibility condition $0 = \int_\Omega \operatorname{div}\mathbf{u}_n\,\mathrm{d}x = \int_{\partial\Omega}\mathbf{u}_n\cdot\mathbf{n}\,\mathrm{d}S$ is trivially satisfied) such that $\mathbf{v}_n \in (C^\infty_0(\Omega))^N$. Moreover, as $\operatorname{div}\mathbf{u}_n \to 0$ in $L^p(\Omega)$, $\mathbf{u}_n \neq \mathbf{v}_n$[1] for infinitely many $n \in \mathbf{N}$. Set $\mathbf{w}_n = \mathbf{u}_n - \mathbf{v}_n$. Then

a) $\operatorname{div}\mathbf{w}_n = \operatorname{div}\mathbf{u}_n - \operatorname{div}\mathbf{v}_n = 0$,

b) $\|\mathbf{w}_n - \mathbf{u}\|_{1,p} \leq \|\mathbf{u}_n - \mathbf{u}\|_{1,p} + \|\mathbf{v}_n\|_{1,p} \longrightarrow 0$,

c) $\mathbf{w}_n \in (C^\infty_0(\Omega))^N$,

i.e., $\mathbf{u} \in \overline{W^{1,p}_{0,\mathrm{div}}(\Omega)}$. ■

Remark. With a certain modification of the proof the same result holds also for Ω exterior domain or $\Omega = \mathbf{R}^N$, see Appendix. However, there are domains, e.g., domains with several exists to infinity, where the spaces differ.

2.3.3 Decomposition of $(L^2(\Omega))^N$. Existence of the pressure.

We will consider spaces of the type

$$\overline{L^2_{0,\mathrm{div}}(\Omega)} = \overline{\{\mathbf{u} \in (C^\infty_0(\Omega))^N; \operatorname{div}\mathbf{u} = 0\}}^{\|\cdot\|_2}.$$

Our aim is to characterize this space and to show that $(L^2(\Omega))^N = \overline{L^2_{0,\mathrm{div}}(\Omega)} \oplus P$, where we further characterize the orthogonal complement P.

Let $1 < p < \infty$. Denote by $W^{1-\frac{1}{p},p}(\partial\Omega)$ the range of the trace operator from $W^{1,p}(\Omega)$. Recall that our space $W^{1-\frac{1}{p},p}(\partial\Omega)$ — with non-integer derivative — is something like intermediate space between $L^p(\partial\Omega)$ and $W^{1,p}(\partial\Omega)$, more precisely

$$\|u\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \|u\|_{L^p(\partial\Omega)} + \Big(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+p-2}}\,\mathrm{d}S_x\mathrm{d}S_y\Big)^{\frac{1}{p}}.$$

[1] except for the case $\mathbf{u} = \mathbf{0}$ which does not require any approximation, or if, by chance, $\operatorname{div}\mathbf{u}_n = 0$ for all n, where we do not need any correction

Denote $W^{-\frac{1}{p'},p'}(\partial\Omega) = (W^{1-\frac{1}{p},p}(\partial\Omega))^*$, $p' = \frac{p}{p-1}$. Let $\mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^N$, $v \in C^\infty(\overline{\Omega})$, $\Omega \in C^{0,1}$. Then

$$\int_\Omega \mathbf{u} \cdot \nabla v \,\mathrm{d}x + \int_\Omega v \operatorname{div} \mathbf{u} \,\mathrm{d}x = \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})\, v \,\mathrm{d}S.$$

As $\Omega \in C^{0,1}$, the normal vector $\mathbf{n}$ exists a.e. at $\partial\Omega$. The left-hand side makes also sense for $\mathbf{u} \in E^p(\Omega)$, $v \in W^{1,p'}(\Omega)$[2]. On the right-hand side, function $v \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$; in a certain sense we will be able to extend this Green formula also for functions with only the above mentioned regularity.

Theorem 2.3.2. *Let $\Omega \in C^{0,1}$, $1 < p < \infty$. Then there exists a continuous linear operator $\gamma_{\boldsymbol{n}}$ from $E^p(\Omega)$ to $W^{-\frac{1}{p},p}(\partial\Omega) = (W^{1-\frac{1}{p'},p'}(\partial\Omega))^*$ such that*

$$\gamma_{\boldsymbol{n}} \boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} \quad \textit{for} \quad \boldsymbol{u} \in (C^\infty(\overline{\Omega}))^N.$$

For $\boldsymbol{u} \in E^p(\Omega)$, $v \in W^{1,p'}(\Omega)$ it holds

$$\int_\Omega \boldsymbol{u} \cdot \nabla v \,\mathrm{d}x + \int_\Omega v \operatorname{div} \boldsymbol{u} \,\mathrm{d}x = \langle \gamma_{\boldsymbol{n}} \boldsymbol{u}, Tv \rangle_{W^{-\frac{1}{p},p}(\partial\Omega), W^{1-\frac{1}{p'},p'}(\partial\Omega)},$$

where Tv is the trace of the function v ($Tv \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$).

Proof. Let $\varphi \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$, $v \in W^{1,p'}(\Omega)$ so that $\varphi = Tv$. For $\mathbf{u} \in E^p(\Omega)$ we set

$$X_{\mathbf{u}}(\varphi) = \int_\Omega (v \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla v) \,\mathrm{d}x.$$

The value $X_{\mathbf{u}}(\varphi)$ does not depend on v, it depends only on its trace $Tv = \varphi$. Indeed, let $v_1, v_2 \in W^{1,p'}(\Omega)$ be such that $Tv_1 = Tv_2 = \varphi$. Set $v = v_1 - v_2$. We show

$$\int_\Omega (v \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla v) \,\mathrm{d}x = 0.$$

For $v \in W_0^{1,p'}(\Omega)$, there exists $v_m \in C_0^\infty(\Omega)$ such that $v_m \to v$ in $W^{1,p'}(\Omega)$, for $\mathbf{u} \in E^p(\Omega)$, there exists $\mathbf{u}_m \in (C^\infty(\overline{\Omega}))^N$ such that $\mathbf{u}_m \to \mathbf{u}$ in $E^p(\Omega)$. Therefore

$$0 = \int_\Omega (v_m \operatorname{div} \mathbf{u}_m + \mathbf{u}_m \cdot \nabla v_m) \,\mathrm{d}x \underset{m\to\infty}{\longrightarrow} \int_\Omega (v \operatorname{div} \mathbf{u} + \mathbf{u} \cdot \nabla v) \,\mathrm{d}x.$$

Due to the inverse trace theorem we have for suitable v (we may take any v, in particular we take that one from the inverse trace theorem)

$$X_{\mathbf{u}}(\varphi) \le \|\mathbf{u}\|_{E^p(\Omega)} \|v\|_{W^{1,p'}(\Omega)} \le C_0 \|\mathbf{u}\|_{E^p(\Omega)} \|\varphi\|_{W^{1-\frac{1}{p'},p'}(\partial\Omega)}.$$

[2] In fact it is enough to have $\operatorname{div} \mathbf{u} \in (W^{1,p}(\Omega))^*$ and $\mathbf{u} \in (L^p(\Omega))^N$, if we understand the duality in the sense

$$\langle \operatorname{div} \mathbf{u}, \varphi \rangle_{(W^{1,p'}(\Omega))^*, W^{1,p'}(\Omega)} := -\int_\Omega \mathbf{u} \cdot \nabla v \,\mathrm{d}x + \langle \mathbf{u} \cdot \mathbf{n}, v \rangle_{W^{-\frac{1}{p},p}(\partial\Omega), W^{1-\frac{1}{p'},p'}(\Omega)},$$

this is, however, not the same as the claim of the theorem.

For fixed $\mathbf{u} \in E^p(\Omega)$ the functional $X_{\mathbf{u}}(\cdot) \in (W^{1-\frac{1}{p'},p'}(\partial\Omega))^*$ and thus there exists $g = g(\mathbf{u}) \in W^{-\frac{1}{p},p}(\partial\Omega)$ such that

$$X_{\mathbf{u}}(\varphi) = \langle g, \varphi \rangle_{W^{-\frac{1}{p},p}(\partial\Omega), W^{1-\frac{1}{p'},p'}(\partial\Omega)} \quad \forall \varphi \in W^{1-\frac{1}{p'},p'}(\partial\Omega).$$

The mapping $\mathbf{u} \mapsto g(\mathbf{u})$ is evidently linear, $\|g\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \le C_0 \|\mathbf{u}\|_{E^p(\Omega)}$. It remains to show that for $\mathbf{u} \in (C^\infty(\overline{\Omega}))^N$ we have $g(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega}$. Let $\mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^N$, $v \in C^\infty(\overline{\Omega})$. Then

$$X_{\mathbf{u}}(Tv) = \int_\Omega \operatorname{div}(v\mathbf{u})\, \mathrm{d}x = \int_{\partial\Omega} v\, \mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = \int_{\partial\Omega} (Tv)\mathbf{u} \cdot \mathbf{n}\, \mathrm{d}S = \langle \mathbf{u} \cdot \mathbf{n}, Tv \rangle .$$

As $T\left(C^\infty(\overline{\Omega})\right)$ is dense in $W^{1-\frac{1}{p'},p'}(\partial\Omega)$ ($W^{1-\frac{1}{p'},p'}(\partial\Omega) = T\big(W^{1,p'}(\Omega)\big)$ and $C^\infty(\overline{\Omega})$ is dense in $W^{1,p'}(\Omega)$), the equality

$$X_{\mathbf{u}}(\varphi) = \langle \mathbf{u} \cdot \mathbf{n}, \varphi \rangle_{W^{-\frac{1}{p},p}(\partial\Omega), W^{1-\frac{1}{p'},p'}(\partial\Omega)} \quad \forall \varphi \in W^{1-\frac{1}{p'},p'}(\partial\Omega)$$

holds true. Therefore

$$g(\mathbf{u}) = \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} \;\text{ for } \mathbf{u} \in \left(C^\infty(\overline{\Omega})\right)^N .$$

■

Before we characterize the space $\overline{L^2_{0,\mathrm{div}}(\Omega)}$, we need to prove the following lemma which basically gives the existence of pressure for steady problems. It holds

Lemma 2.3.4. *Let $\Omega \in C^{0,1}$, $1 < q < \infty$ and let $\boldsymbol{G} \in \left(\left(W_0^{1,q}(\Omega)\right)^N\right)^*$ $\left(= \left(W^{-1,q'}(\Omega)\right)^N\right)$ be such that*

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle_{((W_0^{1,q}(\Omega))^N)^*, (W_0^{1,q}(\Omega))^N} = \langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle = 0 \;\; \forall \boldsymbol{\varphi} \in W^{1,q}_{0,\mathrm{div}}(\Omega).$$

Then $\exists!\, p \in \widetilde{L^{q'}}(\Omega) = \{u \in L^{q'}(\Omega); \int_\Omega u\, \mathrm{d}x = 0\}$ such that

$$\langle \boldsymbol{G}, \boldsymbol{\varphi} \rangle = \int_\Omega p \operatorname{div} \boldsymbol{\varphi}\, \mathrm{d}x \;\; \forall \boldsymbol{\varphi} \in \left(W_0^{1,q}(\Omega)\right)^N.$$

To prove it, we need the following lemma (see, e.g., [3, Théoreme II.18]).

Lemma 2.3.5. *Let A: $X \to Y$ be a bounded linear operator, $D(A) = X$, A^{-1} exist and be continuous. Let X, Y be reflexive Banach spaces.*
Then

$$R(A^*) = (\ker A)^{\perp\, 3} = \big\{f \in X^*; \langle f, u \rangle = 0 \;\; \forall u \in \ker A\big\}.$$

■

[3] here the annihilator

Proof (of Lemma 2.3.4)**.** Take A: $(W_0^{1,q}(\Omega))^N \to \widetilde{L^q}(\Omega)$, $A\mathbf{v} = \operatorname{div}\mathbf{v}$. We consider a special branch of A^{-1}, so called "Bogovskii operator", i.e., the special solution operator to the problem

$$\begin{aligned} \operatorname{div}\mathbf{w} &= \operatorname{div}\mathbf{v} \quad \text{in } \Omega, \\ \mathbf{w}|_{\partial\Omega} &= \mathbf{0}, \\ \|\mathbf{w}\|_{1,q} &\le C\,\|\operatorname{div}\mathbf{v}\|_q\,, \end{aligned}$$

given by Lemma 2.3.2. This operator is linear and bounded, thus also continuous. Therefore we know

$$(\ker A)^{\perp} = R\,(A^*)\,.$$

Evidently

$$\ker A = \left\{\mathbf{u} \in \left(W_0^{1,q}(\Omega)\right)^N;\ \operatorname{div}\mathbf{u} = 0\right\},$$

thus $\mathbf{G} \in (\ker A)^{\perp} = R\,(A^*)$. As $Y = \widetilde{L^q}(\Omega)$, we have

$$Y^* = \left\{L^{q'}(\Omega)|_{\mathbf{R}}\right\}^4.$$

Then, by virtue of $\langle A^*v, u\rangle_{X^*,X} = \langle v, Au\rangle_{Y^*,Y}$, it holds

$$\langle \mathbf{G}, \boldsymbol{\varphi}\rangle = \int_\Omega \underbrace{p}_{p\in\widetilde{L^{q'}}(\Omega)} A\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega p\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x.$$

■

We are now ready to characterize $\overline{L^2_{0,\mathrm{div}}(\Omega)}$:

Theorem 2.3.3. *Let* $\Omega \in C^{0,1}$.
Then

$$\overline{L^2_{0,\mathrm{div}}(\Omega)} = \left\{\boldsymbol{u} \in \left(L^2(\Omega)\right)^N;\ \operatorname{div}\boldsymbol{u} = 0 \ \textit{in}\ \mathcal{D}'(\Omega);\ \gamma_{\boldsymbol{n}}(\boldsymbol{u}) = 0\right\}\left(\equiv L^2_{0,\mathrm{div}}(\Omega)\right)$$

$$\left(\overline{L^2_{0,\mathrm{div}}(\Omega)}\right)^{\perp\,5} = \left\{\boldsymbol{v} \in \left(L^2(\Omega)\right)^N;\ \boldsymbol{v} = \nabla p,\ p \in W^{1,2}(\Omega)\right\}(\equiv P)\,.$$

Proof. *Step 1.* Let $\mathbf{v} \in P$. Then $\forall \mathbf{w} \in \left\{\mathbf{w} \in (C_0^\infty(\Omega))^N; \operatorname{div}\mathbf{w} = 0\right\}$

$$\int_\Omega \mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x = \int_\Omega \mathbf{w}\cdot\nabla p\,\mathrm{d}x = -\int_\Omega p\operatorname{div}\mathbf{w}\,\mathrm{d}x = 0,$$

i.e., $\mathbf{v} \in \left(\overline{L^2_{0,\mathrm{div}}(\Omega)}\right)^{\perp}$. Conversely, let $\mathbf{v} \in \left(\overline{L^2_{0,\mathrm{div}}(\Omega)}\right)^{\perp}$. Then

$$\int_\Omega \mathbf{v}\cdot\mathbf{w}\,\mathrm{d}x = 0 \quad \forall \mathbf{w} \in \overline{L^2_{0,\mathrm{div}}(\Omega)},$$

[4] the quotient space (and can be represented by $\widetilde{L^{q'}}(\Omega)$)
[5] here the orthogonal complement

in particular also $\forall \mathbf{w} \in \overline{\underbrace{\{\mathbf{w} \in (C_0^\infty(\Omega))^N; \operatorname{div} \mathbf{w} = 0\}}_{=W^{1,2}_{0,\mathrm{div}}(\Omega)}}^{\|\cdot\|_{1,2}} \subset \overline{L^2_{0,\mathrm{div}}(\Omega)}$. Due to Lemma 2.3.4 there exists $p \in \widetilde{L^2}(\Omega)$, for which it holds

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x = -\int_\Omega p \operatorname{div} \mathbf{w} \, \mathrm{d}x \quad \forall \mathbf{w} \in (C_0^\infty(\Omega))^N.$$

This implies that $\mathbf{v} = \nabla p$ in $\mathcal{D}'(\Omega)$, i.e., $p \in W^{1,2}(\Omega)$. Thus $\left(\overline{L^2_{0,\mathrm{div}}(\Omega)}\right)^\perp \subset P$ which gives $\left(\overline{L^2_{0,\mathrm{div}}(\Omega)}\right)^\perp = P$.

Step 2. Let $\mathbf{u} \in \overline{L^2_{0,\mathrm{div}}(\Omega)}$. Then there exists a sequence $\mathbf{u}_m \in (C_0^\infty(\Omega))^N$ with $\operatorname{div} \mathbf{u}_m = 0$: $\mathbf{u}_m \to \mathbf{u}$ in $(L^2(\Omega))^N$. Further

$$0 = \int_\Omega \operatorname{div} \mathbf{u}_m \varphi \, \mathrm{d}x = -\int_\Omega \mathbf{u}_m \cdot \nabla\varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega),$$

hence for $m \to \infty$

$$0 = -\int_\Omega \mathbf{u} \cdot \nabla\varphi \, \mathrm{d}x \quad \forall \varphi \in C_0^\infty(\Omega).$$

Thus $\operatorname{div} \mathbf{u} = 0$ in $\mathcal{D}'(\Omega)$. We have $\mathbf{u} \in E^2(\Omega)$, i.e., (recall that $\mathbf{u}_m \to \mathbf{u}$ in $E^2(\Omega)$)

$$0 = \gamma_{\mathbf{n}}(\mathbf{u}_m) \longrightarrow \gamma_{\mathbf{n}}(\mathbf{u}) = 0 \Longrightarrow \mathbf{u} \in L^2_{0,\mathrm{div}}(\Omega).$$

Conversely, let $\overline{L^2_{0,\mathrm{div}}(\Omega)} \subsetneq L^2_{0,\mathrm{div}}(\Omega)$. Let $\mathbf{u} \in H$, where H denotes the orthogonal complement of $\overline{L^2_{0,\mathrm{div}}(\Omega)}$ to $L^2_{0,\mathrm{div}}(\Omega)$ (both spaces are closed!). According to Step 1 there exists $p \in W^{1,2}(\Omega)$ such that $\mathbf{u} = \nabla p$. Therefore,

$$\begin{aligned} \operatorname{div} \mathbf{u} = \operatorname{div}(\nabla p) &= \Delta p = 0 \ \text{ in } \ \mathcal{D}'(\Omega), \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} &= \frac{\partial p}{\partial \mathbf{n}} = 0 \ \text{ in the sense of the operator } \gamma_{\mathbf{n}}(\mathbf{u}), \\ & \quad \big(\gamma_{\mathbf{n}}(\mathbf{u}) \in H^{-\frac{1}{2}}(\partial\Omega)\big). \end{aligned}$$

In $W^{1,2}(\Omega)$, there exists a solution to this problem, unique up to an additive constant. This solution is $p = \mathit{const}$, i.e., $\mathbf{u} = \mathbf{0}$ and $H = \{\mathbf{0}\}$. Thus $\overline{L^2_{0,\mathrm{div}}(\Omega)} = L^2_{0,\mathrm{div}}(\Omega)$. ■

2.4 Stokes problem

Consider the problem:

Find $\mathbf{u} \in (C^2(\Omega))^N \cap (C(\overline{\Omega}))^N$, $p \in C^1(\Omega)$:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

We have two possibilities for the weak formulation:

a) $\mathbf{u} \in W_0^{1,2}(\Omega)$, $p \in L^2(\Omega)$, $\mathbf{f} \in (L^2(\Omega))^N$ (or $\mathbf{f} \in (W^{-1,2}(\Omega))^N$):

$$\int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x = \underbrace{\int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x}_{\text{or}\langle \mathbf{f}, \boldsymbol{\varphi}\rangle} \quad \forall \boldsymbol{\varphi} \in (C_0^\infty(\Omega))^N$$

$\big($or $\forall \boldsymbol{\varphi} \in (W_0^{1,2}(\Omega))^N\big)$,

together with

$$\int_\Omega \mathbf{u} \cdot \nabla \psi \,\mathrm{d}x = 0 \qquad \forall \psi \in W^{1,2}(\Omega).$$

b) $\mathbf{u} \in W_{0,\mathrm{div}}^{1,2}(\Omega)$, $\mathbf{f} \in (L^2(\Omega))^N$ (or $\mathbf{f} \in (W^{-1,2}(\Omega))^N$):

$$\int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x = \underbrace{\int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi} \,\mathrm{d}x}_{\text{or}\langle \mathbf{f}, \boldsymbol{\varphi}\rangle} \quad \forall \boldsymbol{\varphi} \in \mathcal{V} = \{\mathbf{w} \in (C_0^\infty(\Omega))^N ; \operatorname{div} \mathbf{w} = 0\}$$

$\big($or $\forall \boldsymbol{\varphi} \in W_{0,\mathrm{div}}^{1,2}(\Omega)\big)$.

A question appears, whether weak formulation b) does not destroy the information about the pressure. Fortunately, it is not the case. We have from Lemma 2.3.4 for

$$\langle \mathbf{G}, \boldsymbol{\varphi} \rangle \quad = \quad \int_\Omega (\nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - \mathbf{f} \cdot \boldsymbol{\varphi}) \,\mathrm{d}x \tag{2.10}$$

that

- $\mathbf{G} \in (W^{-1,2}(\Omega))^N$
- $\langle \mathbf{G}, \boldsymbol{\varphi} \rangle = 0 \;\; \forall \boldsymbol{\varphi} \in W_{0,\mathrm{div}}^{1,2}(\Omega)$,

and thus $\exists! p \in L^2(\Omega)$, $\int_\Omega p \,\mathrm{d}x = 0$:

$$\int_\Omega (\nabla \mathbf{u} : \nabla \boldsymbol{\varphi} - \mathbf{f} \cdot \boldsymbol{\varphi}) \,\mathrm{d}x = \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x \quad \forall \boldsymbol{\varphi} \in (W_0^{1,2}(\Omega))^N,$$

which is precisely the weak formulation as in a). Thus formulation b) is more suitable, due to

Theorem 2.4.1. *Let $\boldsymbol{f} \in (W^{-1,2}(\Omega))^N$.*

Then there exists the unique weak solution to the Stokes problem in the sense b) above. Moreover,

$$\begin{aligned} \|\nabla \boldsymbol{u}\|_2 &\leq C \,\|\boldsymbol{f}\|_{-1,2} \,, \\ \|p\|_2 &\leq C \,\|\boldsymbol{f}\|_{-1,2} \,, \end{aligned}$$

where p is the pressure constructed above.

Proof. The existence of the unique $\mathbf{u}$, satisfying weak formulation b), together with the estimate, is a consequence of the Lax–Milgram lemma (do the proof carefully!), the existence of the pressure follows from Lemma 2.3.4. Moreover, if we use as test function in weak formulation a) function $\boldsymbol{\varphi}$, solution to

$$\begin{aligned} \operatorname{div} \boldsymbol{\varphi} &= p, \\ \boldsymbol{\varphi}|_{\partial\Omega} &= \mathbf{0} \end{aligned}$$

from Lemma 2.3.2 — check carefully the details! — we have

$$\int_\Omega p^2 \,\mathrm{d}x = -\langle \mathbf{f}, \boldsymbol{\varphi}\rangle + \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x \quad \Rightarrow \quad \|p\|_2 \le C\big(\|\mathbf{f}\|_{-1,2} + \|\nabla \mathbf{u}\|_2 \big).$$

■

Remark. If we take $\mathbf{f} \in \big(W^{1,2}_{0,\mathrm{div}}(\Omega)\big)^*$, then the existence of the unique weak solution according to formulation b) can be shown as above, but it is not clear whether the pressure exists!

