NMFM401 – Mathematics of Non-Life Insurance 1

3. Severity III – Models for the Amount of a Single Payment

A Exercises

1. A classic probability puzzler goes as follows (this version is from Robertson, J. (1994): It's a puzzlement. *The actuarial review*, 21(3), p. 20.) "You are asked to select one of two envelopes and told that one of the envelopes contains twice as much money as the other. You find 100 in the envelope you select. You are offered the opportunity to select the other envelope, but then you must keep only the money in the second envelope. Is it better to switch or stand?" Construct a Bayesian analysis as follows: Let Θ be the smaller of the two amounts and have prior density $\pi(\theta)$. The other envelope contains the amount 2Θ. The amount in the envelope you select is $X = \Theta$ with probability 0.5 and $X = 2\Theta$ with probability 0.5.

(a) Determine the posterior expected value of each of the two strategies.

(b) Show that if the prior density is $\pi(\theta) = 1/\theta$ the two strategies have the same conditional (on the value of X) expected value.

(c) Show that if the prior density is $\pi(\theta) = \exp\{-\theta\}$, there will be some values of X for which switching produces a higher conditional expected value and some for which it is lower. Determine the set of X values for which switching is superior.

2. A random sample of size 100 has been taken from a gamma distribution

$$f(x; \alpha, \theta) = \frac{(x/\theta)^\alpha \exp\{-x/\theta\}}{x\Gamma(\alpha)}, \quad x > 0,\ \alpha > 0,\ \theta > 0$$

with α known to be 2, but θ unknown. For this sample, $\sum_{j=1}^{100} X_j = 30{,}000$. The prior distribution for θ is inverse gamma with β taking the role of α and λ taking the role of θ.

(a) Determine the exact posterior distribution of θ. At this point the values of β and λ have yet to be specified.

(b) The population mean is 2θ. Determine the posterior mean of 2θ first using the prior distribution with $\beta = \lambda = 0$ (this is equivalent to $\pi(\theta) = \theta^{-1}$) and then with $\beta = 2$ and $\lambda = 250$ (which is a prior mean 250). Then, in each case, construct a 95% credibility interval with 2.5% probability on each side.

(c) Determine the posterior variance of 2θ and use the Bayesian CLT to construct a 95% credibility interval for 2θ using each of the two prior distributions given in the previous part.

(d) Determine the maximum likelihood estimate of θ and then use the estimated variance to construct a 95% confidence interval for 2θ.

B Homework

1. The following amounts were paid on a hospital liability policy: 125, 132, 141, 107, 133, 319, 126, 104, 145, and 223. The amount of a single payment has the single-parameter Pareto distribution

$$f(x; \alpha, \theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad x > \theta,\ \alpha > 0,\ \theta > 0$$

with $\theta = 100$ and α unknown. The prior distribution has the gamma distribution with $\alpha = 2$ and $\theta = 1$.

(a) Determine all the relevant Bayesian quantities.

(b) Determine the three Bayes estimates of α considering the squared-error loss, the absolute loss, and the zero-one loss.

(c) Show that if Y is the predictive distribution, then $\log Y - \log 100$ has the Pareto distribution.

2. Let $X_1, \ldots, X_n$ be a random sample from a lognormal distribution

$$f(x; \mu, \sigma) = \frac{\exp\{-z^2/2\}}{x\sqrt{2\pi\sigma^2}}, \quad z = \frac{\log x - \mu}{\sqrt{\sigma^2}}, \quad x > 0,\, \mu \in \mathbb{R},\, \sigma^2 > 0$$

with unknown parameters μ and σ. Let the prior density be $\pi(\mu, \sigma) = \sigma^{-1}$.

(a) Write the posterior pdf of μ and σ up to a constant of proportionality.

(b) Determine the Bayesian estimators of μ and σ by using the posterior mode.

(c) Fix σ at the posterior mode as determined in previous part and then determine the exact (conditional) pdf of μ. The use it to determine a 95% HPD (highest posterior density) credibility interval for μ.

(Version: October 14, 2013)

Source

[1] Klugman, Stuart A., Panjer, Harry H., and Willmont, Gordon E. *Loss Models: From Data to Decisions.* John Wiley & Sons, Inc., New York, NY, 4th Edition, 2012.