Generally (for the proof see, e.g., [12]):

Theorem 2.4.2. *Let $m \ge -1$, $1 < q < \infty$. Let $\boldsymbol{f} \in (W^{m,q}(\Omega))^N$, $\Omega \in C^{\max\{m+2,2\}}$, $\boldsymbol{u}_* \in (W^{m+2-\frac{1}{q},q}(\partial\Omega))^N$, $\int_{\partial\Omega} \boldsymbol{u}_* \cdot \boldsymbol{n} \,\mathrm{d}S = 0$.*

Then there exists the unique weak solution to the Stokes problem with non-homogeneous boundary condition $\boldsymbol{u}_*$ such that

$$\begin{aligned} \boldsymbol{u} &\in (W^{m+2,q}(\Omega))^N, \\ p &\in W^{m+1,q}(\Omega), \int_\Omega p \,\mathrm{d}x = 0 \end{aligned} \tag{2.11}$$

and $\exists C = C(\Omega, N, q)$ such that

$$\|\boldsymbol{u}\|_{m+2,q} + \|p\|_{m+1,q} \le C(\|\boldsymbol{f}\|_{m,q} + \|\boldsymbol{u}_*\|_{m+2-\frac{1}{q},q,\partial\Omega}).$$

■

Remark. The function $\mathbf{u} \in (W^{1,q}(\Omega))^N$ is a (q-)weak solution to the Stokes problem if $\mathbf{u} - \mathbf{u}_* \in (W^{1,q}_0(\Omega))^N$ and

$$\int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \quad \forall \boldsymbol{\varphi} \in \mathcal{V} = \left\{ \mathbf{w} \in (C_0^\infty(\Omega))^N; \operatorname{div} \mathbf{w} = 0 \right\}.$$

Let us return to the case $q = 2$. Denote by Λ the solution operator with homogeneous boundary condition, i.e.,

$$\Lambda : L^2_{0,\mathrm{div}}(\Omega) \to W^{1,2}_{0,\mathrm{div}}(\Omega) \subset (W^{1,2}_0(\Omega))^N,$$

such that

$$\Lambda \mathbf{f} = \mathbf{u},$$

where $\mathbf{u}$ is a weak solution to the Stokes problem. (Recall that arbitrary $\mathbf{f} \in (L^2(\Omega))^N$ can be decomposed

$$\mathbf{f} = \mathbf{f}_1 + \nabla \pi,$$

where $\mathbf{f}_1 \in L^2_{0,\mathrm{div}}(\Omega)$ and π can be absorbed into the pressure, thus assuming the right-hand side directly from $L^2_{0,\mathrm{div}}(\Omega)$ makes sense).

Lemma 2.4.1. *The operator Λ is as operator from $L^2_{0,\text{div}}(\Omega)$ to $L^2_{0,\text{div}}(\Omega)$ selfadjoint and compact.*

Proof. The operator is evidently linear and bounded, $D(\Lambda) = L^2_{0,\text{div}}(\Omega)$, further $R(\Lambda) \subset W^{1,2}_{0,\text{div}}(\Omega) \hookrightarrow\hookrightarrow L^2_{0,\text{div}}(\Omega)$ and thus it is compact.

Let $\mathbf{u}, \mathbf{v} \in L^2_{0,\text{div}}(\Omega)$. Then for $\Lambda\mathbf{u} = \mathbf{f}$, $\Lambda\mathbf{v} = \mathbf{g}$ it holds

$$\int_\Omega \Lambda\mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \overbrace{=}^{\Lambda\mathbf{v}=\mathbf{g}} \int_\Omega \nabla\mathbf{f} : \nabla\mathbf{g}\,\mathrm{d}x \overbrace{=}^{\Lambda\mathbf{u}=\mathbf{f}} \int_\Omega \mathbf{u}\cdot\mathbf{g}\,\mathrm{d}x = \int_\Omega \mathbf{u}\cdot\Lambda\mathbf{v}\,\mathrm{d}x.$$

Thus we have for all $\mathbf{u}$, $\mathbf{v} \in L^2_{0,\text{div}}(\Omega) = D(\Lambda)$ that

$$(\Lambda\mathbf{u}, \mathbf{v})_{L^2_{0,\text{div}}(\Omega)} = (\mathbf{u}, \Lambda\mathbf{v})_{L^2_{0,\text{div}}(\Omega)},$$

i.e., $D(\Lambda) \subseteq D(\Lambda^*)$. As $D(\Lambda) = L^2_{0,\text{div}}(\Omega)$, we know that $D(\Lambda^*) = L^2_{0,\text{div}}(\Omega)$. Hence Λ is selfadjoint. ■

Remark. The eigenfunctions of Λ form an orthonormal basis of the space $L^2_{0,\text{div}}(\Omega)$,

$$\Lambda\mathbf{w}^j = \frac{1}{\lambda_j}\mathbf{w}^j, \qquad j \in \mathbf{N}\ ,\ \lambda_j \to \infty \ \text{ for } \ j \to \infty.$$

Evidently,

$$\int_\Omega \nabla\mathbf{w}^j : \nabla\mathbf{v}\,\mathrm{d}x = \lambda_j \int_\Omega \mathbf{w}^j\cdot\mathbf{v}\,\mathrm{d}x \quad \forall\mathbf{v} \in W^{1,2}_{0,\text{div}}(\Omega),$$

and

$$\int_\Omega \mathbf{w}^j\cdot\mathbf{w}^i\,\mathrm{d}x = \delta_{ij} \Rightarrow \int_\Omega \nabla\mathbf{w}^j : \nabla\mathbf{w}^i\,\mathrm{d}x = \lambda_j\delta_{ij},$$

and thus $\left\{\mathbf{w}^j\right\}_{j=1}^\infty$ form an orthogonal system in $W^{1,2}_{0,\text{div}}(\Omega)$. Moreover, it is also a basis in $W^{1,2}_{0,\text{div}}(\Omega)$ ($\int_\Omega \nabla\mathbf{w}^n : \nabla\boldsymbol{\varphi}\,\mathrm{d}x = 0 \ \ \forall n \Rightarrow \int_\Omega \mathbf{w}^n\cdot\boldsymbol{\varphi} = 0 \ \ \forall n \Rightarrow \boldsymbol{\varphi} = \mathbf{0}$).

Further, due to the regularity of the Stokes problem, if $\Omega \in C^{m+2}$, then $\mathbf{w}^j \in (W^{m+2,2}(\Omega))^N$, $m \geq 0$ (and also $\mathbf{w}^j \in (C^\infty(\Omega))^N$ for arbitrary Ω open).

Chapter 3

Weak solution to evolutionary Navier–Stokes equations

3.1 Existence of a weak solution

Let us recall the classical formulation

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T)\times\Omega,\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } (0,T)\times\Omega,\\
\mathbf{u}|_{\partial\Omega} &= \mathbf{0} \quad \text{in } (0,T),\\
\mathbf{u}(0,x) &= \mathbf{u}_0(x) \quad \text{in } \Omega.
\end{aligned}$$

We get the weak formulation by multiplying the momentum equation by $\boldsymbol{\varphi} \in (C_0^\infty(\Omega))^N$, $\operatorname{div}\boldsymbol{\varphi} = 0$ and integrating over Ω, together with the Gauss formula:

$$\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \nu\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega p\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}\,\mathrm{d}x.$$

First, recall that the pressure term is equal to zero. Further, we will not be able to show that $\frac{\partial \mathbf{u}}{\partial t} \in L^1_{loc}(Q_T)$, thus we replace the scalar product by the duality. One possible representation of this duality was shown in Chapter 1. We will also consider more general right-hand side. We get

Definition 3.1.1. *Let $\Omega \subset \mathbf{R}^N$, $N = 2,3$. Let $\boldsymbol{f} \in L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$, $\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$.*

Then the function $\boldsymbol{u} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega))$ with $\frac{\partial \boldsymbol{u}}{\partial t} \in L^1(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$ is called a weak solution to the Navier–Stokes equations *corresponding to the data $\boldsymbol{f}$ and $\boldsymbol{u}_0$, if*

$$\left\langle \frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{\varphi} \right\rangle_{\left(W^{1,2}_{0,\mathrm{div}}(\Omega)\right)^*, W^{1,2}_{0,\mathrm{div}}(\Omega)} + \int_\Omega (\boldsymbol{u}\cdot\nabla\boldsymbol{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \nu\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x$$

$$= \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle_{\left(W^{1,2}_{0,\mathrm{div}}(\Omega)\right)^*, W^{1,2}_{0,\mathrm{div}}(\Omega)} \qquad \forall \boldsymbol{\varphi}\in W^{1,2}_{0,\mathrm{div}}(\Omega) \text{ and a.a. } t\in(0,T),$$

$$\lim_{t\to 0^+}\int_\Omega \boldsymbol{u}(t,\cdot)\cdot\boldsymbol{\varphi}\,\mathrm{d}x = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\varphi}\,\mathrm{d}x \qquad \forall\boldsymbol{\varphi}\in L^2_{0,\mathrm{div}}(\Omega).$$

□

Remark. The case $N > 3$ can be considered analogously; we will not do it here. It is necessary to take $\boldsymbol{\varphi}$ smooth, so that the convective term makes sense, and consider the time derivative in another spaces.

Remark. Set $V = W^{1,2}_{0,\mathrm{div}}(\Omega)$, $H = L^2_{0,\mathrm{div}}(\Omega)$. According to results of Chapter 2, after possible change on a subset of the time interval of measure zero, $\mathbf{u} \in C([0,T]; V^*) \cap L^\infty(0,T;H)$, and we have $\mathbf{u} \in C([0,T]; H_w)$, due to Lemma 2.2.5. Thus we understand the initial condition in this sense, assuming to have changed the function $\mathbf{u}$ on a set of measure zero, if necessary. Due to Theorem 2.3.3 we even have $\mathbf{u}\in C([0,T];((L^2(\Omega))^N)_w)$. We will see later that for the initial condition we prove a stronger result, i.e., $\lim_{t\to 0^+}\|\mathbf{u}(t)-\mathbf{u}_0\|_2 = 0$.

Remark. Consider a "sufficiently smooth" solution to the Navier–Stokes equations. Multiply the classical formulation by $\mathbf{u}$ and integrate over Ω (or set $\boldsymbol{\varphi} := \mathbf{u}$ in the weak formulation)

$$\int_\Omega \frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{u}\,\mathrm{d}x + \int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x + \nu\int_\Omega\nabla\mathbf{u}:\nabla\mathbf{u}\,\mathrm{d}x = \langle\mathbf{f},\mathbf{u}\rangle,$$

1st term: $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_2^2$

2nd term:
$$\int_\Omega(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x = \frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla|\mathbf{u}|^2\,\mathrm{d}x = -\frac{1}{2}\int_\Omega \underbrace{\operatorname{div}\mathbf{u}}_{=0}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\underbrace{\mathbf{u}\cdot\mathbf{n}}_{=0}|\mathbf{u}|^2\,\mathrm{d}S.$$

Integrating over time

$$\int_\Omega|\mathbf{u}(t)|^2\,\mathrm{d}x + 2\nu\int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_\Omega|\mathbf{u}_0|^2\,\mathrm{d}x + 2\int_0^t\langle\mathbf{f},\mathbf{u}\rangle\,\mathrm{d}\tau, \qquad (3.1)$$

i.e., we got the so called energy equality. However, for $N = 3$ we get only a weaker result, the energy inequality, namely

$$\int_\Omega|\mathbf{u}(t)|^2\,\mathrm{d}x + 2\nu\int_0^t\int_\Omega|\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \int_\Omega|\mathbf{u}_0|^2\,\mathrm{d}x + 2\int_0^t\langle\mathbf{f},\mathbf{u}\rangle\,\mathrm{d}\tau \qquad (3.2)$$

for a.a. $t\in(0,T)$.

Definition 3.1.2. *We call $\mathbf{u}$ the Leray–Hopf weak solution to the Navier–Stokes equations, if $\boldsymbol{u}$ is a weak solution and moreover, it satisfies for a.a. $t\in(0,T)$ inequality (3.2).* □

We aim to prove the following results

Theorem 3.1.1 (weak solution, $N = 2$)**.** *Let $\Omega \subset \mathbf{R}^2$ be a bounded domain, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfy the assumptions of Definition 3.1.1 and $0 < T < \infty$. Then there exists exactly one weak solution to the Navier–Stokes equations. This solution is also the Leray–Hopf weak solution and it fulfils the initial condition in the sense $\lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_2 = 0$. Moreover, $\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}}(\Omega))$ and due to Theorem 2.3.3 also $\boldsymbol{u} \in C([0,T]; (L^2(\Omega))^2)$ and it fulfils also energy equality (3.1).*

Theorem 3.1.2 (weak solution, $N = 3$)**.** *Let $\Omega \subset \mathbf{R}^3$ be a bounded domain, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ satisfy the assumptions of Definition 3.1.1 and $0 < T < \infty$. Then there exists at least one Leray–Hopf weak solution to the Navier–Stokes equations. This solution fulfils the initial condition in the sense $\lim_{t\to 0+} \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_2 = 0$.*

The proof of both theorems will be performed parallelly. Only at the end, we prove the stronger result in two space dimensions. We proceed as follows

(i) Galerkin approximation — formulation

(ii) solvability of Galerkin approximation + a priori estimates of $\mathbf{u}^n$

(iii) a priori estimates of the time derivative

(iv) limit passage

(v) energy inequality

(vi) initial condition

(vii) uniqueness and energy equality for $N = 2$

Step (i) Take $\{\mathbf{w}^i\}_{i=1}^{\infty}$ the orthogonal basis of the space $W^{1,2}_{0,\mathrm{div}}(\Omega)$ formed by the eigenfunctions of the Stokes operator. We further assume that the functions $\{\mathbf{w}^i\}_{i=1}^{\infty}$ are normalized in $(L^2(\Omega))^N$.

Definition 3.1.3. *A function $\boldsymbol{u}^n(t,x) = \sum_{i=1}^{n} c_i^n(t)\boldsymbol{w}^i(x)$ is called the n-th Galerkin approximation, if*

$$\begin{aligned} \int_\Omega \frac{\partial \boldsymbol{u}^n}{\partial t} \cdot \boldsymbol{w}^j \,\mathrm{d}x + \int_\Omega (\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n) \cdot \boldsymbol{w}^j \,\mathrm{d}x + \nu \int_\Omega \nabla \boldsymbol{u}^n : \nabla \boldsymbol{w}^j \,\mathrm{d}x \\ = \langle \boldsymbol{f}, \boldsymbol{w}^j \rangle \qquad \forall j = 1,..,n, \\ \boldsymbol{u}^n(0,x) = \sum_{i=1}^{n} a_i \boldsymbol{w}^i(x), \end{aligned} \tag{3.3}$$

where $a_i = \int_\Omega \boldsymbol{u}_0(x) \cdot \boldsymbol{w}^i(x)\,\mathrm{d}x$ (i.e., $\boldsymbol{u}^n(0,x)$ is the projection of $\boldsymbol{u}_0(x)$ to $Lin\,\{\boldsymbol{w}^i\}_{i=1}^{n}$ in $L^2_{0,\mathrm{div}}(\Omega)$). □

Equality (3.3) can be rewritten to a system of ordinary differential equations for $\{c_i^n(t)\}_{i=1}^{n}$. Recall that $\int_\Omega \mathbf{w}^i \cdot \mathbf{w}^j \,\mathrm{d}x = \delta_{ij}$[1].

$$\dot{c}_j^n(t) + c_k^n(t) c_l^n(t) \int_\Omega \left(\mathbf{w}^k \cdot \nabla \mathbf{w}^l\right) \cdot \mathbf{w}^j \,\mathrm{d}x + \nu \underbrace{\lambda_j c_j^n(t)}_{\text{not summed}} = \left\langle \mathbf{f}, \mathbf{w}^j \right\rangle, \ j = 1,..,n, \tag{3.4}$$

[1] We use the summation convection, i.e., we sum over twice repeated indeces, e.g., $u_i \frac{\partial u_j}{\partial x_i} = \sum_{i=1}^{N} u_i \frac{\partial u_j}{\partial x_i}$ or $c_k^n(t)\mathbf{w}^k(x) = \sum_{k=1}^{n} c_k^n(t)\mathbf{w}^k(x)$.

$$c_j^n(0) = a_j.$$

To simplify the notation, in what follows, we skip the upper index n at $c_j^n(t)$.

Step (ii) We may apply the Carathéodory theory to the system of ordinary differential equations (3.4) (and if $\mathbf{f} \in C([0,T];((W_0^{1,2}(\Omega))^*)^N)$, we could even use the classical theory). Thus there exists (locally in time) exactly one generalized solution — $c_j \in AC\,[0,T_n^*)$ — to system (3.4) $\forall n \in \mathbf{N}$. If the time interval $[0,T_n^*)$ on which this solution exists is such that $T_n^* < T$, then necessarily $\max_{j\in\{1,2,\dots,n\}} |c_j(t)| \xrightarrow[t\to(T_n^*)^-]{} +\infty$. We will exclude this possibility and thus our solution exists on $(0,T)$. Furthermore, as we shall later, the solution can be extended up to $t = T$. Multiply $(3.4)_j$ by $c_j(t)$ and sum over $j = 1,\dots,n$. Integrate over $(0,t)$ (formally it means the same as to take as test function in (3.3) the solution $\mathbf{u}^n$). It yields

$$\int_0^t \frac{1}{2}\sum_{j=1}^n \frac{\mathrm{d}}{\mathrm{dt}}|c_j|^2\,\mathrm{d}\tau + \int_0^t c_k c_l c_j \int_\Omega \left(\mathbf{w}^k\cdot\nabla\mathbf{w}^l\right)\cdot\mathbf{w}^j\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\nu\int_0^t \sum_{j=1}^n |c_j|^2\lambda_j\,\mathrm{d}\tau = \int_0^t \left\langle \mathbf{f}, c_j\mathbf{w}^j\right\rangle \mathrm{d}\tau,$$

or, equivalently

$$\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\mathbf{u}^n(t)\right\|_2^2\,\mathrm{d}\tau + \int_0^t \underbrace{\int_\Omega (\mathbf{u}^n\cdot\nabla\mathbf{u}^n)\cdot\mathbf{u}^n\,\mathrm{d}x}_{=0}\,\mathrm{d}\tau$$
$$+\nu\int_0^t\int_\Omega |\nabla\mathbf{u}^n|^2\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \langle\mathbf{f},\mathbf{u}^n\rangle\,\mathrm{d}\tau$$

and thus

$$\frac{1}{2}\left\|\mathbf{u}^n(t)\right\|_2^2 + \nu\int_0^t \left\|\nabla\mathbf{u}^n\right\|_2^2\,\mathrm{d}\tau$$
$$\le \left\|\mathbf{f}\right\|_{L^2(0,t;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)}\left\|\mathbf{u}^n\right\|_{L^2(0,t;W^{1,2}_{0,\mathrm{div}}(\Omega))} + \frac{1}{2}\left\|\mathbf{u}^n(0)\right\|_2^2.$$

The first term on the right-hand side can be estimated by virtue of the Friedrichs and Young inequalities by

$$C(\nu)\left\|\mathbf{f}\right\|^2_{L^2(0,t;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} + \frac{1}{2}\nu\left\|\nabla\mathbf{u}^n\right\|^2_{L^2(0,t;L^2(\Omega))}$$

and thus we have

$$\left\|\mathbf{u}^n(t)\right\|_2^2 + \nu\int_0^t \left\|\nabla\mathbf{u}^n\right\|_2^2\,\mathrm{d}\tau \le C\left(\mathbf{f},\mathbf{u}_0\right), \tag{3.5}$$

as $\left\|\mathbf{u}^n(0)\right\|_2^2 = \sum_{j=1}^n a_j^2 \le \left\|\mathbf{u}_0\right\|_2^2$. It follows from here that $c_j(\cdot)$ are bounded in time and thus $T_n^* = T$ for all $n \in \mathbf{N}$ as well as that $c_j \in AC[0,T]$,

$j = 1, 2, \ldots, n$. Moreover,

$$\sup_{t\in[0,T)} \|\mathbf{u}^n(t)\|_2^2 + \nu \int_0^T \|\nabla \mathbf{u}^n\|_2^2 \, \mathrm{d}\tau \le C(\mathbf{f}, \mathbf{u}_0). \tag{3.6}$$

It means that the sequence $\mathbf{u}^n$ is bounded in spaces $L^\infty(0,T;(L^2(\Omega))^N)$ and $L^2(0,T;(W^{1,2}(\Omega))^N)$ uniformly with respect to n.

Step (iii) Estimate (3.6) is not sufficient for the limit passage. We have at our disposal the Aubin–Lions lemma, however, to apply it, we need an estimate of the time derivative. We will get different estimates in different space dimensions, thus we first compute the simpler twodimensional case, for $N = 3$ we only show the main difference. Let $\boldsymbol{\varphi} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))$. Then we can write $\boldsymbol{\varphi}(t,x) = \sum\limits_{k=1}^{\infty} a_k(t)\mathbf{w}^k(x)$, $a_k(t) = \int_\Omega \boldsymbol{\varphi}(t,x)\cdot \mathbf{w}^k(x)\,\mathrm{d}x$. Denote $\boldsymbol{\varphi}^n(t,x) = \sum\limits_{k=1}^{n} a_k(t)\mathbf{w}^k(x)$. It is easy to see that (do it carefully!)

$$\|\boldsymbol{\varphi}^n\|_{L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))} \le \|\boldsymbol{\varphi}\|_{L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))}.$$

Thus

$$\left\|\frac{\partial \mathbf{u}^n}{\partial t}\right\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} = \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \left| \int_0^T \int_\Omega \frac{\partial \mathbf{u}^n}{\partial t}\cdot \boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$= \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \left| \int_0^T \int_\Omega \frac{\partial \mathbf{u}^n}{\partial t}\cdot \boldsymbol{\varphi}^n\,\mathrm{d}x\,\mathrm{d}t\right| \underbrace{=}_{\text{we can use Definition 3.1.3}}$$

$$= \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \left| \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi}^n\rangle\,\mathrm{d}t - \int_0^T\int_\Omega (\mathbf{u}^n\cdot\nabla\mathbf{u}^n)\cdot\boldsymbol{\varphi}^n\,\mathrm{d}x\,\mathrm{d}t \right.$$

$$\left. -\nu\int_0^T\int_\Omega \nabla\mathbf{u}^n : \nabla\boldsymbol{\varphi}^n\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\le \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \Big[\Big(\|\mathbf{f}\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)}$$

$$+\nu\|\nabla\mathbf{u}^n\|_{L^2(0,T;(L^2(\Omega))^N)}\Big)\|\boldsymbol{\varphi}^n\|_{L^2(0,T;(W^{1,2}(\Omega))^N)} + \text{C.T.}\Big].$$

Let us estimate the convective term (C.T.)

$$\left| -\int_0^T\int_\Omega (\mathbf{u}^n\cdot\nabla\mathbf{u}^n)\cdot\boldsymbol{\varphi}^n\,\mathrm{d}x\,\mathrm{d}t\right| = \left|\int_0^T\int_\Omega \mathbf{u}^n\cdot(\mathbf{u}^n\cdot\nabla\boldsymbol{\varphi}^n)\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\le \int_0^T \|\nabla\boldsymbol{\varphi}^n\|_2\|\mathbf{u}^n\|_4^2\,\mathrm{d}t \le C\int_0^T \|\nabla\boldsymbol{\varphi}^n\|_2\|\nabla\mathbf{u}^n\|_2\|\mathbf{u}^n\|_2\,\mathrm{d}t$$

$$\le C\|\mathbf{u}^n\|_{L^\infty(0,T;(L^2(\Omega))^2)}\|\nabla\mathbf{u}^n\|_{L^2(0,T;(L^2(\Omega))^4)}\|\nabla\boldsymbol{\varphi}^n\|_{L^2(0,T;(L^2(\Omega))^4)}.$$

Altogether we have

$$\sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \Big|\int_0^T\int_\Omega \frac{\partial \mathbf{u}^n}{\partial t}\cdot\boldsymbol{\varphi}^n \,\mathrm{d}x\,\mathrm{d}t\Big|$$

$$\le \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} C\Big(\|\mathbf{f}\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} + \nu\,\|\nabla\mathbf{u}^n\|_{L^2(0,T;(L^2(\Omega))^4)}$$

$$+\|\nabla\mathbf{u}^n\|_{L^2(0,T;(L^2(\Omega))^4)}\,\|\mathbf{u}^n\|_{L^\infty(0,T;(L^2(\Omega))^2)}\Big)\,\|\boldsymbol{\varphi}^n\|_{L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))}$$

$$\le C(\mathbf{f},\mathbf{u}_0),$$

and hence

$$N=2 \qquad \left\|\frac{\partial \mathbf{u}^n}{\partial t}\right\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C\left(\mathbf{f},\mathbf{u}_0\right). \tag{3.7}$$

In three space dimensions, the only change appears in the convective term. Hence

$$\int_0^T\int_\Omega |\mathbf{u}^n|^2|\nabla\boldsymbol{\varphi}^n|\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \|\nabla\boldsymbol{\varphi}^n\|_2\,\|\mathbf{u}^n\|_4^2\,\mathrm{d}t$$

$$\le C\int_0^T \|\nabla\boldsymbol{\varphi}^n\|_2\,\|\mathbf{u}^n\|_2^{\frac12}\,\|\nabla\mathbf{u}^n\|_2^{\frac32}\,\mathrm{d}t$$

$$\le C\,\|\mathbf{u}^n\|^{\frac12}_{L^\infty(0,T;(L^2(\Omega))^3)}\,\|\nabla\mathbf{u}^n\|^{\frac32}_{L^2(0,T;(L^2(\Omega))^9)}\,\|\nabla\boldsymbol{\varphi}^n\|_{L^4(0,T;(L^2(\Omega))^9)}$$

and we replace the above estimate by

$$\sup_{\substack{\boldsymbol{\varphi}\in L^4(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \Big|\int_0^T\int_\Omega \frac{\partial \mathbf{u}^n}{\partial t}\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\Big| \le C\left(\mathbf{f},\mathbf{u}_0\right),$$

$$N=3 \qquad \left\|\frac{\partial \mathbf{u}^n}{\partial t}\right\|_{L^{\frac43}(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C\left(\mathbf{f},\mathbf{u}_0\right). \tag{3.8}$$

As we will see later, the lower power (in the integrability over the time variable) in this estimate has big consequences. It was explained above that it corresponds to lower integrability in time of the convective term.

Step (iv) We are now ready for the limit passage. Due to the a priori estimates we know that there exists $\mathbf{u}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\cap L^\infty(0,T;(L^2(\Omega))^N)$ with $\frac{\partial\mathbf{u}}{\partial t}\in L^q(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$ $\big(q=2$ for $N=2$, $q=\frac43$ for $N=3\big)$ such that for a suitable subsequence n_k:

$$\mathbf{u}^{n_k} \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in} \quad L^\infty(0,T;(L^2(\Omega))^N),$$

$$\mathbf{u}^{n_k} \rightharpoonup \mathbf{u} \quad \text{in} \quad L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)),$$

$$\frac{\partial\mathbf{u}^{n_k}}{\partial t} \rightharpoonup \frac{\partial\mathbf{u}}{\partial t} \quad \text{in} \quad L^q(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*).$$

If we take in the Aubin–Lions lemma $X_0 = W^{1,2}_{0,\mathrm{div}}(\Omega)$, $X = L^2_{0,\mathrm{div}}(\Omega)$ and $X_1 = (W^{1,2}_{0,\mathrm{div}}(\Omega))^*$, then for Ω bounded

$$X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1,$$

hence, additionally (generally for another subsequence),

$$\mathbf{u}^{n_k} \longrightarrow \mathbf{u} \quad \text{in} \quad L^2(0,T;(L^2(\Omega))^N).$$

Moreover, due to the a priori bounds of $\mathbf{u}^{n_k}$ in $L^\infty(0,T;(L^2(\Omega))^N)$ and in $L^2(0,T;(W^{1,2}(\Omega))^N)$, we have

$$\mathbf{u}^{n_k} \longrightarrow \mathbf{u} \quad \text{in} \quad L^q(0,T;(L^2(\Omega))^N) \quad \forall q < \infty,$$

$$\mathbf{u}^{n_k} \longrightarrow \mathbf{u} \quad \text{in} \quad L^2(0,T;(L^p(\Omega))^N)$$
$$\forall p < \infty \text{ for } N = 2, \quad \forall p < 6 \text{ for } N = 3.$$

Now, take equality (3.3) for a fixed function $\mathbf{w}^j$. Multiply it by $\psi \in C_0^\infty(0,T)$ and integrate over $(0,T)$. We have (instead of n_k we write again n)

$$\int_0^T \Big\langle \frac{\partial \mathbf{u}^n}{\partial t}, \mathbf{w}^j \Big\rangle \psi \,\mathrm{d}t + \int_0^T \int_\Omega (\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \cdot \mathbf{w}^j \,\mathrm{d}x\, \psi \,\mathrm{d}t$$
$$+\nu \int_0^T \int_\Omega \nabla \mathbf{u}^n : \nabla \mathbf{w}^j \,\mathrm{d}x\, \psi \,\mathrm{d}t = \int_0^T \big\langle \mathbf{f}, \mathbf{w}^j \big\rangle \psi \,\mathrm{d}t,$$

where

$$\Big\langle \frac{\partial \mathbf{u}^n}{\partial t}, \mathbf{w}^j \Big\rangle = \Big\langle \frac{\partial \mathbf{u}^n}{\partial t}, \mathbf{w}^j \Big\rangle_{(W^{1,2}_{0,\mathrm{div}}(\Omega))^*, W^{1,2}_{0,\mathrm{div}}(\Omega)} = \int_\Omega \frac{\partial \mathbf{u}^n}{\partial t} \cdot \mathbf{w}^j \,\mathrm{d}x \quad \forall n \in \mathbf{N}.$$

We now let $n \to \infty$. There is no problem in the linear terms, weak convergence is enough. Thus, let us look at the convective term. We have, due to the strong convergence (we proceed for $N = 3$, for $N = 2$ the situation is simpler)

$$\Big| \int_0^T \int_\Omega \big[(\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \cdot \mathbf{w}^j - (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{w}^j \big] \,\mathrm{d}x\, \psi \,\mathrm{d}t \Big|$$
$$= \Big| \int_0^T \int_\Omega \big[(\mathbf{u} \cdot \nabla \mathbf{w}^j) \cdot \mathbf{u} - \big(\mathbf{u}^n \cdot \nabla \mathbf{w}^j\big) \cdot \mathbf{u}^n \big] \,\mathrm{d}x\, \psi \,\mathrm{d}t \Big|$$
$$\leq \Big| \int_0^T \int_\Omega (u_i - u_i^n) \frac{\partial w_k^j}{\partial x_i} u_k \psi \,\mathrm{d}x \,\mathrm{d}t \Big|$$
$$+ \Big| \int_0^T \int_\Omega u_i^n \frac{\partial w_k^j}{\partial x_i} (u_k - u_k^n) \psi \,\mathrm{d}x \,\mathrm{d}t \Big|$$
$$\leq \int_0^T \|\mathbf{u} - \mathbf{u}^n\|_3 \|\mathbf{u}\|_6 \big\|\nabla \mathbf{w}^j\big\|_2 |\psi| \,\mathrm{d}t$$
$$+ \int_0^T \|\mathbf{u} - \mathbf{u}^n\|_3 \|\mathbf{u}^n\|_6 \big\|\nabla \mathbf{w}^j\big\|_2 |\psi| \,\mathrm{d}t \leq$$
$$\leq \|\mathbf{u} - \mathbf{u}^n\|_{L^2(0,T;(L^3(\Omega))^3)} \big\|\nabla \mathbf{w}^j\big\|_{(L^2(\Omega))^9} \|\psi\|_{L^\infty(0,T)} \times$$
$$\times \big(\|\mathbf{u}\|_{L^2(0,T;(L^6(\Omega))^3)} + \|\mathbf{u}^n\|_{L^2(0,T;(L^6(\Omega))^3)} \big) \longrightarrow 0.$$

The limit function $\mathbf{u}$ fulfils

$$\int_0^T \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{w}^j \Big\rangle \psi \,\mathrm{d}t + \int_0^T \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot \mathbf{w}^j \,\mathrm{d}x\psi\,\mathrm{d}t$$
$$+\nu \int_0^T \int_\Omega \left(\nabla\mathbf{u} : \nabla\mathbf{w}^j\right) \mathrm{d}x\psi\,\mathrm{d}t = \int_0^T \left\langle \mathbf{f}, \mathbf{w}^j \right\rangle \psi\,\mathrm{d}t \qquad (3.9)$$
$$\forall j \in \mathbf{N},\ \forall \psi \in C_0^\infty(0,T).$$

Now, let $\boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, thus $\boldsymbol{\varphi} \in \overline{Lin\,\{\mathbf{w}^j\}_{j=1}^\infty}$ and therefore (formally, it is just another limit passage $\mathbf{w}^n$, $n \to \infty$) equality (3.9) is satisfied for all test functions from $W^{1,2}_{0,\mathrm{div}}(\Omega)$. As the equality is satisfied for all $\psi \in C_0^\infty(0,T)$, it holds

$$\Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \Big\rangle + \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x + \nu\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x = \langle \mathbf{f}, \boldsymbol{\varphi}\rangle$$
$$\forall \boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega) \quad \text{for a.a. } t \in (0,T).$$

Step (v) Take the equality

$$\frac{1}{2}\left\|\mathbf{u}^n(t)\right\|_2^2 + \nu\int_0^t \int_\Omega \left|\nabla\mathbf{u}^n\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \langle \mathbf{f}, \mathbf{u}^n\rangle \,\mathrm{d}\tau - \frac{1}{2}\left\|\mathbf{u}^n(0)\right\|_2^2 = 0,$$

multiply it by $\psi \in C_0^\infty(0,T)$, $\psi \geq 0$ on $[0,T]$ and integrate over $[0,T]$. We have

$$\int_0^T \Big[\frac{1}{2}\left\|\mathbf{u}^n(t)\right\|_2^2 \psi + \nu\int_0^t\int_\Omega \left|\nabla\mathbf{u}^n\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau\psi$$
$$- \int_0^t \langle \mathbf{f}, \mathbf{u}^n\rangle \,\mathrm{d}\tau\psi - \frac{1}{2}\left\|\mathbf{u}^n(0)\right\|_2^2 \psi\Big] \,\mathrm{d}t = 0$$

and let $n \to \infty$. The first term goes due to the strong convergence $\mathbf{u}^n \to \mathbf{u}$ in $L^2(0,T;(L^2(\Omega))^N)$ to $\int_0^T \frac{1}{2}\left\|\mathbf{u}\right\|_2^2 \psi\,\mathrm{d}t$. In the second term we use the lower semicontinuity of the norm and the Fatou lemma. As

$$\liminf_{n\to\infty} \int_0^t \int_\Omega \left|\nabla\mathbf{u}^n\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau \geq \int_0^t\int_\Omega \left|\nabla\mathbf{u}\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau,$$

the function $\psi \geq 0$, we have

$$\liminf_{n\to\infty} \int_0^T \Big(\int_0^t\int_\Omega \left|\nabla\mathbf{u}^n\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)\psi\,\mathrm{d}t$$
$$\geq \int_0^T \Big(\liminf_{n\to\infty}\int_0^t\int_\Omega \left|\nabla\mathbf{u}^n\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau\Big)\psi\,\mathrm{d}t \geq \int_0^T\int_0^t\int_\Omega \left|\nabla\mathbf{u}\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau\psi\,\mathrm{d}t.$$

The third term is simple — weak convergence is enough and the last term goes to $\int_0^T -\frac{1}{2}\left\|\mathbf{u}(0)\right\|_2^2 \psi\,\mathrm{d}t$, due to the completeness of the orthogonal system $\left\{\mathbf{w}^i\right\}_{i=1}^\infty$ in $L^2_{0,\mathrm{div}}(\Omega)$. Altogether we have

$$\int_0^T \Big[\frac{1}{2}\left\|\mathbf{u}(t)\right\|_2^2 + \nu\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \langle \mathbf{f}, \mathbf{u}\rangle\,\mathrm{d}\tau$$
$$-\frac{1}{2}\left\|\mathbf{u}_0\right\|_2^2\Big]\psi(t)\,\mathrm{d}t \leq 0 \quad \forall\psi \in C_0^\infty(0,T), \quad \psi \geq 0 \text{ in } [0,T].$$

By a suitable choice $\psi = \omega_\varepsilon$ — mollifier in time — after the limit $\varepsilon \to 0^+$ we get

$$\frac{1}{2}\left\|\mathbf{u}(t)\right\|_2^2 + \nu \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq \frac{1}{2}\left\|\mathbf{u}_0\right\|_2^2 + \int_0^t \langle \mathbf{f}, \mathbf{u}\rangle \, \mathrm{d}\tau$$

for a.a. $t \in (0,T)$, which is the energy inequality.

Step (vi) Let us now check in which sense the initial condition is satisfied. We proceed as in the existence proof. However, we take $\psi \in C^\infty[0,T]$, $\psi(T) = 0$, we integrate by parts over the time interval $(0,T)$ and get

$$\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{u}^n \cdot \mathbf{w}^j \frac{\partial \psi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}^n(0) \cdot \mathbf{w}^j \psi(0) \, \mathrm{d}x \\
&+ \int_0^T \int_\Omega (\mathbf{u}^n \cdot \nabla \mathbf{u}^n) \cdot \mathbf{w}^j \psi \, \mathrm{d}x \, \mathrm{d}t \\
&+\nu \int_0^T \int_\Omega \nabla \mathbf{u}^n : \nabla \mathbf{w}^j \psi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \left\langle \mathbf{f}, \mathbf{w}^j \right\rangle \psi \, \mathrm{d}t.
\end{aligned}$$

We let $n \to \infty$. Due to the completeness of $\{\mathbf{w}^j\}_{j=1}^\infty$ we get (actually, we proceed in two steps, as in Step (iv))

$$\begin{aligned}
&-\int_0^T \int_\Omega \mathbf{u} \cdot \boldsymbol{\varphi} \frac{\partial \psi}{\partial t} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi} \psi(0) \, \mathrm{d}x + \int_0^T \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\varphi} \psi \, \mathrm{d}x \, \mathrm{d}t \\
&+\nu \int_0^T \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \psi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi} \rangle \psi \, \mathrm{d}t.
\end{aligned}$$

Recall now that

$$\begin{aligned}
\int_0^T \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \Big\rangle \psi \, \mathrm{d}t &= \int_0^T \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{u}, \boldsymbol{\varphi} \rangle}_{= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x} \psi \, \mathrm{d}t \\
&= -\int_0^T \Big(\int_\Omega \mathbf{u} \cdot \boldsymbol{\varphi} \, \mathrm{d}x \Big) \frac{\partial \psi}{\partial t} \, \mathrm{d}t - \underbrace{\int_\Omega \mathbf{u}(0) \cdot \boldsymbol{\varphi} \, \mathrm{d}x \psi(0)}_{\mathbf{u} \in C([0,T];(L^2_w(\Omega))^N)}.
\end{aligned}$$

Choosing $\psi(0) \neq 0$ it yields

$$\int_\Omega \mathbf{u}(0) \cdot \boldsymbol{\varphi} \, \mathrm{d}x = \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\varphi} \, \mathrm{d}x,$$

thus, as $\mathbf{u} \in C([0,T]; (L^2_{0,\mathrm{div}}(\Omega))_w)$,

$$\mathbf{u}(t) \rightharpoonup \mathbf{u}_0 \text{ in } (L^2(\Omega))^N \text{ for } t \longrightarrow 0^+.$$

In particular,

$$\liminf_{t \to 0^+} \left\|\mathbf{u}(t)\right\|_2^2 \geq \left\|\mathbf{u}_0\right\|_2^2 .$$

On the other hand, the energy inequality yields

$$\limsup_{t\to 0^+} \|\mathbf{u}(t)\|_2^2 \le \|\mathbf{u}_0\|_2^2 \Longrightarrow \lim_{t\to 0^+} \|\mathbf{u}(t)\|_2^2 = \|\mathbf{u}_0\|_2^2 .$$

The Hilbert structure of $L^2(\Omega)$ implies $\lim_{t\to 0^+} \|\mathbf{u}(t) - \mathbf{u}_0\|_2^2 = 0$. Note that in two space dimensions we have due to Lemma 2.2.4 that our solution $\mathbf{u} \in C([0,T]; L^2_{0,\mathrm{div}}(\Omega))$ and thus the strong convergence follows immediately.

Step (vii) Let $\mathbf{u}_1, \mathbf{u}_2$ be two different solutions to the Navier–Stokes equations in two space dimensions, corresponding to the initial condition $\mathbf{u}_0$ and the right-hand side $\mathbf{f}$. Then

$$\Big\langle \frac{\partial \mathbf{u}_i}{\partial t}, \boldsymbol{\varphi} \Big\rangle + \nu \int_\Omega \nabla \mathbf{u}_i : \nabla \boldsymbol{\varphi}\,\mathrm{d}x + \int_\Omega (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) \cdot \boldsymbol{\varphi}\,\mathrm{d}x = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle ,$$

$i = 1, 2.$

Subtracting yields

$$\Big\langle \frac{\partial (\mathbf{u}_1 - \mathbf{u}_2)}{\partial t}, \boldsymbol{\varphi} \Big\rangle + \nu \int_\Omega \nabla (\mathbf{u}_1 - \mathbf{u}_2) : \nabla \boldsymbol{\varphi}\,\mathrm{d}x$$
$$+ \int_\Omega (\mathbf{u}_1 \cdot \nabla \mathbf{u}_1 - \mathbf{u}_2 \cdot \nabla \mathbf{u}_2) \cdot \boldsymbol{\varphi}\,\mathrm{d}x = 0.$$

Recall that the difference $\mathbf{u}_1 - \mathbf{u}_2$ belongs to $L^2(0,T; (W^{1,2}_{0,\mathrm{div}}(\Omega))^N) \cap L^\infty(0,T; (L^2(\Omega))^N)$. It can be shown as above that the time derivative $\frac{\partial}{\partial t}(\mathbf{u}_1 - \mathbf{u}_2) \in L^2(0,T; (W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$ and thus

$$\mathbf{u}_1 - \mathbf{u}_2 \in C([0,T]; L^2_{0,\mathrm{div}}(\Omega)) \quad \text{and}$$
$$\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 = 2 \Big\langle \frac{\partial (\mathbf{u}_1 - \mathbf{u}_2)}{\partial t}, \mathbf{u}_1 - \mathbf{u}_2 \Big\rangle,$$

see Lemma 2.2.4. Thus the function $\mathbf{u}_1 - \mathbf{u}_2$ can be used as test function. It reads

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 + \nu \int_\Omega |\nabla (\mathbf{u}_1 - \mathbf{u}_2)|^2 \,\mathrm{d}x$$
$$= \int_\Omega (\mathbf{u}_2 \cdot \nabla \mathbf{u}_2 - \mathbf{u}_1 \cdot \nabla \mathbf{u}_1) \cdot (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}x.$$

Let us rewrite the right-hand side

$$(\text{R.H.S.}) = \underbrace{\int_\Omega -\big(\mathbf{u}_2 \cdot \nabla (\mathbf{u}_1 - \mathbf{u}_2)\big) \cdot (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}x}_{=0}$$
$$+ \int_\Omega (\mathbf{u}_2 - \mathbf{u}_1) \cdot \nabla \mathbf{u}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}x$$
$$\le \|\mathbf{u}_1 - \mathbf{u}_2\|_4^2 \|\nabla \mathbf{u}_1\|_2 \le C \|\mathbf{u}_1 - \mathbf{u}_2\|_2 \|\nabla (\mathbf{u}_1 - \mathbf{u}_2)\|_2 \|\nabla \mathbf{u}_1\|_2 .$$

We have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 + \nu \|\nabla (\mathbf{u}_1 - \mathbf{u}_2)\|_2^2 \le \frac{\nu}{2} \|\nabla (\mathbf{u}_1 - \mathbf{u}_2)\|_2^2$$
$$+ C(\nu) \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 \|\nabla \mathbf{u}_1\|_2^2 , \tag{3.10}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 + \nu \int_\Omega |\nabla(\mathbf{u}_1 - \mathbf{u}_2)|^2 \, \mathrm{d}x \le C(\nu) \|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 \|\nabla \mathbf{u}_1\|_2^2 .$$

As $\|\nabla \mathbf{u}_1\|_2^2 \in L^1(0,T)$ and $(\mathbf{u}_1 - \mathbf{u}_2)(0) = \mathbf{0}$, Gronwall's inequality implies

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_2^2 (t) = 0 \text{ a.e. in } (0,T), \text{ i.e., } \mathbf{u}_1 = \mathbf{u}_2 \text{ a.e. in } (0,T) \times \Omega.$$

Note finally that due to the same arguments as above we can use as test function in two space dimensions the solutions itself which results into the energy equality.

■

3.2 Reconstruction of the pressure

The aim of this part is to find out whether the weak formulation did not destroy the information about the pressure, i.e., whether there exists $p \in \mathcal{D}'((0,T) \times \Omega)$ (or more regular) such that

$$\left\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \right\rangle + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \nu \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x + \langle \nabla p, \boldsymbol{\varphi} \rangle = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle$$
$$\forall \boldsymbol{\varphi} \in (C_0^\infty(\Omega))^N \text{ and a.a. } t \in (0,T). \tag{3.11}$$

Generally, if only $\mathbf{f} \in L^2(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$, it is not evident and the pressure may not exist, see, e.g., paper [35].

We may try to use for $\mathbf{f} \in L^2(0,T;(W^{-1,2}(\Omega))^N)$ the previously proved lemma about the existence of the pressure in the steady case. Consider the functional

$$\left\langle \mathbf{F}, \boldsymbol{\varphi} \right\rangle = \left\langle \frac{\partial \mathbf{u}}{\partial t} - \mathbf{f}, \boldsymbol{\varphi} \right\rangle + \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\varphi} \, \mathrm{d}x + \nu \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \, \mathrm{d}x.$$

However, generally it is not clear, whether $\mathbf{F}$ is a distribution! The reason is that the time derivative $\frac{\partial \mathbf{u}}{\partial t} \in L^q(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)$, but we have no information about it in the space $L^q(0,T;((W_0^{1,2}(\Omega))^*)^N)$.

Remark. Using other boundary conditions, e.g., if only $\mathbf{u} \cdot \mathbf{n} = 0$ (together with, e.g., the slip boundary condition), we would have

$$\boldsymbol{\varphi} \in (W_0^{1,2}(\Omega))^N \Longrightarrow \boldsymbol{\varphi} = \underbrace{\boldsymbol{\varphi}_1}_{\in (W^{1,2}(\Omega))^N,\, \operatorname{div} \boldsymbol{\varphi}_1 = 0,\, \boldsymbol{\varphi}_1 \cdot \mathbf{n} = 0 \text{ on } \partial\Omega} + \nabla \pi,$$
$$\left\langle \frac{\partial \mathbf{u}}{\partial t}, \nabla \pi \right\rangle = 0,$$

and now, we have to verify that $\boldsymbol{\varphi}_1$ is an appropriate test function — considering $\mathbf{u} \cdot \mathbf{n} = 0$ it works. Thus $\frac{\partial \mathbf{u}}{\partial t}$ is a distribution and we can use Lemma on the existence of the pressure (Lemma 2.3.4). For the Cauchy problem or for the periodic boundary conditions we can proceed differently. We can apply on the momentum equation the operator divergence and get the following equation

$$\Delta p = \operatorname{div} \mathbf{f} - \operatorname{div} \operatorname{div}(\mathbf{u} \otimes \mathbf{u}).$$

This problem is in the corresponding spaces uniquely solvable. However, e.g., for the Dirichlet boundary condition on the velocity we would miss a boundary condition for the pressure, thus this approach fails. We can see that the Dirichlet boundary condition complicates the problem of the existence of the pressure.

Nevertheless, it holds

Theorem 3.2.1. *Let $\boldsymbol{u}$ be the weak solution to the Navier–Stokes equations constructed by the Galerkin method, $\Omega \in C^{0,1}$, $N = 2, 3$.*

Then there exists P: $(0,T) \times \Omega \to \mathbf{R}$ such that $P(t) \in L^2(\Omega)$ $\forall t \in (0,T)$ and it satisfies

$$\int_0^t \Big(-\nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{\chi} \,\mathrm{d}x - \int_\Omega (\boldsymbol{u} \cdot \nabla \boldsymbol{u}) \cdot \boldsymbol{\chi} \,\mathrm{d}x + \langle \boldsymbol{f}, \boldsymbol{\chi} \rangle \Big) \,\mathrm{d}\tau$$
$$= \int_\Omega P(t) \operatorname{div} \boldsymbol{\chi} \,\mathrm{d}x + \int_\Omega \boldsymbol{u}(t) \cdot \boldsymbol{\chi} \,\mathrm{d}x - \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\chi} \,\mathrm{d}x \quad \forall \boldsymbol{\chi} \in (W_0^{1,2}(\Omega))^N.$$

Proof. Let us take the formula for the Galerkin approximation, integrate over the time and the term with the time derivative integrate by parts:

$$\int_0^t \int_\Omega \frac{\partial \mathbf{u}^m}{\partial t} \cdot \mathbf{w}^i \,\mathrm{d}x \,\mathrm{d}\tau = \int_\Omega \mathbf{u}^m(t) \cdot \mathbf{w}^i \,\mathrm{d}x - \int_\Omega \mathbf{u}^m(0) \cdot \mathbf{w}^i \,\mathrm{d}x.$$

We have

$$\int_0^t \Big(-\nu \int_\Omega \nabla \mathbf{u}^m : \nabla \mathbf{w}^i \,\mathrm{d}x - \int_\Omega (\mathbf{u}^m \cdot \nabla \mathbf{u}^m) \cdot \mathbf{w}^i \,\mathrm{d}x + \big\langle \mathbf{f}, \mathbf{w}^i \big\rangle \Big) \,\mathrm{d}\tau$$
$$= \int_\Omega \mathbf{u}^m(t) \cdot \mathbf{w}^i \,\mathrm{d}x - \int_\Omega \mathbf{u}^m(0) \cdot \mathbf{w}^i \,\mathrm{d}x \quad \forall \mathbf{w}^i\ ,\ i = 1, .., m.$$

By the limit passage $m \to \infty$ (recall that $\mathbf{u} \in V = \big\{ \mathbf{v} \in L^2(0,T; (W^{1,2}(\Omega)^N)) \cap L^\infty(0,T; (L^2(\Omega))^N)\ \frac{\partial \mathbf{v}}{\partial t} \in L^q(0,T; (W^{1,2}_{0,\mathrm{div}}(\Omega))^*) \big\} \hookrightarrow C([0,T]; (L^2_{0,\mathrm{div}})_w)$) and further by the limit passage "$\mathbf{w}^i \to \boldsymbol{\chi}$" we have (we use the density of finite linear combinations of the basis functions in $W^{1,2}_{0,\mathrm{div}}(\Omega)$)

$$F(\boldsymbol{\chi}) = \int_0^t \Big\{ -\nu \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\chi} - \int_\Omega (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \boldsymbol{\chi} + \langle \mathbf{f}, \boldsymbol{\chi} \rangle \Big\} \,\mathrm{d}\tau$$
$$- \int_\Omega \mathbf{u}(t) \cdot \boldsymbol{\chi} \,\mathrm{d}x + \int_\Omega \mathbf{u}_0 \cdot \boldsymbol{\chi} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{\chi} \in W^{1,2}_{0,\mathrm{div}}(\Omega).$$

Moreover, $F(\boldsymbol{\chi})$ is defined $\forall \boldsymbol{\chi} \in (W_0^{1,2}(\Omega))^N$, $\forall t \in (0,T)$, thus Lemma 2.3.4 yields that

$$\forall t \in (0,T)\ \exists P(t) \in L^2(\Omega): \tag{3.12}$$
$$F(\boldsymbol{\chi}) = \int_\Omega P(t) \operatorname{div} \boldsymbol{\chi} \,\mathrm{d}x \quad \forall \boldsymbol{\chi} \in (W_0^{1,2}(\Omega))^N, \quad N = 2, 3.$$

■

Remark. Generally it is not true that $P(t) = \int_0^t p(\tau) \,\mathrm{d}\tau$, it is not clear that our "pressure" is really a primitive function to the real pressure. Thus this result is not very satisfactory.

In the case when Ω is smooth, it is possible to strengthen this result:[2]

Theorem 3.2.2. *Let $\Omega \in C^2$, the function $\boldsymbol{u} \in L^q(0,T;(L^s(\Omega))^N)$, $\operatorname{div} \boldsymbol{u} = 0$ in the weak sense and the functions $\boldsymbol{H}_i \in L^{q_i}(0,T;(L^{s_i}(\Omega))^{N^2})$, $i = 1,2$ be such that*

$$-\int_0^T \int_\Omega \boldsymbol{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega (\boldsymbol{H}_1 + \boldsymbol{H}_2) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \tag{3.13}$$

for all $\boldsymbol{\varphi} \in (C_0^\infty((0,T)\times\Omega))^N$ with $\operatorname{div} \boldsymbol{\varphi} = 0$. Then there exist scalar functions $p_i \in L^{q_i}(0,T;L^{s_i}(\Omega))$, $i = 1,2$ and a scalar harmonic function $p_h \in L^q(0,T;L^{s^*}(\Omega))$ with $\nabla p_h \in L^q(0,T;(L^s(\Omega))^N)$, $s^* = \frac{Ns}{N-s}$ for $s < N$, $s^* \in [1,\infty)$ for $s = N$ a $s^* \in [1,\infty]$ for $s > N$ such that

$$\begin{aligned} -\int_0^T \int_\Omega \boldsymbol{u} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^T \int_\Omega (\boldsymbol{H}_1 + \boldsymbol{H}_2) : \nabla \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_\Omega (p_1 + p_2) \operatorname{div} \boldsymbol{\varphi} \,\mathrm{d}x\,\mathrm{d}t &+ \int_0^T \int_\Omega \nabla p_h \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.14}$$

for all $\boldsymbol{\varphi} \in (C_0^\infty((0,T)\times\Omega))^N$. Moreover,

$$\|p_i\|_{L^{q_i}(0,T;L^{s_i}(\Omega))} \le C \|\boldsymbol{H}_i\|_{L^{q_i}(0,T;(L^{s_i}(\Omega))^{N^2})}, \quad i = 1,2,$$
$$\|\nabla p_h\|_{L^q(0,T;(L^s(\Omega))^N)} \le C \|\boldsymbol{u}\|_{L^q(0,T;(L^s(\Omega))^N)}.$$

Remark. We can use this theorem in such a way that we take for $\mathbf{H}_1$ the convective term $\mathbf{u} \otimes \mathbf{u}$ and for $\mathbf{H}_2$ the function $-\nu \nabla u - \mathbf{F}$ with $\mathbf{f} = \operatorname{div} \mathbf{F}$. This theorem can be applied for quite general right-hand sides, however, it shows again that the pressure does not behave in the way we could naively expect.

Proof. Choose $t_0 \in (0,T)$, arbitrarily in such a way that t_0 is a Lebesgue point for $\mathbf{u}$, i.e.,

$$\lim_{r \to 0^+} \frac{1}{2r} \int_{t_0 - r}^{t_0 + r} \mathbf{u}(\tau) \,\mathrm{d}\tau = \mathbf{u}(t_0)$$

in $(L^s(\Omega))^N$. We define for $i = 1,2$

$$\widetilde{\mathbf{H}}_i(t) = \int_{t_0}^t \mathbf{H}_i(\tau) \,\mathrm{d}\tau$$

and consider the following Stokes problems

$$\begin{aligned} -\Delta \mathbf{v}_i &= -\nabla \pi_i - \operatorname{div} \widetilde{\mathbf{H}}_i(t) \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v}_i &= 0 \quad \text{in } \Omega, \\ \mathbf{v}_i|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Due to the regularity of the Stokes problem we have for a.a. t and a.a. $h \in (0, T-t)$

$$\frac{1}{h} \|\pi_i(t+h) - \pi_i(t)\|_{s_i} \le \frac{C}{h} \|\widetilde{\mathbf{H}}_i(t+h) - \widetilde{\mathbf{H}}_i(t)\|_{s_i}.$$

[2] Part of this theorem can be shown also for less regular domains, however, it requires deep results from the regularity theory for the Stokes problem in Lipschitz domains.

Therefore $\pi_i \in W^{1,q_i}(0,T;L^{s_i}(\Omega))$ and it holds

$$\Big\|\frac{\partial \pi_i}{\partial t}\Big\|_{L^{q_i}(0,T;L^{s_i}(\Omega))} \le C\|\mathbf{H}_i\|_{L^{q_i}(0,T;L^{s_i}(\Omega))}.$$

Further, for a.e. $t \in (0,T)$ we consider the Stokes problem

$$\begin{aligned} -\Delta \mathbf{v}_h &= -\nabla \pi_h + \mathbf{u}(t_0) - \mathbf{u}(t) \quad \text{in } \Omega,\\ \operatorname{div} \mathbf{v}_h &= 0 \quad \text{in } \Omega,\\ \mathbf{v}_h|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Again, using the regularity of the Stokes problem and integrating over time

$$\|\nabla \pi_h\|_{L^q(0,T;L^s(\Omega))} \le C\|\mathbf{u}\|_{L^q(0,T;L^s(\Omega))}.$$

Evidently, $\Delta \pi_h = 0$ on $(0,T)\times\Omega$. Summing up the Stokes problems above we have for a.e. $t \in (0,T)$

$$-\Delta(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_h) = -\nabla(\pi_1 + \pi_2 + \pi_h) - \operatorname{div}(\widetilde{\mathbf{H}_1} + \widetilde{\mathbf{H}_2}) + \mathbf{u}(t_0) - \mathbf{u}(t). \quad (3.15)$$

If we take in (3.13) as test function $\boldsymbol{\varphi}^n \in C_0^\infty((0,T)\times\Omega)^N$ such that $\boldsymbol{\varphi}^n \to \boldsymbol{\varphi}$, where

$$\boldsymbol{\varphi}(\tau, x) = \begin{cases} \boldsymbol{\psi}(x) \in (C_0^\infty(\Omega))^N & \tau \in (t_0, t),\\ \mathbf{0} & \tau \in (0,T)\setminus(t_0,t), \end{cases}$$

we have

$$\int_\Omega (\mathbf{u}(t) - \mathbf{u}(t_0))\cdot \boldsymbol{\psi}\,\mathrm{d}x = \int_\Omega (\widetilde{\mathbf{H}_1} + \widetilde{\mathbf{H}_2}) : \nabla \boldsymbol{\psi}\,\mathrm{d}x$$

for all $\boldsymbol{\psi} \in (C_0^\infty(\Omega))^N$, $\operatorname{div}\boldsymbol{\psi} = 0$ and due to Lemma 2.3.4 there exists $\pi \in L^r(\Omega)$, $r > 1$ such that

$$\mathbf{u}(t) - \mathbf{u}(t_0) = -\operatorname{div}(\widetilde{\mathbf{H}_1} + \widetilde{\mathbf{H}_2}) + \nabla\pi \qquad \text{in } \mathcal{D}'(\Omega). \quad (3.16)$$

Therefore (3.15) and (3.16) imply

$$\begin{aligned} -\Delta(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_h) &= -\nabla(\pi_1 + \pi_2 + \pi_h - \pi) \quad \text{in } \Omega,\\ \operatorname{div}(\mathbf{v}_1 + \mathbf{v}_2 + \boldsymbol{v}_h) &= 0 \quad \text{in } \Omega,\\ \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_h|_{\partial\Omega} &= \mathbf{0} \end{aligned}$$

and due to the uniqueness of the solution to the steady Stokes problem (for the pressure up to an additive constant; we assume zero integral mean of each of them to avoid this problem) we deduce

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_h = \mathbf{0} \qquad \pi_1 + \pi_2 + \pi_h = \pi.$$

Thus in $\mathcal{D}'(0,T)$

$$p = \frac{\partial \pi}{\partial t} = p_1 + p_2 + \frac{\partial \pi_h}{\partial t},$$

where $p_i = \frac{\partial \pi_i}{\partial t}$. To conclude, we set $p_h = \pi_h$ and use its spatial regularity. ∎

Let us mention another possibility to reconstruct the pressure. To this aim we will have to consider the nonstationary Stokes problem

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p &= \mathbf{g} \qquad \text{in } (0,T)\times\Omega, \\ \operatorname{div} \mathbf{u} &= 0 \qquad \text{in } (0,T)\times\Omega, \\ \mathbf{u} &= \mathbf{0} \qquad \text{on } (0,T)\times\partial\Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 \qquad \text{in } \Omega. \end{aligned}$$

The weak formulation is analogous to the weak formulation for the Navier–Stokes equations. We look for $\mathbf{u} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^\infty(0,T;(L^2(\Omega))^N)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$ such that

$$\Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi} \Big\rangle + \nu \int_\Omega \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} \,\mathrm{d}x = \langle \mathbf{g}, \boldsymbol{\varphi}\rangle \qquad \forall \boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega) \text{ and a.a. } t \in (0,T),$$

and

$$\mathbf{u}(t) \rightharpoonup \mathbf{u}_0 \qquad \text{in } L^2_{0,\mathrm{div}}(\Omega)$$

for $t \to 0^+$. Indeed, it holds (the proof follows the same ideas as the proof of the existence of a solution to the Navier–Stokes equations, it is only slightly simpler)

Theorem 3.2.3. *Let $\boldsymbol{g} \in L^2(0,T;(W^{-1,2}(\Omega))^N)$, $\Omega \subset \mathbf{R}^N$, $\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$.*
Then there exists the unique solution to the nonstationary Stokes problem. Moreover, $\boldsymbol{u} \in C([0,T];L^2_{0,\mathrm{div}}(\Omega))$, hence $\lim_{t\to 0^+} \|\boldsymbol{u}(t) - \boldsymbol{u}_0\|_{(L^2(\Omega))^N} = 0$. ■

The following theorem is in the same spirit as Theorem 3.2.2 (for the proof, see [19]):

Theorem 3.2.4. *Let the initial condition be sufficiently smooth, $\Omega \in C^2$ is convex and let $\boldsymbol{g} = \operatorname{div} \boldsymbol{F}$, $\boldsymbol{F} \in (L^p((0,T)\times\Omega))^{N^2}$, $1 < p < \infty$.*
Then the unique solution to the Stokes problem $\boldsymbol{u} \in L^p(0,T;W^{1,p}_{0,\mathrm{div}}(\Omega)) \cap W^{\frac{1}{2},p}(0,T;(L^p(\Omega))^N)$. Moreover, the pressure

$$\pi = p_1 + \frac{\partial P}{\partial t},$$

where P is a harmonic function, $p_1 \in L^p((0,T)\times\Omega)$, $P \in L^p\big(0,T;W^{2,p}(\Omega)\big)$, $\nabla P \in W^{\frac{1}{2},p}(0,T;(L^p(\Omega))^N)$ and it holds

$$\begin{aligned} &\|\boldsymbol{u}\|_{W^{\frac{1}{2}}(0,T;(L^p(\Omega))^N)} + \|\nabla \boldsymbol{u}\|_{L^p(0,T;(L^p(\Omega))^{N^2})} + \|p_1\|_{L^p(0,T;L^p(\Omega))} \\ &+ \|\nabla P\|_{W^{\frac{1}{2}}(0,T;(L^p(\Omega))^N)} + \|\nabla P\|_{L^p(0,T;(W^{1,p}(\Omega))^N)} \\ &\le C\,\|\boldsymbol{F}\|_{L^p(0,T;(L^p(\Omega))^{N^2})} + C_1(\boldsymbol{u}_0). \end{aligned}$$

Moreover,

$$\|p\|_{W^{1-\frac{1}{2p}-\frac{r}{2},p}(0,T;W_p^{\frac{1}{p}+r}(\Omega))} \le C(\boldsymbol{u}_0, \|\boldsymbol{F}\|_{L^p(0,T;(L^p(\Omega))^{N^2})})\,,\ r \in (0, 1-\tfrac{1}{p}].$$

■

The above result seems to be optimal, i.e., we are not able to avoid the presence of a harmonic part of the pressure which has very low regularity in time. If $\mathbf{f} \in L^t\left(0,T;L^s(\Omega)\right)$, we get much better result, for the proof see [14]

Theorem 3.2.5 (Solonnikov, Giga, Sohr)**.** *Let the initial condition be sufficiently smooth, $\Omega \in C^2$, and let $\boldsymbol{g} \in L^t\left(0,T;(L^s(\Omega))^N\right)$.*

Then the unique solution to the Stokes problem satisfies $\nabla^2\boldsymbol{u}$, $\frac{\partial \boldsymbol{u}}{\partial t}$, $\nabla p \in L^t(0,T;(L^s(\Omega))^k)$ and it holds

$$\left(\left\|\nabla^2\boldsymbol{u}\right\|_X + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_X + \left\|\nabla p\right\|_X \right) \leq C\left(\boldsymbol{u}_0, \|\boldsymbol{g}\|_X\right),$$

where $X = L^t\left(0,T;(L^s(\Omega))^k\right)$, $1 < t, s < \infty$, $k = N^2$ or N, respectively. ■

Remark. This result was originally shown by V.A. Solonnikov for $t = s$, the paper cited above is the extension for $t \neq s$.

We can use these estimates in the following way. We shift the convective term to the right-hand side. Thus

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} - \mathbf{u}\cdot\nabla\mathbf{u}, \\ \operatorname{div}\mathbf{u} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned} \tag{3.17}$$

As our solution to the nonlinear problem $\mathbf{u}$ exists and the solution to the non-stationary Stokes problem is uniquely determined, it is clear that we may apply the estimates from Theorems 3.2.4 a 3.2.5 to our solution. Recall that the result from Theorem 3.2.4 is not very suitable for us, the pressure is not an L^p-function, as the harmonic pressure has low regularity in time and cannot be differentiated with respect to time. Thus we use rather Theorem 3.2.5. The assumptions on $\mathbf{f}$ are not so important, we may take the force term as regular as we need. Let us check in which spaces we control the convective term:

a) $N = 2$

$$\int_\Omega |\mathbf{u}\cdot\nabla\mathbf{u}|^s \,\mathrm{d}x \leq \|\nabla\mathbf{u}\|_2^s \|\mathbf{u}\|_{\frac{2s}{2-s}}^s, \quad 1 < s < 2,$$
$$\|\mathbf{u}\|_{\frac{2s}{2-s}} \leq C\|\mathbf{u}\|_2^{\frac{2-s}{s}} \|\mathbf{u}\|_{1,2}^{\frac{2s-2}{s}},$$

i.e.,

$$\left(\int_0^T \left(\int_\Omega |\mathbf{u}\cdot\nabla\mathbf{u}|^s \,\mathrm{d}x\right)^{\frac{t}{s}} \mathrm{d}\tau\right)^{\frac{1}{t}} \leq C\left(\int_0^T \|\nabla\mathbf{u}\|_2^{t+(2s-2)\frac{t}{s}} \|\mathbf{u}\|_2^{\frac{t}{s}(2-s)}\right)^{\frac{1}{t}}$$
$$\leq C\|\mathbf{u}\|_{L^\infty(0,T;(L^2(\Omega))^2)}^{\frac{1}{s}(2-s)} \|\mathbf{u}\|_{L^2(0,T;(W^{1,2}(\Omega))^2)}^{\frac{3s-2}{s}}$$

assuming that

$$t + (2s-2)\frac{t}{s} = 2 \implies \frac{2}{t} + \frac{2}{s} = 3,\ s < 2.$$

b) $N = 3$

$$\frac{2s}{2-s} \leq 6, \text{ i.e., } s \leq \frac{3}{2}.$$

Then

$$\|\mathbf{u}\|_{\frac{2s}{2-s}} \leq C \, \|\mathbf{u}\|_2^{\frac{3-2s}{s}} \, \|\mathbf{u}\|_{1,2}^{\frac{3s-3}{s}}$$
$$\Big(\frac{2-s}{2s} = \frac{\alpha}{2} + \frac{1-\alpha}{6} \Longrightarrow \alpha = \frac{3-2s}{s} \; , \; 1-\alpha = \frac{3s-3}{s}\Big)$$

and

$$\Big(\int_0^T \big(\int_\Omega |\mathbf{u} \cdot \nabla \mathbf{u}|^s \,\mathrm{d}x\big)^{\frac{t}{s}} \,\mathrm{d}t\Big)^{\frac{1}{t}} \leq C\Big(\int_0^T \|\mathbf{u}\|_{1,2}^{t+(3s-3)\frac{t}{s}} \, \|\mathbf{u}\|_2^{\frac{1}{s}(3-2s)} \,\mathrm{d}t\Big)^{\frac{1}{t}}$$
$$\leq C \, \|\mathbf{u}\|_{L^\infty(0,T;(L^2(\Omega))^3)}^{\frac{1}{s}(3-2s)} \, \|\mathbf{u}\|_{L^2(0,T;(W^{1,2}(\Omega))^3)}^{\frac{4s-3}{s}} \, ,$$

if

$$t + (3s-3)\frac{t}{s} = 2 \Longrightarrow \frac{2}{t} + \frac{3}{s} = 4, \; s \leq \frac{3}{2}.$$

Therefore we have

Theorem 3.2.6. *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations and let $\Omega \in C^2$, $\boldsymbol{f}$ and $\boldsymbol{u}_0$ be sufficiently smooth.*

Then there exists a scalar function, the pressure, and the Navier–Stokes equations are satisfied a.e. in the time-space. Moreover,

$$\nabla^2 \boldsymbol{u}, \; \frac{\partial \boldsymbol{u}}{\partial t}, \; \nabla p \in L^t(0,T;(L^s(\Omega))^k), \; 1 < s < \frac{N}{N-1}, \; k = N^2 \text{ or } N,$$
$$\frac{2}{t} + \frac{N}{s} = N+1, \; N = 2,3.$$

■

Remark. The same result holds true for $N > 3$.

Let us check, what kind of information we got in three space dimensions for the pressure. To fix uniquely the pressure, we will suppose that $\int_\Omega p(x,t)\,\mathrm{d}x = 0$ for a.a. $t \in (0,T)$. We know that

$$\nabla p \in L^t\left(0,T;L^s\right) \qquad \frac{2}{t} + \frac{3}{s} = 4 \; , \; s < \frac{3}{2},$$

i.e.,

$$p \in L^t(0,T;L^{s^*}(\Omega)), \quad s^* = \frac{3s}{3-s}, \qquad \frac{2}{t} + \frac{3}{s^*} = \frac{2}{t} + \frac{3-s}{s}$$
$$\Longrightarrow \frac{2}{t} + \frac{3}{s^*} = 3.$$

If we want to have $t = s^* \Rightarrow \frac{2}{t} + \frac{3}{t} = 3 \Rightarrow t = \frac{5}{3}$, i.e., $p \in L^{\frac{5}{3}}\left((0,T)\times\Omega\right)$. (Check that for $N = 2$ we have $p \in L^q\left((0,T)\times\Omega\right)$ for any $q < 2$!)

3.3 Regularity ($N = 2$)

Let us show now that our uniquely determined weak solution to the Navier–Stokes equations in two space dimensions is more regular provided the data are so. We will prove the following

Theorem 3.3.1. *Let $\Omega \in C^2$, $\boldsymbol{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, $\boldsymbol{f} \in L^2(0,T;(L^2(\Omega))^2)$.*

Then the weak solution to the Navier–Stokes equations in two space dimensions fulfils

$$\nabla^2 \boldsymbol{u}, \frac{\partial \boldsymbol{u}}{\partial t}, \nabla p \in L^2(0,T;(L^2(\Omega))^k), \nabla \boldsymbol{u} \in L^\infty(0,T;(L^2(\Omega))^4),\ k = 4 \,\text{or}\, 2,$$

$$\left\|\nabla^2 \boldsymbol{u}\right\|_{L^2(0,T;(L^2(\Omega))^4)} + \left\|\frac{\partial \boldsymbol{u}}{\partial t}\right\|_{L^2(0,T;(L^2(\Omega))^2)} + \|\nabla p\|_{L^2(0,T;(L^2(\Omega))^2)}$$
$$+ \|\nabla \boldsymbol{u}\|_{L^\infty(0,T;(L^2(\Omega))^4)} \le C(\|\boldsymbol{f}\|_{L^2(0,T;(L^2(\Omega))^2)}, \|\boldsymbol{u}_0\|_{1,2}).$$

In particular, $\boldsymbol{u} \in C([0,T];(W^{1,2}(\Omega))^2)$. If $\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$ only, then the above mentioned estimates hold true on $[\delta,T]$, $\delta > 0$, arbitrarily small.

First, let us prove one lemma

Lemma 3.3.1. *Denote by P the projector from $(L^2(\Omega))^N$ to $L^2_{0,\mathrm{div}}(\Omega)$ (it is sometimes called the Leray projector). Let $\Omega \in C^2$.*

Then

$$\exists C_1, C_2 \colon \forall \boldsymbol{u} \in W^{1,2}_{0,\mathrm{div}}(\Omega) \cap (W^{2,2}(\Omega))^N$$

it holds

$$C_1 \|\boldsymbol{u}\|_{2,2} \le \|P\Delta \boldsymbol{u}\|_2 \le C_2 \|\boldsymbol{u}\|_{2,2}.$$

Proof. Let us consider the Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Without loss of generality we assume $\mathbf{f} \in L^2_{0,\mathrm{div}}(\Omega)$. The problem can be equivalently rewritten as

$$\begin{aligned} -P\Delta \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u}|_{\partial\Omega} &= \mathbf{0}. \end{aligned}$$

Due to the regularity of the solutions to the Stokes problem we know that

$$\|\mathbf{u}\|_{2,2} \le C \|\mathbf{f}\|_2$$

and thus

$$\|\mathbf{u}\|_{2,2} \le C \|\mathbf{f}\|_2 = C \|P\Delta \mathbf{u}\|_2.$$

On the other hand, as P is the projector,

$$\|P\Delta \mathbf{u}\|_2 \le \|\Delta \mathbf{u}\|_2 \le C \|\mathbf{u}\|_{2,2}.$$

■

Proof (of Theorem 3.3.1)**.** Recall that we constructed the solution by the Galerkin approximation. Take the j-th equation, multiply it by $\dot{c}_j^m(t)$, sum over j and integrate over time (i.e., we use as test function for $\mathbf{u}^m$ the time derivative $\frac{\partial \mathbf{u}^m}{\partial t}$). We have

$$\begin{aligned} &\int_0^t \int_\Omega \left|\frac{\partial \mathbf{u}^m}{\partial t}\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\nu \int_0^t \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |\nabla \mathbf{u}^m|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ = &\int_0^t \int_\Omega \mathbf{f}\cdot \frac{\partial \mathbf{u}^m}{\partial t} \,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_\Omega (\mathbf{u}^m \cdot \nabla \mathbf{u}^m)\cdot \frac{\partial \mathbf{u}^m}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

Next, multiply the j-th equation by $\lambda_j c_j^m(t)$, sum over j and integrate over time. Recall that

$$\int_\Omega \nabla \mathbf{w}^j : \nabla \boldsymbol{\varphi}\,\mathrm{d}x = \lambda_j \int_\Omega \mathbf{w}^j \cdot \boldsymbol{\varphi}\,\mathrm{d}x \qquad \forall \boldsymbol{\varphi} \in W^{1,2}_{0,\mathrm{div}}(\Omega),$$

i.e., we use as test function $-P\Delta\mathbf{u}^m$. We have

$$\begin{aligned} &\frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega |\nabla \mathbf{u}^m|^2 \,\mathrm{d}x\,\mathrm{d}\tau + \nu \int_0^t\int_\Omega \nabla \mathbf{u}^m : \Big(\sum_{j=1}^m c_j(\tau)\lambda_j \nabla \mathbf{w}^j\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ = &\int_0^t \int_\Omega \mathbf{f}\cdot \underbrace{\sum_{j=1}^m c_j(\tau)\lambda_j \mathbf{w}^j}_{=-P\Delta\mathbf{u}^m} \,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t \int_\Omega (\mathbf{u}^m\cdot\nabla\mathbf{u}^m)\cdot \Big(\sum_{j=1}^m c_j(\tau)\lambda_j \mathbf{w}^j\Big)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

Compute

$$\begin{aligned} &\int_0^t\int_\Omega \Big(\nabla \mathbf{u}^m : \sum_{j=1}^m c_j(\tau)\lambda_j \nabla\mathbf{w}^j\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\ &= \underbrace{-\int_0^t\int_\Omega \Big(\Delta \mathbf{u}^m : \sum_{j=1}^m c_j(\tau)\lambda_j \mathbf{w}^j\Big)\,\mathrm{d}x\,\mathrm{d}\tau}_{-\int_0^t\int_\Omega (P\Delta\mathbf{u}^m + \nabla z)\left(\sum_{j=1}^m c_j(\tau)\lambda_j\mathbf{w}^j\right)\,\mathrm{d}x\,\mathrm{d}\tau}\,. \end{aligned}$$

Altogether,

$$\begin{aligned} &\int_0^t \Big\|\frac{\partial \mathbf{u}^m}{\partial t}\Big\|_2^2 \,\mathrm{d}\tau + \nu\int_0^t \|\nabla^2\mathbf{u}^m\|_2^2\,\mathrm{d}\tau + \|\nabla\mathbf{u}^m\|_2^2(t) \\ &\le C\Big(\int_0^t \Big\|\frac{\partial \mathbf{u}^m}{\partial t}\Big\|_2^2\,\mathrm{d}\tau + \nu\int_0^t \|P\Delta\mathbf{u}^m\|_2^2\,\mathrm{d}\tau + \frac{1}{2}(1+\nu)\|\nabla\mathbf{u}^m\|_2^2(t)\Big) \\ &\le C\int_0^t \|\mathbf{f}\|_2\Big(\Big\|\frac{\partial \mathbf{u}^m}{\partial t}\Big\|_2 + \|P\Delta\mathbf{u}^m\|_2\Big)\,\mathrm{d}\tau \qquad (3.18) \\ &+C\int_0^t \Big(\int_\Omega |\nabla\mathbf{u}^m|^2|\mathbf{u}^m|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega\Big(\Big(\frac{\partial\mathbf{u}^m}{\partial t}\Big)^2 + (P\Delta\mathbf{u}^m)^2\Big)\,\mathrm{d}x\Big)^{1/2}\,\mathrm{d}\tau \\ &+C\|\nabla\mathbf{u}^m(0)\|_2^2 \le \frac{1}{2}\int_0^t \Big\|\frac{\partial \mathbf{u}^m}{\partial t}\Big\|_2^2\,\mathrm{d}\tau + \frac{1}{2}\nu\int_0^t \|\nabla^2\mathbf{u}^m\|_2^2\,\mathrm{d}\tau \\ &+C(\nu)\int_0^t \|\mathbf{f}\|_2^2\,\mathrm{d}\tau + C\,\|\nabla\mathbf{u}^m(0)\|_2^2 + C(\nu)\underbrace{\int_0^t\int_\Omega |\nabla\mathbf{u}^m|^2\,|\mathbf{u}^m|^2\,\mathrm{d}x\,\mathrm{d}\tau}_{I}\,. \end{aligned}$$

We need to estimate the convective term

$$I \leq C \int_0^t \|\nabla \mathbf{u}^m\|_4^2 \|\mathbf{u}^m\|_4^2 \, \mathrm{d}\tau \leq C \int_0^t \|\nabla \mathbf{u}^m\|_{1,2} \|\nabla \mathbf{u}^m\|_2^2 \|\mathbf{u}^m\|_2 \, \mathrm{d}\tau$$
$$\leq \frac{1}{4}\nu \int_0^t \|\nabla^2 \mathbf{u}^m\|_2^2 + C(\nu) \int_0^t \|\nabla \mathbf{u}^m\|_2^4 \, \mathrm{d}\tau.$$

Thus

$$\frac{1}{2} \int_0^t \left\| \frac{\partial \mathbf{u}^m}{\partial t} \right\|_2^2 \mathrm{d}\tau + \frac{\nu}{4} \int_0^t \|\nabla^2 \mathbf{u}^m\|_2^2 \, \mathrm{d}\tau + \|\nabla \mathbf{u}^m\|_2^2 (t)$$
$$\leq \|\nabla \mathbf{u}^m(0)\|_2^2 + C(\nu) \int_0^t \|\mathbf{f}\|_2^2 \, \mathrm{d}\tau + C(\nu) \int_0^t \|\nabla \mathbf{u}^m\|_2^2 \|\nabla \mathbf{u}^m\|_2^2 \, \mathrm{d}\tau.$$

It follows from the integral form of the Gronwall lemma

$$f(t) \leq f(0) + \int_0^t g(\tau) \, \mathrm{d}\tau + \int_0^t h(\tau) f(\tau) \, \mathrm{d}\tau \Longrightarrow$$
$$f(t) \leq \Big(f(0) + \int_0^t g(\tau) \, \mathrm{d}\tau\Big) \mathrm{e}^{\int_0^t h(\tau) \, \mathrm{d}\tau},$$

choosing $f = \|\nabla \mathbf{u}^m(t)\|_2^2$, $h = \|\nabla \mathbf{u}^m(t)\|_2^2 \in L^1(0,T)$ that

$$\sup_{(0,T)} \|\nabla \mathbf{u}^m(t)\|_2 \leq C\big(\|\nabla \mathbf{u}^m(0)\|_2 , \, \|\mathbf{f}\|_{L^2(0,T;(L^2(\Omega))^2)}, \, T\big).$$

Substitute it back to the estimate above,

$$\int_0^T \left\| \frac{\partial \mathbf{u}^m}{\partial t} \right\|_2^2 \mathrm{d}\tau + \nu \int_0^T \|\nabla^2 \mathbf{u}^m\|_2^2 \, \mathrm{d}\tau + \sup_{(0,T)} \|\nabla \mathbf{u}^m(t)\|_2^2$$
$$\leq C\Big(\|\nabla \mathbf{u}^m(0)\|_2 , \int_0^T \|\mathbf{f}\|_2^2 \, \mathrm{d}\tau, T\Big) \leq C\left(\mathbf{u}_0, \mathbf{f}, T\right),$$

as

$$\|\nabla \mathbf{u}^m(0)\|_2 \leq \|\nabla \mathbf{u}_0\|_2 .$$

Now, use Lemma 3.3.1 and pass with $m \to \infty$. If the information about the initial condition is not sufficient, take

$$\begin{aligned} g(t) &:= 0 \qquad 0 < t < \frac{\delta}{2}, \\ g(t) &:= 1 \qquad t > \delta, \\ g \in C^1\left([0,T]\right), &\qquad g \geq 0 \end{aligned}$$

and before integrating over time, multiply the inequality by g. Then

$$\int_0^t g(\tau) \frac{\mathrm{d}}{\mathrm{dt}} \|\nabla \mathbf{u}^m(\tau)\|_2^2 \, \mathrm{d}\tau = g(t) \|\nabla \mathbf{u}^m(t)\|_2^2 - \int_0^t g'(\tau) \|\nabla \mathbf{u}^m(\tau)\|_2^2 \, \mathrm{d}\tau.$$

We transfer the second term on the right-hand side and continue as above. Due to the properties of g we "lose" the information about the behaviour for times near zero, on the other hand, we do not need to know anything about the gradient of the initial condition. The theorem is proved. ∎

We could also study higher regularity. It is possible to show that for div $\mathbf{u}_0 = 0$, $\mathbf{u}_0 = 0$ on $\partial\Omega$ and $\mathbf{f} \in (C^\infty\left((0,T)\times\overline{\Omega}\right))^N$, $\Omega \in C^\infty$ also the solution $\mathbf{u} \in (C^\infty\left((0,T)\times\overline{\Omega}\right))^2$, $p \in C^\infty\left((0,T)\times\overline{\Omega}\right)$ — attention, not up to time instant 0 — this requires certain compatibility conditions between $\mathbf{u}_0$ and $\mathbf{f}$ and regularity of $\mathbf{u}_0$. We will not continue in this direction, we rather switch to the regularity and uniqueness problem in three space dimensions.

3.4 Uniqueness ($N = 3$)

First, let us recall that we do not know whether all weak solution to the Navier–Stokes equations in three space dimensions satisfy the energy inequality. However, it holds

Lemma 3.4.1. *Let $\boldsymbol{u}$ be a weak solution, which additionally belongs to the space $L^4\left(0,T;(L^4(\Omega))^N\right)$.* [3]

Then $\boldsymbol{u}$ fulfils the energy equality.

Proof. Let us show that if $\mathbf{u}$ is a weak solution to the Navier–Stokes equations and belongs additionally to $L^4\left(0,T;(L^4(\Omega))^N\right)$, then $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$. Indeed,

$$
\begin{aligned}
&\sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \Big| \int_0^T \big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}\big\rangle \,\mathrm{d}\tau \Big| = \\
&\sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \Big| \int_0^T \Big(-\nu \int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x + \langle \mathbf{f},\boldsymbol{\varphi}\rangle - \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\Big)\,\mathrm{d}\tau\Big| \\
&\le \sup_{\substack{\boldsymbol{\varphi}\in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))\\ \|\boldsymbol{\varphi}\|\le 1}} \int_0^T \Big(\|\nabla\mathbf{u}\|_2 \|\nabla\boldsymbol{\varphi}\|_2 + \|\mathbf{f}\|_{-1,2}\|\boldsymbol{\varphi}\|_{1,2} + \|\nabla\boldsymbol{\varphi}\|_2 \|\mathbf{u}\|_4^2\Big)\,\mathrm{d}\tau \\
&\le C.
\end{aligned}
$$

Thus we are allowed to take as test function the solution $\mathbf{u}$ itself as all integrals are finite. It yields (see also Lemma 2.2.4)

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}\|_2^2 + \nu\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x = \underbrace{-\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x}_{=0} + \langle \mathbf{f},\mathbf{u}\rangle\,.
$$

Moreover, $\mathbf{u} \in C([0,T];L^2_{0,\mathrm{div}}(\Omega))$, and thus, integrating over time

$$
\frac{1}{2}\|\mathbf{u}(t)\|_2^2 + \nu\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\|\mathbf{u}_0\|_2^2 + \int_0^t \langle\mathbf{f},\mathbf{u}\rangle\,\mathrm{d}\tau \qquad \forall t\in[0,T]\,.
$$

■

[3]This conditions has been in a certain sense relaxed, see [5]. The condition on the additional regularity can be expressed using the Sobolev–Slobodetskii spaces, i.e., with noninteger derivative, however it is on a weaker scale than the condition from our theorem.

Remark. Recall that in two space dimensions the weak solution to the Navier–Stokes equations belongs to $\mathbf{u} \in L^4(0,T;(L^4(\Omega))^2)$ and thus it satisfies not only the energy inequality, but also the energy equality.

It is generally not known whether the class of weak solutions in three space dimensions is also the uniqueness class. However, we have

Theorem 3.4.1. *Let $\boldsymbol{u}, \boldsymbol{v}$ be two weak solutions to the Navier–Stokes equations corresponding to the same data. Let $\boldsymbol{u}$ satisfy the energy inequality and let $\boldsymbol{v}$ satisfy additionally*

$$\boldsymbol{v} \in L^t(0,T;(L^s(\Omega))^3),\ \frac{2}{t} + \frac{3}{s} = 1,\ s \in [3,\infty]\,.$$

Then $\boldsymbol{u} = \boldsymbol{v}$ a.e. in $(0,T)\times\Omega$.

Remark. It is a uniqueness result of the type strong solution = weak solution. It indicates that the uniqueness and the regularity are closely connected. The conditions on $\mathbf{v}$ from Theorem 3.4.1 are often called the Prodi–Serrin conditions.

Proof (of Theorem 3.4.1)**.** We will perform the proof for $s > 3$. The case $L^\infty(0,T;(L^3(\Omega))^3)$ is technically more complicated. Let us first proceed formally.

Take as test function for $\mathbf{u}$ and $\mathbf{v}$ the difference $\mathbf{u}-\mathbf{v}$ (which we are not allowed to) and subtract the resulting inequalities. We have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}-\mathbf{v}\|_2^2 + \nu\int_\Omega |\nabla(\mathbf{u}-\mathbf{v})|^2\,\mathrm{d}x = \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{u})\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\\
&= \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x - \underbrace{\int_\Omega (\mathbf{u}\cdot\nabla(\mathbf{u}-\mathbf{v}))\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x}_{=0} \qquad (3.19)\\
&- \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x = -\int_\Omega ((\mathbf{u}-\mathbf{v})\cdot\nabla\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\,\mathrm{d}x\\
&= \int_\Omega (\mathbf{u}-\mathbf{v})\otimes\mathbf{v} : \nabla(\mathbf{u}-\mathbf{v})\,\mathrm{d}x.
\end{aligned}$$

We estimate the term on the right-hand side

$$\begin{aligned}
&|\mathrm{C.T.}| \le \int_\Omega \underbrace{|\nabla(\mathbf{u}-\mathbf{v})|}_{2}\,\underbrace{|\mathbf{u}-\mathbf{v}|}_{\frac{2s}{s-2}}\,\underbrace{|\mathbf{v}|}_{s} \le \|\nabla(\mathbf{u}-\mathbf{v})\|_2\,\|\mathbf{u}-\mathbf{v}\|_{\frac{2s}{s-2}}\,\|\mathbf{v}\|_s\\
&\le \|\nabla(\mathbf{u}-\mathbf{v})\|_2^{\frac{s+3}{s}}\,\|\mathbf{u}-\mathbf{v}\|_2^{\frac{s-3}{s}}\,\|\mathbf{v}\|_s\,,\\
&\|\mathbf{u}-\mathbf{v}\|_{\frac{2s}{s-2}} \le \|\mathbf{u}-\mathbf{v}\|_2^{\frac{s-3}{s}}\,\|\mathbf{u}-\mathbf{v}\|_6^{\frac{3}{s}}\,,\\
&\Longrightarrow |\mathrm{C.T.}| \le \frac{1}{2}\nu\,\|\nabla(\mathbf{u}-\mathbf{v})\|_2^2 + C(\nu)\,\|\mathbf{u}-\mathbf{v}\|_2^2\,\|\mathbf{v}\|_s^{\frac{2s}{s-3}}\,.
\end{aligned}$$

(If $s = 3$, this proof does not work.) For $s = \infty$ the convective term can be estimated by

$$\|\mathbf{v}\|_\infty\,\|\mathbf{u}-\mathbf{v}\|_2\,\|\nabla(\mathbf{u}-\mathbf{v})\|_2 \le \frac{1}{2}\nu\,\|\nabla(\mathbf{u}-\mathbf{v})\|_2^2 + C(\nu)\,\|\mathbf{u}-\mathbf{v}\|_2^2\,\|\mathbf{v}\|_\infty^2\,.$$

Thus, altogether

$$\frac{\mathrm{d}}{\mathrm{dt}} \|\mathbf{u}-\mathbf{v}\|_2^2 \leq C \|\mathbf{u}-\mathbf{v}\|_2^2 \|\mathbf{v}\|_s^t,$$

and as $(\mathbf{u}-\mathbf{v})(0) = \mathbf{0}$, it follows from Gronwall's lemma that $\mathbf{u}-\mathbf{v} = \mathbf{0}$.

Let us now try to deduce relation (3.19) rigorously; as a matter of fact, we deduce the form integrated over time and the equality will be replaced by inequality. However, it will be sufficient to conclude the proof.

The first relation we have at our disposal is the energy inequality for $\mathbf{u}$:

$$\frac{1}{2} \|\mathbf{u}(t)\|_2^2 + \nu \int_0^t \int_\Omega |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{1}{2} \|\mathbf{u}_0\|_2^2 + \int_0^t \langle \mathbf{f}, \mathbf{u}\rangle \,\mathrm{d}\tau. \tag{3.20}$$

Further $\mathbf{v} \in L^4(0,T;(L^4(\Omega))^3)$, which follows simply by interpolation and thus according to Lemma 3.4.1

$$\frac{1}{2} \|\mathbf{v}(t)\|_2^2 + \nu \int_0^t \int_\Omega |\nabla \mathbf{v}|^2 \,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2} \|\mathbf{u}_0\|_2^2 + \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle \,\mathrm{d}\tau. \tag{3.21}$$

We have to show that we can take as a test function for $\mathbf{v}$ the function $\mathbf{u}$ and vice versa. From the proof of Lemma 3.4.1 we know that $\frac{\partial \mathbf{v}}{\partial t} \in L^2(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$, thus we may use $\mathbf{u} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega))$ as test function. We have

$$\begin{aligned} &-\int_0^t \Big\langle \frac{\partial \mathbf{v}}{\partial t}, \mathbf{u}\Big\rangle - \int_0^t \int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau - \nu\int_0^t\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau \\ &= -\int_0^t \langle \mathbf{f}, \mathbf{u}\rangle \,\mathrm{d}\tau. \end{aligned} \tag{3.22}$$

It remains the last step, to test the equation for $\mathbf{u}$ by the function $\mathbf{v}$. Let us show first that $\frac{\partial \mathbf{u}}{\partial t} \in \Big(L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^t(0,T;(L^s(\Omega))^3)\Big)^*$:

$$\begin{aligned} \int_0^T \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \boldsymbol{\varphi}\Big\rangle \,\mathrm{d}\tau &= -\nu\int_0^T\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^T\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\quad - \int_0^T \langle \mathbf{f}, \boldsymbol{\varphi}\rangle \,\mathrm{d}\tau. \end{aligned}$$

The first and the third term are estimated in a standard way, for the convective term we have (see above)

$$\begin{aligned} &\Big|\int_0^T\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}\tau\Big| \leq \int_0^T \|\nabla\mathbf{u}\|_2 \|\boldsymbol{\varphi}\|_s \|\mathbf{u}\|_{\frac{2s}{s-2}} \,\mathrm{d}\tau \\ &\leq \Big(\int_0^T \|\nabla\mathbf{u}\|_2^2 \,\mathrm{d}\tau\Big)^{\frac{s+3}{2s}} \Big(\int_0^T \|\boldsymbol{\varphi}\|_s^{\frac{2s}{s-3}} \,\mathrm{d}\tau\Big)^{\frac{s-3}{2s}} \|\mathbf{u}\|^{1-\frac{3}{s}}_{L^\infty(0,T;(L^2(\Omega))^3)}. \end{aligned}$$

The case $s=\infty$ is left as an exercise to the kind reader.

Thus we may test the equation for $\mathbf{u}$ by the function $\mathbf{v}$:

$$\begin{aligned} &-\int_0^t \Big\langle \frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\Big\rangle \,\mathrm{d}\tau - \int_0^t\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}\tau \\ &-\nu\int_0^t\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x\,\mathrm{d}\tau = -\int_0^t \langle \mathbf{f}, \mathbf{v}\rangle \,\mathrm{d}\tau. \end{aligned} \tag{3.23}$$

If we proceed as in the proof of Lemma 2.2.4, we can show (by means of a suitable approximation; see also remark before the proof of the lemma)

$$\int_0^t \Big(\Big\langle\frac{\partial \mathbf{u}}{\partial t},\mathbf{v}\Big\rangle + \Big\langle\frac{\partial \mathbf{v}}{\partial t},\mathbf{u}\Big\rangle\Big)\,\mathrm{d}\tau = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u}\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_\Omega (\mathbf{u}\cdot\mathbf{v})(t)\,\mathrm{d}x - \int_\Omega (\mathbf{u}\cdot\mathbf{v})(0)\,\mathrm{d}x. \tag{3.24}$$

Note that $\mathbf{u} \in C([0,T];(L^2_{0,\mathrm{div}}(\Omega))_w)$ and $\mathbf{v} \in C([0,T];L^2_{0,\mathrm{div}}(\Omega))$, hence the value at zero is well-defined. If we sum (3.20)–(3.23) and use (3.24), we get

$$\frac{1}{2}\,\|\mathbf{u}-\mathbf{v}\|_2^2\,(t) + \nu\int_0^t\int_\Omega |\nabla(\mathbf{u}-\mathbf{v})|^2\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t\int_\Omega (\mathbf{v}\cdot\nabla\mathbf{v})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}\tau$$

and we proceed further as in the formal part of the proof. ■

3.5 Global-in-time conditional regularity ($N = 3$)

We aim to prove the following

Theorem 3.5.1. *Let $\Omega \subset \mathbf{R}^3$, $\Omega \in C^2$, let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations with the initial condition $\boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega)$ and the right-hand side $\boldsymbol{f} \in L^2(0,T;(L^2(\Omega))^3)$. Let additionally $\boldsymbol{u} \in L^t(0,T;(L^s(\Omega))^3)$, $\frac{2}{t}+\frac{3}{s} \le 1$, $s > 3$ or $\|\boldsymbol{u}\|_{L^\infty(0,T;(L^3(\Omega))^3)}$ be sufficiently small.*

Then the weak solution $\boldsymbol{u} \in L^2(\varepsilon,T;(W^{2,2}(\Omega))^3) \cap L^\infty(\varepsilon,T;(W^{1,2}(\Omega))^3)$, $\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(\varepsilon,T;(L^2(\Omega))^3)$ $\forall \varepsilon > 0$. If $\boldsymbol{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, then we may take $\varepsilon = 0$.

We prove the theorem in two steps. Consider the problem

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \nabla\pi &= \mathbf{f}, \qquad (3.25)\\ \operatorname{div}\mathbf{v} &= 0,\\ \mathbf{v}(0,x) &= \mathbf{u}_0(x),\\ \mathbf{v}|_{\partial\Omega} &= \mathbf{0}\end{aligned}$$

(in the weak sense). We first prove

Lemma 3.5.1. *Let $\boldsymbol{u}$, $\boldsymbol{u}_0$, $\boldsymbol{f}$ and Ω fulfil the assumptions of Theorem 3.5.1. Let $\boldsymbol{u} \in L^2(0,T;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^\infty(0,T;L^2_{0,\mathrm{div}}(\Omega))$.*

Then there exists a solution to (3.25) in the weak sense. Furthermore, $\boldsymbol{v} \in L^2\left(\varepsilon,T;(W^{2,2}(\Omega))^3\right) \cap L^\infty\left(\varepsilon,T;(W^{1,2}(\Omega))^3\right)$, $\frac{\partial \boldsymbol{v}}{\partial t} \in L^2(\varepsilon,T;(L^2(\Omega))^3)$. If $\boldsymbol{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, then we may take $\varepsilon = 0$.

Next we show

Lemma 3.5.2. *Let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to the data $\boldsymbol{u}_0$, $\boldsymbol{f}$ and let $\boldsymbol{v}$ be a weak solution to (3.25) corresponding to the same data. Let the assumptions of Theorem 3.5.1 be fulfilled.*

Then $\boldsymbol{u} = \boldsymbol{v}$ a.a. in $(0,T)\times\Omega$.

Indeed, Lemma 3.5.1 and Lemma 3.5.2 imply the proof of Theorem 3.5.1. Recall only (see [8], [9], [29]) that for $\Omega = \mathbf{R}^3$, $\Omega = \mathbf{R}^3_+$ or $\Omega \in C^2$ it is enough to assume $\mathbf{u} \in L^\infty(0,T;(L^3(\Omega))^3)$. We will prove our theorem for $s > 3$, the case $s = 3$ with additional smallness assumptions follows the same lines and is left as an exercise for the kind reader. Note also that having proved Theorem 3.5.1, we could proceed as in the two-dimensional case and get the full regularity. In particular, if the right-hand side and the domain Ω are C^∞, then also the solution is C^∞, however, in general not up to the time instant 0.

Proof (of Lemma 3.5.1)**.** The existence of a solution is shown by means of the Galerkin method. We take the basis formed by eigenvectors of the Stokes problem and we construct the weak solution as in the proof of the existence of a solution to the Navier–Stokes equations. We show ($\int_\Omega \left(\mathbf{u}\cdot\nabla\mathbf{v}^k\right)\cdot\mathbf{v}^k\,\mathrm{d}x = 0!$) that

$$\begin{aligned}&\left\|\mathbf{v}^k\right\|_{L^\infty(0,T;(L^2(\Omega))^3)} + \nu\left\|\nabla\mathbf{v}^k\right\|_{L^2(0,T;(L^2(\Omega))^{3\times3})}\\ &\le C\left(\|\mathbf{f}\|_{L^2(0,T;(L^2(\Omega))^3)}, \|\mathbf{u}_0\|_2\right).\end{aligned}$$

Moreover, as shown several times before, we can prove that

$$\left\|\frac{\partial\mathbf{v}^k}{\partial t}\right\|_{L^2(0,T;(W^{1,2}_{0,\mathrm{div}}(\Omega))^*)} \le C\left(\|\mathbf{f}\|_{L^2(0,T;(L^2(\Omega))^3)}, \|\mathbf{u}_0\|_2\right).$$

Next, exactly as in the proof of the regularity in two space dimensions, we use as test functions $\frac{\partial\mathbf{v}^k}{\partial t}$ and $-P\Delta\mathbf{v}^k$ (i.e., we multiply the j-th equation by $\lambda_j c_j^k(t)$ and $\frac{\mathrm{d}}{\mathrm{dt}}c_j^k(t)$), respectively. We get (see the 2D case)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\nabla\mathbf{v}^k\right\|_2^2 + \nu\left\|P\Delta\mathbf{v}^k\right\|_2^2 &= \int_\Omega\left(\mathbf{u}\cdot\nabla\mathbf{v}^k\right)\cdot P\Delta\mathbf{v}^k\,\mathrm{d}x - \int_\Omega P\Delta\mathbf{v}^k\cdot\mathbf{f}\,\mathrm{d}x\\ \frac{1}{2}\nu\frac{\mathrm{d}}{\mathrm{dt}}\left\|\nabla\mathbf{v}^k\right\|_2^2 + \left\|\frac{\partial\mathbf{v}^k}{\partial t}\right\|_2^2 &= -\int_\Omega\left(\mathbf{u}\cdot\nabla\mathbf{v}^k\right)\cdot\frac{\partial\mathbf{v}^k}{\partial t}\,\mathrm{d}x + \int_\Omega\mathbf{f}\cdot\frac{\partial\mathbf{v}^k}{\partial t}\,\mathrm{d}x.\end{aligned}$$

The term with $\mathbf{f}$ does not cause any troubles, we have to estimate the convective term.

$$\begin{aligned}&\int_\Omega\left(\mathbf{u}\cdot\nabla\mathbf{v}^k\right)\cdot\mathbf{a}\,\mathrm{d}x \le \|\mathbf{a}\|_2\|\mathbf{u}\|_s\left\|\nabla\mathbf{v}^k\right\|_2^{\frac{s-3}{s}}\left\|\nabla\mathbf{v}^k\right\|_{1,2}^{\frac{3}{s}}\\ &\le \varepsilon\|\mathbf{a}\|_2^2 + \varepsilon\left\|P\Delta\mathbf{v}^k\right\|_2^2 + C(\varepsilon)\|\mathbf{u}\|_s^{\frac{2s}{s-3}}\left\|\nabla\mathbf{v}^k\right\|_2^2,\\ &\left(\frac{1}{2}+\frac{3}{2s}+\frac{1}{q} = 1 \Rightarrow q = \frac{2s}{s-3}\right).\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{dt}}\left\|\nabla\mathbf{v}^k\right\|_2^2 + \nu\left\|P\Delta\mathbf{v}^k\right\|_2^2 + \left\|\frac{\partial\mathbf{v}^k}{\partial t}\right\|_2^2 \le C_1\|\mathbf{u}\|_s^{\frac{2s}{s-3}}\left\|\nabla\mathbf{v}^k\right\|_2^2 + C_2\|\mathbf{f}\|_2^2.$$

Now, as in the two-dimensional case, we deduce from the Gronwall inequality (we use a suitable cut-off function in time) the estimates for $\nabla\mathbf{v}^k$, $P\Delta\mathbf{v}^k$ and $\frac{\partial\mathbf{v}^k}{\partial t}$ on (ε,T). The limit passage in the equations is simple as we have stronger estimates than for the Navier–Stokes equations and our system is only linear. If $\mathbf{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, then we may take $\varepsilon = 0$. ∎

Proof (of Lemma 3.5.2)**.** We take $\varepsilon > 0$ fixed. Then

$$\int_\varepsilon^t \int_\Omega \frac{\partial \mathbf{v}}{\partial t} \cdot \boldsymbol{\varphi}_1 \,\mathrm{d}x\,\mathrm{d}\tau + \nu \int_\varepsilon^t \int_\Omega \nabla \mathbf{v} : \nabla \boldsymbol{\varphi}_1 \,\mathrm{d}x\,\mathrm{d}\tau + \int_\varepsilon^t \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{v}) \cdot \boldsymbol{\varphi}_1 \,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_\varepsilon^t \int_\Omega \mathbf{f} \cdot \boldsymbol{\varphi}_1 \,\mathrm{d}x\,\mathrm{d}\tau \qquad \forall \boldsymbol{\varphi}_1 \in W^{1,2}_{0,\mathrm{div}}(\Omega). \tag{3.26}$$

The assumptions on $\mathbf{u}$ ensure that $\mathbf{u} \in (L^4((0,T)\times\Omega))^3$, hence $\mathbf{u}$ fulfils the energy equality. Thus

$$\frac{1}{2}\|\mathbf{u}(t)\|_2^2 - \frac{1}{2}\|\mathbf{u}(\varepsilon)\|_2^2 + \nu \int_\varepsilon^t \|\nabla \mathbf{u}\|_2^2 \,\mathrm{d}\tau = \int_\varepsilon^t \int_\Omega \mathbf{f}\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{3.27}$$

Further,

$$\int_\varepsilon^t \frac{\mathrm{d}}{\mathrm{dt}}\Big(\int_\Omega \mathbf{u}\cdot\boldsymbol{\varphi}_2 \,\mathrm{d}x\Big)\mathrm{d}\tau - \int_\varepsilon^t \int_\Omega \mathbf{u}\cdot\frac{\partial \boldsymbol{\varphi}_2}{\partial t}\,\mathrm{d}x\,\mathrm{d}\tau + \nu\int_\varepsilon^t \int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}_2 \,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \int_\varepsilon^t \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\boldsymbol{\varphi}_2 \,\mathrm{d}x\,\mathrm{d}\tau = \int_\varepsilon^t \int_\Omega \mathbf{f}\cdot\boldsymbol{\varphi}_2\,\mathrm{d}x\,\mathrm{d}\tau \tag{3.28}$$
$$\forall \boldsymbol{\varphi}_2 \in L^2(\varepsilon,T;W^{1,2}_{0,\mathrm{div}}(\Omega)\cap(W^{2,2}(\Omega))^3);\ \frac{\partial\boldsymbol{\varphi}_2}{\partial t} \in L^2(\varepsilon,T;(L^2(\Omega))^3).$$

Choosing

$$\boldsymbol{\varphi}_1 := \mathbf{v}-\mathbf{u},$$
$$\boldsymbol{\varphi}_2 := -\mathbf{v},$$

and summing up (3.26) + (3.27) + (3.28), we have for $\mathbf{w} := \mathbf{u}-\mathbf{v}$

$$\frac{1}{2}\|\mathbf{w}(t)\|_2^2 + \nu\int_\varepsilon^t \int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}x\,\mathrm{d}\tau = \frac{1}{2}\|\mathbf{w}(\varepsilon)\|_2^2$$

(as

$$\int_\varepsilon^t \int_\Omega [(\mathbf{u}\cdot\nabla\mathbf{v})\cdot(\mathbf{v}-\mathbf{u}) - (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{v}]\,\mathrm{d}x\,\mathrm{d}\tau$$
$$= \int_\varepsilon^t \int_\Omega [\mathbf{u}\cdot\nabla\mathbf{v} - \mathbf{u}\cdot\nabla\mathbf{u}]\cdot(\mathbf{v}-\mathbf{u})\,\mathrm{d}x\,\mathrm{d}\tau = \int_\varepsilon^t \int_\Omega (\mathbf{u}\cdot\nabla\mathbf{w})\cdot\mathbf{w}\,\mathrm{d}x\,\mathrm{d}\tau = 0$$

and the term is well defined.). Now we pass with $\varepsilon \to 0^+$. As $\mathbf{u}$ satisfies the energy (in)equality, $\lim_{\varepsilon\to 0^+}\|\mathbf{u}(\varepsilon)\|_2^2 = \|\mathbf{u}_0\|_2^2$. Note that due to the construction we also have ($\mathbf{v}$ fulfils the energy inequality) $\lim_{\varepsilon\to 0^+}\|\mathbf{v}(\varepsilon)\|_2^2 = \|\mathbf{u}_0\|_2^2$. Thus, using also the weak continuity, $\lim_{t\to 0^+}\|\mathbf{u}(t)-\mathbf{u}_0\|_2 = \lim_{t\to 0^+}\|\mathbf{v}(t)-\mathbf{u}_0\|_2 = 0$ which implies $\lim_{\varepsilon\to 0^+}\|(\mathbf{u}-\mathbf{v})(\varepsilon)\|_2^2 = 0$, i.e., $\mathbf{w} = \mathbf{0}$ a.e. in $(0,T)\times\Omega$. ■

3.6 Local-in-time regularity, regularity for small data and concluding remarks ($N = 3$)

Let $\mathbf{f} \in L^2(0,T;(L^2(\Omega))^3)$, $\mathbf{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$. Let us show that $\exists T^* > 0$ such that the solution to the Navier–Stokes equations belongs to $L^\infty(0,T^*;W^{1,2}_{0,\mathrm{div}}(\Omega)) \cap L^2(0,T^*;(W^{2,2}(\Omega))^3)$ and $\frac{\partial\mathbf{u}}{\partial t} \in L^2(0,T^*;(L^2(\Omega))^3)$ (similarly for ∇p).

Theorem 3.6.1. *Let $\Omega \subset \mathbf{R}^3$, $\Omega \in C^2$, $\boldsymbol{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, $\boldsymbol{f} = \boldsymbol{0}$ (for simplicity). Then $\exists T^* = T^*(\nu, \|\boldsymbol{u}_0\|_{1,2}, \Omega)$ such that on $(0, T^*)$ there exists exactly one "regular" solution to the Navier–Stokes equations, especially $T^* \geq \frac{C\nu^3}{\|\nabla \boldsymbol{u}_0\|_2^4}$, $C = C(\Omega)$. Moreover, $\exists G = G(\xi)$, $\xi > 0$ such that for $\|\boldsymbol{u}_0\|_2 \leq G(\|\nabla \boldsymbol{u}_0\|_2)$, T^* can be arbitrary large number. For Ω bounded $G = \frac{C\nu^2}{\|\nabla \boldsymbol{u}_0\|_2}$, $C = C(\Omega)$.*

Remark. The second result of this theorem can be interpreted in two ways. The conclusion holds true, if either the initial condition in $L^2(\Omega)$ is sufficiently small or the viscosity ν is sufficiently large.

Proof. a) "short time"

We proceed as in the construction by means of the Galerkin method in the last theorem. We have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\nabla \mathbf{u}^k\right\|_2^2 + \nu \int_\Omega \left|P\Delta \mathbf{u}^k\right|^2 \,\mathrm{d}x = \int_\Omega \left(\mathbf{u}^k \cdot \nabla \mathbf{u}^k\right) P\Delta \mathbf{u}^k \,\mathrm{d}x.$$

Let us estimate the convective term:

$$\begin{aligned} |\mathrm{C.T.}| &\leq \left\|\mathbf{u}^k\right\|_6 \left\|\nabla \mathbf{u}^k\right\|_3 \left\|P\Delta \mathbf{u}^k\right\|_2 \leq \widetilde{C}(\Omega) \left\|\nabla \mathbf{u}^k\right\|_2^{\frac{3}{2}} \left\|P\Delta \mathbf{u}^k\right\|_2^{\frac{3}{2}} \\ &\leq \frac{1}{2}\nu \left\|P\Delta \mathbf{u}^k\right\|_2^2 + \frac{C(\Omega)}{2}\nu^{-3} \left\|\nabla \mathbf{u}^k\right\|_2^6. \end{aligned}$$

If we set $y = \left\|\nabla \mathbf{u}^k\right\|_2^2$, we have

$$\frac{\mathrm{d}y}{\mathrm{dt}} \leq \underbrace{C(\Omega)\nu^{-3}}_{K} y^3 \Rightarrow \frac{1}{y^2} = -2Kt + \frac{1}{y_0^2} \Rightarrow y^2 = \frac{y_0^2}{1 - 2Kty_0^2}.$$

The solution exists, provided

$$1 - 2KT^* y_0^2 > 0 \Rightarrow T^* < \frac{1}{2Ky_0^2} = \frac{\nu^3}{2C(\Omega)\left\|\nabla \mathbf{u}_0\right\|_2^4}.$$

Testing by $\frac{\partial \mathbf{u}^k}{\partial t}$ and using the estimate on $\Delta \mathbf{u}^k$ we get the required estimates on the time derivative.

b) "long time"

We now estimate the convective term differently

$$\begin{aligned} &|\mathrm{C.T.}| \leq \left\|\mathbf{u}^k\right\|_3 \left\|\nabla \mathbf{u}^k\right\|_6 \left\|P\Delta \mathbf{u}^k\right\|_2 \leq C \left\|\mathbf{u}^k\right\|_2^{\frac{1}{2}} \left\|\nabla \mathbf{u}^k\right\|_2^{\frac{1}{2}} \left\|P\Delta \mathbf{u}^k\right\|_2^2, \\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\nabla \mathbf{u}^k\right\|_2^2 + \left(\nu - C(\Omega)\left\|\mathbf{u}^k\right\|_2^{\frac{1}{2}} \left\|\nabla \mathbf{u}^k\right\|_2^{\frac{1}{2}}\right) \left\|P\Delta \mathbf{u}^k\right\|_2^2 \leq 0. \end{aligned}$$

If $\nu - C(\Omega)\left\|\mathbf{u}(t)\right\|_2^{\frac{1}{2}} \left\|\nabla \mathbf{u}(t)\right\|_2^{\frac{1}{2}} > 0$, then

$$\left\|\nabla \mathbf{u}^k(t)\right\|_2 \leq \left\|\nabla \mathbf{u}_0\right\|_2.$$

But as

$$\left\|\mathbf{u}^k(t)\right\|_2 \leq \left\|\mathbf{u}_0\right\|_2,$$

from the assumption

$$\nu - C(\Omega)\left\|\mathbf{u}_0\right\|_2^{\frac{1}{2}} \left\|\nabla \mathbf{u}_0\right\|_2^{\frac{1}{2}} > 0$$

the inequality

$$\nu - C(\Omega)\left\|\mathbf{u}^k(t)\right\|_2^{\frac{1}{2}}\left\|\nabla\mathbf{u}^k(t)\right\|_2^{\frac{1}{2}} > 0 \qquad \forall t > 0$$

follows.

Thus we get the estimate for $\nabla\mathbf{u}^k$ in $L^\infty(0,T;(L^2(\Omega))^{3\times 3})$ and for $\mathbf{u}^k$ in $L^2(0,T;(W^{2,2}(\Omega))^3)$. The estimate for the time derivative can be shown easily as mentioned above.

■

Remark. If $\Omega \in C^\infty$, $\mathbf{u}_0 \in (C^\infty(\overline{\Omega}))^3$ and $\mathbf{f} \in C^\infty([0,\infty)\times\overline{\Omega})^3$, then $\mathbf{u} \in C^\infty\left((0,T)\times\overline{\Omega}\right)$ (and it is enough $\mathbf{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$ for local regularity in time). However, we cannot expect that $\mathbf{u} \in C^\infty\left([0,T)\times\overline{\Omega}\right)$. Why?

$$\mathbf{u}|_{\partial\Omega\times(0,T)} = \mathbf{0}, \text{ thus necessarily on } \partial\Omega$$
$$\underbrace{\frac{\partial\mathbf{u}_0}{\partial t}}_{=0} + \underbrace{\mathbf{u}_0\cdot\nabla\mathbf{u}_0}_{=0} - \nu\Delta\mathbf{u}_0 + \nabla p(0,x) = \mathbf{f}(0,x).$$

Simultaneously it must hold in Ω

$$\Delta p(0,x) = -\operatorname{div}\operatorname{div}(\mathbf{u}_0\otimes\mathbf{u}_0) + \operatorname{div}\mathbf{f}(0,x),$$

and at $\partial\Omega$

$$\frac{\partial p(0,x)}{\partial\mathbf{n}} = \Delta\mathbf{u}_0\cdot\mathbf{n} + \mathbf{f}(0,x)\cdot\mathbf{n}$$

and there is no reason to hold on $\partial\Omega$: $-\nu\Delta\mathbf{u}_0 + \nabla p(0,x) = \mathbf{f}(0,x)$! The elliptic problem would be overdetermined.

Remark. Inequality of the type

$$\frac{1}{2}\|\mathbf{u}(t)\|_2^2 + \nu\int_\sigma^t \|\nabla\mathbf{u}\|_2^2\,\mathrm{d}\tau \le \frac{1}{2}\|\mathbf{u}(\sigma)\|_2^2$$

for a.a. $\sigma \ge 0$, including $\sigma = 0$, and for all $t \in (\sigma, T]$ is called the strong energy inequality (before we had $\sigma = 0$). For example, for bounded domains such a solution exists and we could basically get it by our construction, but we would have to be more careful in the limit passage in the energy inequality. In such a case it holds

Theorem 3.6.2. *Let $\Omega \in C^\infty$, let $\boldsymbol{u}$ be a weak solution to the Navier–Stokes equations corresponding to $\boldsymbol{f} = \boldsymbol{0}$ and let $\boldsymbol{u}$ fulfil the strong energy inequality.*

Then there exists $\mathcal{T}$ — a union of disjoint time intervals such that

(a) $|(0,\infty)\setminus\mathcal{T}|_1 = 0 \quad \left(\mathcal{H}^{\frac{1}{2}}((0,\infty)\setminus\mathcal{T}) = 0\right)$,

(b) $\boldsymbol{u} \in (C^\infty(\mathcal{T}\times\overline{\Omega}))^3$,

(c) $\exists T^* \in (0,\infty) : (T^*,\infty) \subset \mathcal{T}$,

(d) *If* $\boldsymbol{u}_0 \in W^{1,2}_{0,\mathrm{div}}(\Omega)$, *then* $\exists T_1 > 0 : (0,T_1) \subset \mathcal{T}$.

Chapter 4

Appendix: Solvability of problem $\operatorname{div}\mathbf{u} = f$

4.1 Integral operators

Before we start with the proof of Lemma 2.3.2, we recall several basic results concerning integral operators.

Definition 4.1.1. *Let Ω be a bounded domain and let*

$$K(x, x-y) = \begin{cases} \dfrac{\Theta\left(x, \frac{x-y}{|x-y|}\right)}{|x-y|^\lambda}, & (x,y) \in \Omega \times \Omega, x \neq y, \\ 0, & \textit{otherwise,} \end{cases}$$

where $\Theta \in L^\infty(\Omega \times \partial B_1)$. Let $0 < \lambda < N$. Then

$$T\colon f \mapsto \int_\Omega K(x, x-y) f(y)\,\mathrm{d}y$$

is called a weakly singular integral operator.

It holds (see [37] or [12])

Theorem 4.1.1. *Let $1 < q < \infty$, $\Omega \subset \mathbf{R}^N$ be a bounded domain. Then T: $L^q(\Omega) \to L^q(\Omega)$ and*

$$\|Tf\|_q \leq C(N, \lambda, q)|\Omega|^{\frac{N-\lambda}{N}} \|\Theta\|_{L^\infty(\Omega\times\partial B_1)}\|f\|_q.$$

Definition 4.1.2. *Let*

$$K(x,z) = \frac{\Theta(x, \frac{z}{|z|})}{|z|^N},$$

where $\Theta \in L^\infty(\mathbf{R}^N \times \partial B_1)$. Let

$$\int_{|z|=1} \Theta(x,z)\,\mathrm{d}S_z = 0 \qquad \forall x \in \mathbf{R}^N.$$

Then

$$[Tf](x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|\geq\varepsilon} K(x, x-y) f(y)\,\mathrm{d}y$$

is called a singular integral operator of Calderón–Zygmund type, K is a singular kernel of Calderón–Zygmund type.

It holds (see [37])

Theorem 4.1.2. *Let $1 < q < \infty$ and let T be a singular integral operator of Calderón–Zygmund type. Then $T\colon L^q(\mathbf{R}^N) \to L^q(\mathbf{R}^N)$ and*

$$\|[Tf]\|_q \le C(q,N)\|\Theta\|_{L^\infty(\mathbf{R}^N\times\partial B_1)}\|f\|_q.$$

4.2 Bogovskii operator in bounded domains

4.2.1 Homogeneous boundary condition

We consider problem

$$\begin{aligned} \operatorname{div}\mathbf{v} &= f \qquad && \text{in } \Omega, \\ \mathbf{v} &= 0 && \text{at } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain. We assume $f \in L^q(\Omega)$ for a certain $q > 1$ and Ω sufficiently regular; hence

$$\int_\Omega \operatorname{div}\mathbf{v}\,\mathrm{d}x = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\,\mathrm{d}S = 0,$$

i.e.,

$$\int_\Omega f\,\mathrm{d}x = 0$$

is the necessary condition for the existence of a solution to our problem.

Denote

$$\overline{L^p(\Omega)} = \Big\{f \in L^p(\Omega); \int_\Omega f\,\mathrm{d}x = 0\Big\}.$$

The main result is

Theorem 4.2.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with Lipschitz boundary. Then there exists a linear operator $\mathcal{B}_\Omega = (\mathcal{B}^1_\Omega, \mathcal{B}^2_\Omega, \dots, \mathcal{B}^N_\Omega)$ such that:*

(i)

$$\mathcal{B}_\Omega : \overline{L^p(\Omega)} \to (W^{1,p}_0(\Omega))^N, \quad 1 < p < \infty$$

(ii) For $f \in \overline{L^p(\Omega)}$

$$\operatorname{div}(\mathcal{B}_\Omega(f)) = f \qquad \textit{a.e. in } \Omega$$

(iii) $\exists C = C(p,N,\Omega)$: $\forall f \in \overline{L^p(\Omega)}$ we have

$$\|\nabla\mathcal{B}_\Omega(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty$$

(iv) If $f = \operatorname{div}\boldsymbol{g}$, $\boldsymbol{g} \in E^p_0(\Omega)$, then

$$\|\mathcal{B}_\Omega(f)\|_p \le C(p,N,\Omega)\|\boldsymbol{g}\|_p, \quad 1 < p < \infty$$

(v) If $f \in W_0^{m,p}(\Omega) \cap \overline{L^p(\Omega)}$, $m \geq 0$, *then*

$$\|\nabla \mathcal{B}_\Omega(f)\|_{m,p} \leq C(p,N,\Omega)\|f\|_{m,p}, \quad 1 < p < \infty$$

(vi) If $f \in C_0^\infty(\Omega)$ *(and, indeed,* $\int_\Omega f \,\mathrm{d}x = 0$*), then* $\mathcal{B}_\Omega(f) \in (C_0^\infty(\Omega))^N$.

Remark. Note that Lemma 2.3.2 is a direct consequence of the theorem above. Note also that from the proof it follows that the operator $\mathcal{B}_\Omega$ is linear and independent of p.

Instead of Theorem 4.2.1 we prove another result, where the assumption about the Lipschitz boundary is replaced by the assumption that Ω is star-shaped with respect to a ball B_R. More precisely,

Lemma 4.2.1. *Let* $\Omega \subset \mathbf{R}^N$ *be star-shaped with respect to a ball* $B_R(x_0)$, *where* $\overline{B_R(x_0)} \subset \Omega$. *Then there exists a linear operator*

$$\mathcal{B}_\Omega : \overline{C_0^\infty(\Omega)} = \{f \in C_0^\infty(\Omega); \int_\Omega f \,\mathrm{d}x = 0\} \to (C_0^\infty(\Omega))^N$$

such that

$$\begin{aligned} \operatorname{div} \mathcal{B}_\Omega(f) &= f, & f &\in \overline{C_0^\infty(\Omega)} \\ \|\nabla \mathcal{B}_\Omega(f)\|_q &\leq C(q,N,\Omega)\|f\|_q, & 1 &< q < \infty. \end{aligned}$$

Moreover, if $f = \operatorname{div} \boldsymbol{g}$, $\boldsymbol{g} \in (C_0^\infty(\Omega))^N$, *then*

$$\|\mathcal{B}_\Omega(f)\|_q \leq C(q,N,\Omega)\|\boldsymbol{g}\|_q, \quad 1 < q < \infty.$$

The constant C *has the form*

$$C = C_0(q,N)\Big(\frac{\operatorname{diam}\Omega}{R}\Big)^N \Big(1 + \frac{\operatorname{diam}\Omega}{R}\Big),$$

where $\operatorname{diam}\Omega = \sup_{x,y\in\Omega}|x-y|$.

Remark. Theorem 4.2.1 can be shown using Lemma 4.2.1 as follows. Due to Lemma 2.3.1 we decompose $\Omega \in C^{0,1}$ into several subdomains which are star-shaped with respect to balls, lying inside these subdomains. We decompose function $f \in C_0^\infty(\Omega)$ with zero mean into a sum of functions $f_i \in C_0^\infty(\Omega_i)$ with zero mean over Ω_i and in each Ω_i, we construct $\mathcal{B}_{\Omega_i}$. Then

$$\mathcal{B}_\Omega(f) = \sum_{i=1}^{r+m} \mathcal{B}_{\Omega_i}(f_i)$$

and due to Lemma 2.3.1 the estimates from Lemma 4.2.1 remain valid in the Lipschitz domain. Finally we use the density of compactly supported smooth functions in $L^q(\Omega)$ or in $E_0^p(\Omega)$, respectively. Analogously, using the density of these functions in $W_0^{m,q}(\Omega)$ and after a minor change of the proof of Lemma 4.2.1 we show also the estimate for higher order derivatives, i.e.,

Lemma 4.2.2. *Let* $f \in W_0^{m,q}(\Omega) \cap \overline{L^q(\Omega)}$, *where* $\Omega \subset \mathbf{R}^N$ *is star-shaped with respect to a ball* $B_R(x_0)$ *such that* $\overline{B_R(x_0)} \subset \Omega$ *Then the operator* $\mathcal{B}_\Omega$ *from Lemma 4.2.1 also satisfies*

$$\|\nabla \mathcal{B}_\Omega\|_{m,q} \leq C(q,N,\Omega)\|f\|_{m,q}, \quad m \in \mathbf{N}_0,\ 1 < q < \infty.$$

Proof. The proof is a homework for the kind reader; it is just a modification of the proof of Lemma 4.2.1 below. ■

Proof (of Lemma 4.2.1)**.** The operator div is invariant on translations. Thus it is enough to consider domain Ω, star-shaped with respect to a ball $B_R(0)$ (with the center at the origin). The candidate for the solution is

$$\mathbf{v}(x) = \mathcal{B}_\Omega(f)(x) = \int_\Omega f(y)\frac{x-y}{|x-y|^N}\Big[\int_{|x-y|}^\infty \omega_R\Big(y + s\frac{x-y}{|x-y|}\Big)s^{N-1}\,\mathrm{d}s\Big]\,\mathrm{d}y, \tag{4.1}$$

where

$$\omega_R(x) = \frac{1}{R^N}\omega\Big(\frac{x}{R}\Big),$$

and ω is the standard mollifier, i.e., $\omega \in C_0^\infty(\mathbf{R}^N)$, $\operatorname{supp}\omega \subset \overline{B_1(0)}$ and

$$\int_{\mathbf{R}^N} \omega\,\mathrm{d}x = 1.$$

Then $\operatorname{supp}\omega_R \subset \overline{B_R(0)}$, $\int_\Omega \omega_R\,\mathrm{d}x = 1$ and

$$\|\omega_R\|_{C^0(\mathbf{R}^N)} \le \frac{1}{R^N}\|\omega\|_{C^0(\mathbf{R}^N)}, \qquad \|\nabla\omega_R\|_{C^0(\mathbf{R}^N)} \le \frac{1}{R^{N+1}}\|\nabla\omega\|_{C^0(\mathbf{R}^N)}.$$

We rewrite (4.1) into several equivalent forms. Using the change of variable $r = \frac{s}{|x-y|}$ we have

$$\mathbf{v}(x) = \int_\Omega f(y)(x-y)\Big(\int_1^\infty \omega_R\big(y + r(x-y)\big)r^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}y, \tag{4.2}$$

and the change of variables $s = |x-y| + r$ leads to

$$\mathbf{v}(x) = \int_\Omega f(y)\frac{(x-y)}{|x-y|^N}\Big(\int_0^\infty \omega_R\Big(x + r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}y. \tag{4.3}$$

As $f \in C_0^\infty(\Omega)$, we can replace the integration over Ω by the integration over $\mathbf{R}^N$. The change of variables $z = x - y$ in (4.2) gives

$$\mathbf{v}(x) = \int_{\mathbf{R}^N} f(x-z)z\Big(\int_1^\infty \omega_R(x - z + zr)r^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}z. \tag{4.4}$$

In the proof we can use any of the above given equivalent forms. Taking arbitrary derivative with respect to x in (4.4) it follows that we have $\mathcal{B}_\Omega(f) \in (C^\infty(\Omega))^N$. Let us show that $\operatorname{supp}\mathcal{B}_\Omega(f) \subset A$, where

$$A = \{z \in \Omega; z = \lambda z_1 + (1-\lambda)z_2, z_1 \in \operatorname{supp} f, z_2 \in \overline{B_R(0)}, \lambda \in [0,1]\}.$$

(Recall that Ω is star-shaped with respect to all points z_2 and thus the line (z_1, z_2) is contained in Ω.) Let $x \in \Omega\setminus A$. Then $y + r(x-y) \notin \overline{B_R(0)}$ for $r \ge 1$, $y \in \operatorname{supp} f$, as for $w = y + r(x-y)$ we have $x = y(1-\frac{1}{r}) + w\frac{1}{r}$. Thus $\mathcal{B}_\Omega(f)(x) = 0$ for $x \in \Omega \setminus A$. As A is a compact set, we have shown that $\mathcal{B}_\Omega(f) \in (C_0^\infty(\Omega))^N$.

We now compute derivatives.

$$\begin{aligned}\frac{\partial v_j(x)}{\partial x_i} &= \int_{\mathbf{R}^N} \frac{\partial f(x-z)}{\partial x_i} z_j \Big(\int_1^\infty \omega_R(x-z+zr) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}z \\ &+ \int_{\mathbf{R}^N} f(x-z) z_j \Big(\int_1^\infty \frac{\partial \omega_R}{\partial x_i}(x-z+rz) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}z \\ &= \int_{B_\varepsilon(0)} \Big\{ \frac{\partial f(x-z)}{\partial x_i} z_j \Big(\int_1^\infty \omega_R(x-z+zr) r^{N-1}\,\mathrm{d}r \Big) \\ &+ f(x-z) z_j \Big(\int_1^\infty \frac{\partial \omega_R}{\partial x_i}(x-z+rz) r^{N-1}\,\mathrm{d}r \Big) \Big\}\,\mathrm{d}z \\ &+ \int_{B^\varepsilon(0)} f(x-z) \Big\{ \delta_{ij} \Big(\int_1^\infty \omega_R(x-z+zr) r^{N-1}\,\mathrm{d}r \Big) \\ &+ z_j \Big(\int_1^\infty \frac{\partial \omega_R}{\partial x_i}(x-z+rz) r^{N}\,\mathrm{d}r \Big) \Big\}\,\mathrm{d}z \\ &+ \int_{\partial B_\varepsilon(0)} f(x-z) z_j \frac{z_i}{|z|} \Big(\int_1^\infty \omega_R(x-z+zr) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}S_z \\ &= (I^1_\varepsilon)_{ij}(x) + (I^2_\varepsilon)_{ij}(x) + (I^3_\varepsilon)_{ij}(x).\end{aligned}$$

Evidently

$$\lim_{\varepsilon\to 0^+} (I^1_\varepsilon)_{ij} = 0.$$

In the second integral, we use change of variables $y = x - z$ and then similar procedure as from (4.1) to (4.3):

$$\begin{aligned}&(I^2_\varepsilon)_{ij}(x) = \\ &\int_{B^\varepsilon(x)} \Big[f(y) \frac{\delta_{ij}}{|x-y|^N} \Big(\int_0^\infty \omega_R\Big(x + r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1}\,\mathrm{d}r \Big) \Big]\,\mathrm{d}y + \\ &\int_{B^\varepsilon(x)} \Big[f(y) \frac{x_j - y_j}{|x-y|^{N+1}} \Big(\int_0^\infty \frac{\partial}{\partial \xi_i}\omega_R\Big(x + r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N}\,\mathrm{d}r \Big) \Big]\,\mathrm{d}y.\end{aligned}$$

We rewrite the term $(|x-y|+r)^N$ and analogous term in the first integral by means of binomial theorem. We write separately the term without $|x-y|$. As

$$0 < r < (R + \operatorname{diam}\Omega) \le 2\operatorname{diam}\Omega, \qquad |x-y| < \operatorname{diam}\Omega,$$

it is possible to estimate

$$\begin{aligned}&\int_0^\infty \frac{\partial}{\partial \xi_i}\omega_R\Big(x + r\frac{x-y}{|x-y|}\Big) r^{N-k}|x-y|^{k-1}\,\mathrm{d}r \le C \max_{x\in B_R} |\nabla\omega_R| (\operatorname{diam}\Omega)^N, \\ &\int_0^\infty \omega_R\Big(x + r\frac{x-y}{|x-y|}\Big) r^{N-1-k}|x-y|^{k-1}\,\mathrm{d}r \le C \max_{x\in B_R} |\omega_R| (\operatorname{diam}\Omega)^{N-1}.\end{aligned}$$

Altogether,

$$(I^2_\varepsilon)_{ij} = \int_{B^\varepsilon(x)} K_{ij}(x, x-y) f(y)\,\mathrm{d}y + \int_{B^\varepsilon(x)} G_{ij}(x,y) f(y)\,\mathrm{d}y,$$

where $K_{ij}(x,z) = \frac{\Theta_{ij}(x, \frac{z}{|z|})}{|z|^N}$ with

$$\Theta_{ij}\Big(x, \frac{z}{|z|}\Big) = \delta_{ij} \int_0^\infty \omega_R\Big(x + r\frac{z}{|z|}\Big) r^{N-1}\,\mathrm{d}r + \frac{z_j}{|z|} \int_0^\infty \frac{\partial}{\partial \xi_i}\omega_R\Big(x + r\frac{z}{|z|}\Big) r^N\,\mathrm{d}r$$

and

$$|G_{ij}(x,y)| \le \frac{C(N)}{|x-y|^{N-1}} \frac{(\operatorname{diam}\Omega)^{N-1}}{R^N}\Big(1+\frac{\operatorname{diam}\Omega}{R}\Big), \tag{4.5}$$

$x, y \in \Omega$.

We can rewrite the third integral

$$\begin{aligned}(I_\varepsilon^3)_{ij}(x) \\ = \int_{\partial B_\varepsilon(0)} (f(x-z)-f(x)) z_j \frac{z_i}{|z|}\Big(\int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}S_z \\ + f(x)\int_{\partial B_\varepsilon(0)} z_j \frac{z_i}{|z|}\Big(\int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}S_z.\end{aligned}$$

First, let us look at the second term. We change the variables $z = \varepsilon w$

$$\begin{aligned}{}_2(I_\varepsilon^3)_{ij} &= f(x)\varepsilon^N \int_{\partial B_1(0)} w_j w_i \Big(\int_1^\infty \omega_R(x-\varepsilon w+r\varepsilon w) r^{N-1}\,\mathrm{d}r\Big)\,\mathrm{d}S_w \\ [\varepsilon(r-1)=t] &= f(x)\varepsilon^{N-1}\int_{\partial B_1(0)} w_j w_i \Big(\int_0^\infty \omega_R(x+tw)\Big(\frac{t}{\varepsilon}+1\Big)^{N-1}\,\mathrm{d}t\Big)\,\mathrm{d}S_w \\ &= f(x)\int_{\mathbf{R}^N} \frac{w_i w_j}{|w|^2}\omega_R(x+w)\,\mathrm{d}w + o(1) \quad \text{for } \varepsilon \to 0^+.\end{aligned}$$

As the first term contains additional term tending to zero ($|f(x-z)-f(x)| \le C|z| \to 0$ for $\varepsilon \to 0^+$), we have

$$\lim_{\varepsilon\to 0^+} (I_\varepsilon^3)_{ij}(x) = f(x) H_{ij}(x),$$

where

$$H_{ij}(x) = \int_{\mathbf{R}^N} \frac{w_i w_j}{|w|^2}\omega_R(x+w)\,\mathrm{d}w.$$

Altogether

$$\frac{\partial v_j(x)}{\partial x_i} = \lim_{\varepsilon\to 0^+}\int_{B^\varepsilon(x)} K_{ij}(x,x-y) f(y)\,\mathrm{d}y + \int_{\mathbf{R}^N} G_{ij}(x,y) f(y)\,\mathrm{d}y + f(x) H_{ij}(x),$$

$x \in \Omega$. As

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\Big[\omega_R\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\Big] = \frac{x_k-y_k}{|x-y|}\frac{\partial}{\partial \xi_k}\omega_R\Big(x+r\frac{x-y}{|x-y|}\Big)\times \\ \times(|x-y|+r)^N + N\omega_R\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^{N-1},\end{aligned}$$

we get

$$\begin{aligned}(I_\varepsilon^2)_{ii}(x) \\ = \int_{B^\varepsilon(x)} \frac{f(y)}{|x-y|^N}\Big(\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\Big[\omega_R\Big(x+r\frac{x-y}{|x-y|}\Big)(|x-y|+r)^N\Big]\,\mathrm{d}r\Big)\,\mathrm{d}y \\ = -\omega_R(x)\int_{B^\varepsilon(x)} f(y)\,\mathrm{d}y \underset{\varepsilon\to 0^+}{\longrightarrow} -\omega_R(x)\int_\Omega f(y)\,\mathrm{d}y = 0.\end{aligned}$$

Further

$$H_{ii}(x) = \int_{\mathbf{R}^N} \omega_R(x+w)\,\mathrm{d}w = 1,$$

and thus

$$\operatorname{div}\mathbf{v}(x) = f(x), \qquad x \in \Omega.$$

It remains to show the estimates. Due to (4.5) G_{ij} is a weakly singular kernel and due to Theorem 4.1.1 ($|\Omega|^{\frac{1}{N}} \le \operatorname{diam}\Omega$)

$$\begin{aligned}\Big\| \int_{\mathbf{R}^N} G_{ij}(\cdot,y) f(y)\,\mathrm{d}y \Big\|_q &\le \Big\| \int_{\Omega} G_{ij}(\cdot,y) f(y)\,\mathrm{d}y \Big\|_q \\ &\le C(q,N)\Big(\frac{\operatorname{diam}\Omega}{R}\Big)^N \Big(1 + \frac{\operatorname{diam}\Omega}{R}\Big)\|f\|_q.\end{aligned}$$

Next

$$\begin{aligned} &\int_{|z|=1} \Theta_{ij}(x,z)\,\mathrm{d}S_z \\ = &\int_{|z|=1} \Big(\delta_{ij}\int_0^\infty \omega_R(x+zr) r^{N-1}\,\mathrm{d}r + z_j \int_0^\infty \frac{\partial}{\partial \xi_i}\omega_R(x+rz) r^N\,\mathrm{d}r\Big)\,\mathrm{d}S_z \\ &= \int_{\mathbf{R}^N} \Big[\delta_{ij}\omega_R(x+y) + y_j \frac{\partial}{\partial y_i}\omega_R(x+y)\Big]\,\mathrm{d}y = 0. \end{aligned}$$

Similarly as in the estimates of the weakly singular integrals

$$\sup_{x,z\in\mathbf{R}^N} \Big|\Theta\Big(x, \frac{z}{|z|}\Big)\Big| \le C(N)\|\omega\|_{C^1(\mathbf{R}^N)}\Big(\frac{\operatorname{diam}\Omega}{R}\Big)^N \Big(1 + \frac{\operatorname{diam}\Omega}{R}\Big);$$

hence

$$\begin{aligned} &\lim_{\varepsilon\to 0^+} \Big\| \int_{B^\varepsilon(\cdot)} K_{ij}(\cdot,\cdot - y) f(y)\,\mathrm{d}y \Big\|_q \\ \le\ & C(q,N,\omega)\Big(\frac{\operatorname{diam}\Omega}{R}\Big)^N \Big(1 + \frac{\operatorname{diam}\Omega}{R}\Big)\|f\|_q. \end{aligned}$$

The last term

$$\sup_{x\in\Omega}|H_{ij}(x)| = \sup_{x\in\Omega}\int_{\mathbf{R}^N} \frac{w_i w_j}{|w|^2}\omega_R(x+w)\,\mathrm{d}w \le \int_{\mathbf{R}^N} \omega_R(y)\,\mathrm{d}y = 1,$$

and thus the estimate

$$\|\nabla \mathcal{B}_\Omega(f)\|_q \le c_0(q,N)\Big(\frac{\operatorname{diam}\Omega}{R}\Big)^N \Big(1 + \frac{\operatorname{diam}\Omega}{R}\Big)\|f\|_q$$

is proved.

It remains to show the estimate for $f = \operatorname{div}\mathbf{g}$, $\mathbf{g} \in (C_0^\infty(\Omega))^N$. We plug in the form of f to (4.4). We have

$$\begin{aligned} v_j(x) = &\int_{B_\varepsilon(0)} \mathrm{div}_x\, \mathbf{g}(x-z) z_j \Big(\int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}z \\ &+ \int_{B^\varepsilon(0)} \mathrm{div}_x\, \mathbf{g}(x-z) z_j \Big(\int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}z \\ = &\int_{B_\varepsilon(0)} \mathrm{div}_x\, \mathbf{g}(x-z) z_j \Big(\int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}z \\ &+ \int_{B^\varepsilon(0)} \mathbf{g}(x-z) \Big[\delta_{ij} \int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r \\ &+ z_j \int_1^\infty \frac{\partial}{\partial \xi_i} \omega_R(x-z+rz)(r-1) r^{N-1}\,\mathrm{d}r \Big]\,\mathrm{d}z \\ &+ \int_{\partial B_\varepsilon(0)} \Big(g_i(x-z) z_j \frac{z_i}{|z|} \int_1^\infty \omega_R(x-z+rz) r^{N-1}\,\mathrm{d}r \Big)\,\mathrm{d}S_z \\ = &(J^1_\varepsilon)_{ij}(x) + (J^2_\varepsilon)_{ij}(x) + (J^3_\varepsilon)_{ij}(x), \quad \varepsilon > 0. \end{aligned}$$

Similarly as above we conclude

$$v_j(x) = \lim_{\varepsilon \to 0^+} \int_{B^\varepsilon(x)} K_{ij}(x, x-y) g_i(y)\,\mathrm{d}y + \int_{\mathbf{R}^N} \widetilde{G}_{ij}(x,y) g_i(y)\,\mathrm{d}y + H_{ij}(x) g_i(x),$$

$x \in \Omega$, where K_{ij} is the same as above, $\widetilde{G}_{ij}$ is a weakly singular kernel satisfying the same estimate as G_{ij}. Hence we get the same estimate as in the previous case. ■

Remark. In our case, i.e., for a bounded domain, we can show for $p < N$ estimate

$$\|\mathbf{v}\|_{\frac{Np}{N-p}} \le C \|f\|_p, \qquad 1 < p < N,$$

hence (for $p \ge N$ using Friedrichs inequality)

$$\|\mathbf{v}\|_{1,p} \le C \|f\|_p, \qquad 1 < p < \infty.$$

However, the constant C depends on Ω through the constant from Friedrichs inequality or from the embedding $L^{\frac{Np}{N-p}}(\Omega) \hookrightarrow L^p(\Omega)$.

4.2.2 Inhomogeneous boundary condition

We solve the problem

$$\begin{aligned} \mathrm{div}\, \mathbf{u} &= f \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{a} \quad \text{on } \partial\Omega \end{aligned} \tag{4.6}$$

with the compatibility condition

$$\int_\Omega f\,\mathrm{d}x = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\,\mathrm{d}S.$$

Theorem 4.2.2. *Let $\Omega \in C^{0,1}$ be a bounded domain. Thus there exists a linear operator $\widetilde{\mathcal{B}}_\Omega = (\widetilde{\mathcal{B}}^1_\Omega, \widetilde{\mathcal{B}}^2_\Omega, \ldots, \widetilde{\mathcal{B}}^N_\Omega)$ such that for $f \in L^p(\Omega)$, $\boldsymbol{a} \in (W^{1-\frac{1}{p},p}(\partial\Omega))^N$, satisfying the compatibility condition*

$$\int_\Omega f\,\mathrm{d}x = \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{n}\,\mathrm{d}S$$

it holds

$$\operatorname{div} \widetilde{\mathcal{B}}_\Omega(f, \boldsymbol{a}) = f \quad \text{a.e. in } \Omega, \qquad T(\widetilde{\mathcal{B}}_\Omega(f, \boldsymbol{a})) = \boldsymbol{a},$$

where T is the trace operator. Moreover, there exists a constant C, dependent only on the dimension N, exponent p and domain Ω such that

$$\|\widetilde{\mathcal{B}}_\Omega(f, \boldsymbol{a})\|_{1,p} \le C(N, p, \Omega)\big(\|f\|_p + \|\boldsymbol{a}\|_{1-\frac{1}{p},p,\partial\Omega}\big), \quad 1 < p < \infty.$$

Proof. We denote by $\mathbf{A}$ the extension of $\mathbf{a}$ to $(W^{1,p}(\Omega))^N$ due to the inverse trace theorem, i.e., $\mathrm{T}\,\mathbf{A} = \mathbf{a}$, $\|\mathbf{A}\|_{1,p,\Omega} \le C(p, N, \Omega)\|\mathbf{a}\|_{1-\frac{1}{p},p,\partial\Omega}$. We set

$$\mathbf{v} = \widetilde{\mathcal{B}}_\Omega(f, \mathbf{a}) = \mathcal{B}_\Omega(f - \operatorname{div} \mathbf{A}) + \mathbf{A},$$

i.e., we look for the solution in the form $\mathbf{v} = \mathbf{w} + \mathbf{A}$, where $\operatorname{div} \mathbf{w} = f - \operatorname{div} \mathbf{A}$, $\mathbf{w} = \mathbf{0}$ at $\partial\Omega$. Evidently

$$\operatorname{div} \mathbf{v} = f, \qquad \mathrm{T}\,\mathbf{v} = \mathrm{T}\,\mathbf{A} = \mathbf{a}$$

and it holds

$$\begin{aligned}\|\mathbf{v}\|_{1,p} &\le C(p, N, \Omega)\big(\|\mathbf{A}\|_{1,p,\Omega} + \|f - \operatorname{div} \mathbf{A}\|_{p,\Omega}\big) \\ &\le C(p, N, \Omega)\big(\|f\|_p + \|\mathbf{a}\|_{1-\frac{1}{p},p,\partial\Omega}\big).\end{aligned}$$

■

Recall that for the inhomogeneous boundary condition the Lipschitz boundary is important in order to define the trace of a function. For the homogeneous boundary condition certain regularity is also needed. The following construction is due to Luc Tartar and shows that for domains with cusps the solution may not exist, even for $p = 2$.

Consider $\Omega \subset \mathbf{R}^2$ with the boundary formed by two parabolas $y = x^2$ and $y = -x^2$, $0 < x < 1$, and by the arc of the circle $y^2 + (x-1)^2 = 1$, $1 \le x \le 2$. Let $\mathbf{u} = (u_1, u_2)$ be a solution to

$$\operatorname{div} \mathbf{u} = f,$$

where $f \in \overline{L^2(\Omega)}$. We show that in general, $|\nabla \mathbf{u}|$ may not be in $L^2(\Omega)$. Assume the contrary and we aim to get a contradiction. We set

$$g(x) = \int_{-x^2}^{x^2} \Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\Big)\,\mathrm{d}y = \int_{-x^2}^{x^2} \frac{\partial u_1}{\partial x}(x, y)\,\mathrm{d}y,$$

as $u_2 \in W_0^{1,2}(\Omega)$. Further we define for a.a. $x \in (0, 1)$

$$A(x) = \int_{-x^2}^{x^2} u_1(x, y)\,\mathrm{d}y.$$

Then, since $u_1 \in W_0^{1,2}(\Omega)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}A(x) &= \int_{-x^2}^{x^2} \frac{\partial}{\partial x}u_1(x, y)\,\mathrm{d}y + 2x(u_1(x, x^2) + u_1(x, -x^2)) \\ &= \int_{-x^2}^{x^2} \frac{\partial}{\partial x}u_1(x, y)\,\mathrm{d}y.\end{aligned}$$

Thus $g(x) = A'(x)$. But $A(0) = 0$, hence

$$A(x) = \int_0^x g(s)\,\mathrm{d}s. \tag{4.7}$$

Further, as $u_1(x, -x^2) = 0$ for a.a. $x \in (0,1)$,

$$u_1(x,y) = \int_{-x^2}^{y} \frac{\partial}{\partial \tau} u_1(x,\tau)\,\mathrm{d}\tau.$$

Using Fubini's theorem

$$\begin{aligned} A(x) = &\int_{-x^2}^{x^2} \Big(\int_{-x^2}^{y} \frac{\partial}{\partial \tau} u_1(x,\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}y = \int_{-x^2}^{x^2} \Big(\int_{\tau}^{x^2} \frac{\partial}{\partial \tau} u_1(x,\tau)\,\mathrm{d}y\Big)\,\mathrm{d}\tau \\ = &\int_{-x^2}^{x^2} (x^2 - \tau) \frac{\partial}{\partial \tau} u_1(x,\tau)\,\mathrm{d}\tau, \end{aligned}$$

thus

$$|A(x)| \le \frac{2^{3/2}}{3^{1/2}} x^3 \Big(\int_{-x^2}^{x^2} \Big(\frac{\partial u_1}{\partial y}(x,y)\Big)^2\,\mathrm{d}y\Big)^{\frac{1}{2}}.$$

Therefore

$$\frac{A(x)}{x^3} \in L^2(0,1),$$

as

$$\int_0^1 \Big(\frac{A(x)}{x^3}\Big)^2\,\mathrm{d}x \le \frac{8}{3} \int_0^1 \int_{-x^2}^{x^2} \Big(\frac{\partial u_1}{\partial y}(x,y)\Big)^2\,\mathrm{d}y\,\mathrm{d}x$$

and $u_1 \in W^{1,2}(\Omega)$.

Take $f(x) = x^\alpha$, $x \in (0,1)$, extended into Ω in such a way that it holds $\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$. As $f \in L^2(\Omega)$, we must have

$$\int_0^1 \int_{-x^2}^{x^2} x^{2\alpha}\,\mathrm{d}y\,\mathrm{d}x < \infty,$$

i.e.,

$$\int_0^1 x^{2\alpha+2}\,\mathrm{d}x < \infty,$$

and thus $2\alpha + 2 > -1$. Hence our f belongs to $L^2(\Omega)$ if and only if $\alpha > -\frac{3}{2}$.

On the other hand,

$$g(x) = -\int_{-x^2}^{x^2} \operatorname{div} \mathbf{u}(x,y)\,\mathrm{d}y = \int_{-x^2}^{x^2} f(x,y)\,\mathrm{d}y = 2x^{\alpha+2}$$

for $x \in (0,1)$. Hence, due to (4.7),

$$A(x) = \int_0^x 2y^{\alpha+2}\,\mathrm{d}y = \frac{2}{\alpha+3} x^{\alpha+3}$$

if $\alpha > -3$ which is fulfilled due to assumptions on f. The condition $\frac{A(x)}{x^3} \in L^2(0,1)$ gives

$$x^\alpha \in L^2(0,1)$$

which is fulfilled for $\alpha > -\frac{1}{2}$ and this is a stronger restriction than the condition following from $f \in L^2(\Omega)$ (i.e., $\alpha > -\frac{3}{2}$). It means that the condition $f \in \overline{L^2(\Omega)}$ is not sufficient to guarantee the existence of $\mathbf{u} \in (W_0^{1,2}(\Omega))^2$ solving our problem.

4.3 Unbounded domains

Denote for $1 \le p < \infty$

$$D_0^{1,p}(\Omega) = \overline{\{u \in C_0^\infty(\Omega)\}}^{\|\nabla \cdot\|_p}.$$

Note that for Ω bounded we get the space $W_0^{1,p}(\Omega)$, hence a nontrivial situation appears for Ω unbounded. Further, for $1 \le p < N$ and $\partial\Omega \in C^{0,1}$ (if non-empty) we have (see, e.g., [12])

$$D_0^{1,p}(\Omega) = \{u \in L^1_{loc}(\Omega); \nabla u \in (L^p(\Omega))^N; u \in L^{\frac{Np}{N-p}}(\Omega); \mathrm{T}\, u|_{\partial\Omega} = 0\}.$$

If $p \ge N$, then

$$D_0^{1,p}(\Omega) = \{u \in L^p_{loc}(\Omega); \nabla u \in (L^p(\Omega))^N; \mathrm{T}\, u|_{\partial\Omega} = 0\}.$$

For $\Omega = \mathbf{R}^N$, $p \ge N$ it is

$$D_0^{1,p}(\mathbf{R}^N) = \{u = \{\tilde{u} + C\}_{C \in \mathbf{R}}; \tilde{u} \in L^p_{loc}(\mathbf{R}^N); \nabla \tilde{u} \in (L^p(\mathbf{R}^N))^N\}.$$

4.3.1 Whole space

In this case the solution to $\operatorname{div} \mathbf{u} = f$ is extremely simple. We can look for the solution in the form $\mathbf{u} = \nabla\psi$, i.e.,

$$\Delta\psi = f \qquad \text{in } \mathbf{R}^N;$$

hence

$$\mathcal{B}_{\mathbf{R}^N}(f) = \nabla\mathcal{E} * f,$$

where $\mathcal{E}$ is the fundamental solution to the Laplace equation.

We have (the proof is easy and is left as exercise)

Theorem 4.3.1. *The operator* $\mathcal{B}_{\mathbf{R}^N} \colon L^p(\mathbf{R}^N) \to (D_0^{1,p}(\mathbf{R}^N))^N$, $1 < p < \infty$. *For* $f \in L^p(\mathbf{R}^N)$ *it holds*

$$\operatorname{div} \mathcal{B}_{\mathbf{R}^N}(f) = f \qquad a.e.\ in\ \mathbf{R}^N$$

and

$$\|\nabla\mathcal{B}_{\mathbf{R}^N}(f)\|_p \le C(p,N)\|f\|_p, \quad 1 < p < \infty.$$

If $f \in C_0^\infty(\mathbf{R}^N)$, *then* $\mathcal{B}_{\mathbf{R}^N}(f) \in (C^\infty(\mathbf{R}^N))^N$ *and*

$$|\mathcal{B}_{\mathbf{R}^N}(f)|(x) \le \frac{C(p,N,R)}{|x|^{N-1}}$$

for all $x \in B^R(0)$, $R > 0$.

4.3.2 Exterior domains

Theorem 4.3.2. *Let* Ω *be an exterior domain with Lipschitz boundary. Then there exists a linear operator* $\mathcal{B}_\Omega = (\mathcal{B}^1_\Omega, \mathcal{B}^2_\Omega, \dots, \mathcal{B}^N_\Omega)$ *such that*

(i) $\mathcal{B}_\Omega \colon L^p(\Omega) \to (D_0^{1,p}(\Omega))^N$, $1 < p < \infty$

(ii) $\operatorname{div} \mathcal{B}_\Omega(f) = f$ *a.e. in* Ω, $f \in L^p(\Omega)$

(iii) $\|\nabla \mathcal{B}_\Omega(f)\|_p \leq C(p,\Omega)\|f\|_p$, $1 < p < \infty$

(iv) If $f \in C_0^\infty(\Omega)$, *then* $\mathcal{B}_\Omega(f) \in (C^\infty(\Omega))^N$ *and* $|\mathcal{B}_\Omega(f)|(x) \leq \frac{C(p,N,R)}{|x|^{N-1}}$ *for* $x \in B^R(0)$, $R > R_0$ *such that* $\Omega^c \subset B_{R_0}(0)$.

Proof. Without loss of generality assume $f \in C_0^\infty(\Omega)$, i.e., we use the density property. We extend f by zero in such a way that $f \in C_0^\infty(\mathbf{R}^N)$. We set

$$\mathbf{v} = \mathbf{u} + \mathbf{w},$$

where

$$\mathbf{u} = \left\{ \begin{array}{rr} \nabla \mathcal{E} * f, & 1 < p < N \\ \nabla \mathcal{E} * f - \dfrac{1}{|\Omega_{R_0}|} \displaystyle\int_{\Omega_{R_0}} \nabla \mathcal{E} * f \, \mathrm{d}x, & N \leq p < \infty \end{array} \right\} \in (D_0^{1,p}(\mathbf{R}))^N,$$

i.e., $\operatorname{div} \mathbf{u} = f$ in $\mathbf{R}^N$. Thus $\operatorname{div} \mathbf{w} = 0$ in Ω and we have to choose $\mathbf{w}$ to eliminate the nonzero value of $\mathbf{u}$ at $\partial\Omega$. Hence

$$\begin{aligned} \operatorname{div} \mathbf{w} = 0 \qquad & \text{in } \Omega_{R_0} = \Omega \cap B_{R_0}(0), \\ \mathbf{w} = -\mathbf{u} \qquad & \text{at } \partial\Omega, \\ \mathbf{w} = \mathbf{0} \qquad & \text{at } \partial B_{R_0}(0). \end{aligned}$$

As

$$\int_{\partial\Omega} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S + \int_{\partial B_{R_0}} 0 \, \mathrm{d}S = -\int_{\partial\Omega^c} \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = -\int_{\Omega^c} \operatorname{div} \mathbf{u} \, \mathrm{d}x = 0,$$

the compatibility condition is fulfilled and $\mathbf{w}$ exists due to Theorem 4.2.2. We extend $\mathbf{w}$ by zero outside $B_{R_0}(0)$. We have fulfilled (ii), (iv) and it remains to show the estimates. Evidently,

$$\|\nabla \mathbf{u}\|_{p,\mathbf{R}^N} \leq C\|f\|_p.$$

Further, due to the Poincaré inequality,

$$\begin{aligned} \|\nabla \mathbf{w}\|_{p,\mathbf{R}^N} \leq \|\nabla \mathbf{w}\|_{p,\Omega_{R_0}} &\leq C(p,N,\Omega_{R_0})\|\operatorname{Tr} \mathbf{u}\|_{1-\frac{1}{p},p,\partial\Omega} \\ &\leq C\|\mathbf{u}\|_{1,p,\Omega_{R_0}} \leq C\|\nabla \mathbf{u}\|_{p,\Omega_{R_0}} \leq C\|f\|_p. \end{aligned}$$

■

4.3.3 Domains with noncompact boundaries

Consider the domain

$$\Omega = \{x \in \mathbf{R}^N; x^N > F(x_1, \ldots, x_{N-1}) = F(x')\},$$

where F is *globally* Lipschitz function. Without loss of generality assume $F(\mathbf{0}) = 0$. As an example we may take

$$\Omega = \{x \in \mathbf{R}^N; x^N > (|x'| + 1)^\alpha - 1\}, \quad \alpha \leq 1.$$

Hence domain Ω contains inside a cone. Denote for $M > 0$

$$C_\vartheta^+ = \{x \in \mathbf{R}^N; x_N > M|x'|\}.$$

Due to assumptions above domain Ω contains such a cone for a certain M. It holds (the following theorem is taken from [36])

Theorem 4.3.3 (Solonnikov)**.** *Let Ω be a domain of the type above. Then there exists operator $\mathcal{B}_\Omega = (\mathcal{B}^1_\Omega, \mathcal{B}^2_\Omega, \dots, \mathcal{B}^N_\Omega)$ such that*

(i) $\mathcal{B}_\Omega : L^p(\Omega) \to (D_0^{1,p}(\Omega))^N$, $1 < p < \infty$

(ii) $\operatorname{div} \mathcal{B}_\Omega(f) = f$ *a.e. in* Ω, $f \in L^p(\Omega)$

(iii) $\|\nabla \mathcal{B}_\Omega(f)\|_p \le C(p,N)\|f\|_p$, $1 < p < \infty$

(iv) If $f \in C_0^\infty(\Omega)$, *then* $\mathcal{B}_\Omega(f) \in (C^\infty(\Omega))^N$ *and* $|\mathcal{B}_\Omega(f)|(x) \le \frac{C(p,N,\Omega)}{|x|^{N-1}}$ *for* $x \in \Omega^R$, $R > 0$.

Proof. Set $C_x = C^-_{x,\vartheta} = \{y \in \mathbf{R}^N; y - x \in C^-_{0,\vartheta} = -C^+_{0,\vartheta}\}$. Then $C_x \subset \mathbf{R}^N \setminus \Omega$. Let $f \in C_0^\infty(\Omega)$. Extend f by zero outside Ω and set

$$\begin{aligned}\mathbf{v}(x) = \mathcal{B}_\Omega(f)(x) &= \int_{C_x} \frac{x-y}{|x-y|^N}\omega\Big(\frac{x-y}{|x-y|}\Big) f(y)\,\mathrm{d}y \\ &= \int_{C^+_{0,\vartheta}} \frac{z}{|z|^N}\omega\Big(\frac{z}{|z|}\Big) f(x-z)\,\mathrm{d}z,\end{aligned}$$

where $\omega \in C_0^1(\partial B_1(0) \cap C^+_{0,\vartheta})$, $\int_{\partial B_1(0)} \omega \,\mathrm{d}S = 1$.

The rest of the proof is analogous as above and thus it is left as exercise for interested reader. ∎

4.3.4 Applications

We want to prove the density of smooth functions with compact support in $W^{1,p}_{0,\mathrm{div}}(\Omega)$; for Ω exterior domain the proof is more complicated than for Ω bounded domain. We give the proof for $\Omega = \mathbf{R}^N$, when all complications coming from unboundedness of the domain appear, we only save the problems near the boundary which we have solved in the case of bounded domain before.

Let $\mathbf{u} \in W^{1,p}_{0,\mathrm{div}}(\mathbf{R}^N)$. Take $R \gg 1$ and set $\mathbf{u}_R = \mathbf{u}\eta_R$, where η_R is a cut-off function such that

$$\eta_R(x) = \begin{cases} 1 & x \in B_R(0), \\ 0 & x \in B^{2R}(0), \end{cases}$$

$0 \le \eta_R \le 1$, $\nabla \eta_R \le \frac{C}{R}$. Evidently, $\lim_{R\to\infty} \mathbf{u}_R = \mathbf{u}$ in $(W^{1,p}(\mathbf{R}^N))^N$. The function u_R has already compact support, but it has generally non-zero divergence. Hence we set

$$\begin{aligned}\operatorname{div} \mathbf{v}_R = \operatorname{div} \mathbf{u}_R \qquad &\text{in } B_{2R}(0) \setminus B_R(0), \\ \mathbf{v}_R|_{\partial B_R(0)} = \mathbf{v}_R|_{\partial B_{2R}(0)} = \mathbf{0}&.\end{aligned}$$

Such a solution evidently exists and fulfils

$$\|\nabla \mathbf{v}_R\|_{p,B_{2R}(0)\setminus B_R(0)} \le C\|\operatorname{div} \mathbf{u}_R\|_{p,B_{2R}(0)\setminus B_R(0)},$$

where C is independent of R. Thus

$$\|\nabla \mathbf{v}_R\|_{p,B_{2R}(0)\setminus B_R(0)} \le C\|\mathbf{u}\cdot\nabla\eta_R\|_{p,B_{2R}(0)\setminus B_R(0)} \le \frac{C}{R}\|\mathbf{u}\|_{p,B_{2R}(0)\setminus B_R(0)}.$$

The Poincaré inequality implies

$$\|\mathbf{v}_R\|_{p,B_{2R}(0)\setminus B_R(0)} \le CR\|\nabla \mathbf{v}_R\|_{p,B_{2R}(0)\setminus B_R(0)} \le C\|\mathbf{u}\|_{p,B_{2R}(0)\setminus B_R(0)}.$$

Set $\mathbf{w}_R = \mathbf{u}_R - \mathbf{v}_R$. Thus $\mathbf{w}_R$ has compact support $(B_{2R}(0))$, fulfils

$$\operatorname{div} \mathbf{w}_R = \operatorname{div} \mathbf{u}_R - \operatorname{div} \mathbf{v}_R = 0$$

in $\mathbf{R}^N$ and it holds

$$\begin{aligned}\|\mathbf{w}_R - \mathbf{u}\|_{1,p,\mathbf{R}^N} &\leq \|\mathbf{u}(1-\eta_R)\|_{1,p,B_{2R}(0)\setminus B_R(0)} + \|\mathbf{v}_R\|_{1,p,B_{2R}(0)\setminus B_R(0)} \\ &\leq C\|\mathbf{u}\|_{1,p,B_{2R}(0)\setminus B_R(0)} \longrightarrow 0\end{aligned}$$

for $R \to +\infty$. Now it is enough to take mollification

$$\mathbf{w}_{R,n} = \omega_{\frac{1}{n}} * \mathbf{w}_R.$$

For a suitable sequence $R_n \to +\infty$ we have $\mathbf{w}_{R_n,n} \in (C_0^\infty(\mathbf{R}^N))^N$, $\operatorname{div} \mathbf{w}_{R_n,n} = 0$ (the mollification commutes with the divergence), and

$$\lim_{n\to\infty} \mathbf{w}_{R_n,n} = \mathbf{u}$$

in $(W^{1,p}(\mathbf{R}^N))^N$.

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