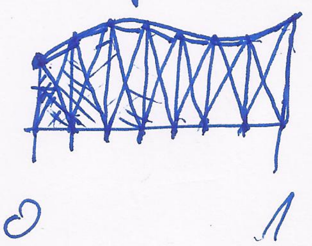
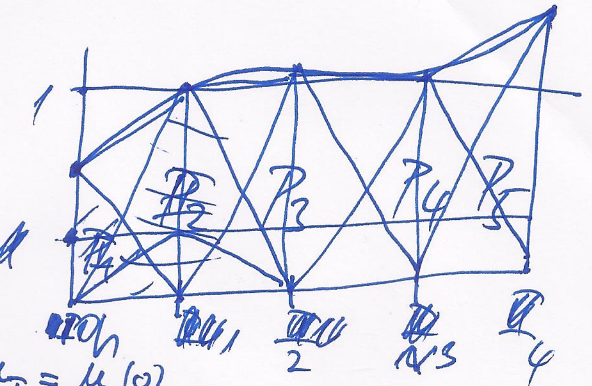
$\cdot$ Aproximace fcí $u(x)$ na $[0,1]$





$u_I = u(0), \ldots$

$$u_h = \sum_{I=0}^{4} u_I P_I \quad \text{po částech lineární}$$

$\text{span}\{P_I\} = V_h$

$\cdot$ Poissonova rce 1D

$$\frac{\partial^2 u(x)}{\partial x^2} = f(x) \quad \text{na } \Omega = [0,1]$$

O.P.: $u(0) = \alpha$, $u(1) = \beta$,

Hledáme $u \in V$, např. fce u: $\int u^2 dx < \infty$, $\int (\partial_x u)^2 dx < \infty$

$L^2(\Omega)$ (Sobolevův pr. H^1)

Lebesgueův p.

$$\frac{\partial^2 u}{\partial x^2} \cdot v(x) = f(x) v(x) \quad \forall v(x) \text{ fcí } v \in V, v = 0 \text{ na } \partial\Omega$$

$v(0) = v(1) = 0$

v každé z 1 bezpeč. chyba (nerovnom. nenulová)

$$\int_0^1 \frac{\partial^2 u}{\partial x^2} v \, dx = \int_0^1 f(x) v \, dx$$

$\Uparrow$: chyba $\varepsilon(x) = \frac{\partial^2 u}{\partial x^2} - f(x)$

$\Rightarrow \int \varepsilon v \neq 0$

$\int \varepsilon v = 0 \; \forall v \Rightarrow \varepsilon = 0$

|| P.P.

$$\underbrace{\left[\frac{\partial u}{\partial x} v\right]_0^1}_{0 \text{ 𝜖 } v=0 \text{ na } \partial\Omega} - \int_0^1 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx$$

vyhovující než $\frac{\partial^2 u}{\partial x^2}$, protože přesné

ar pro p.č. lin. fce

Hledej $u \in V$: $\int_0^1 -\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dx = \int_0^1 f(x) v(x) dx \quad \forall v \in V, v = 0 \text{ na } \partial\Omega$

$u(0) = \alpha$, $u(1) = \beta$

Slabá formulace

• MKP: $u \approx u_h \in V_h$ $\qquad h \to 0$ V_h se blíží V

$\Uparrow$ $\qquad u_h \to u$

V $\qquad f \approx f_h,\ v \approx v_h$

$$\int_0^1 -\frac{\partial u_h}{\partial x}\cdot\frac{\partial v_h}{\partial x}\,dx = \int_0^1 f_h(x)\, v_h(x)\,dx \qquad \forall v_h \in V_h \Leftrightarrow \forall v_h = P_I\ \forall I$$

báze

$u_h = \sum_J u_J P_J$

$$\int_0^1 -\sum_{J=0}^{N} u_J \frac{\partial P_J(x)}{\partial x}\,\frac{\partial P_I(x)}{\partial x}\,dx = \int_0^1 \sum_k f_k P_k(x) P_I(x)$$

$$= \sum_{J=0}^{N} \underbrace{\int_0^1 -\frac{\partial P_J}{\partial x}\cdot\frac{\partial P_I}{\partial x}\,dx}_{A_{IJ}}\ u_J = \int_0^1 \sum f_k P_k P_I$$

$$= \sum_{J=1}^{N-1} A_{IJ} u_J + A_{I0}\alpha + A_{IN}\beta$$

$A \cdot \vec{u} = b$ $\qquad$ soustava lin. rovnic $\to$ vyřešit

$(u_1, u_2, u_3, \dots, u_{N-1})$

neobsahuje hraniční hodnoty

$$\sum_{J=1}^{N-1} A_{IJ} u_J = \underbrace{\sum_{k=0}^{N}\int_0^1 f_k P_k P_I\,dx - A_{I0}\alpha - A_{IN}\beta}_{b_I}$$

$I \in 1 \dots N-1$ f. $u = 0$ na $\partial\Omega$, aby A_{IJ} byla čtvercová a soustava šla řešit

Shrnutí: $\frac{\partial^2 u}{\partial x^2} = f$ $\to$ slabá formulace $\to$ aproximace prostorů

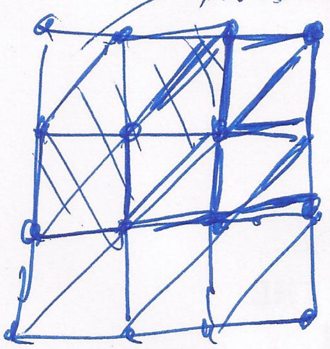
$\downarrow$

$A \cdot \vec{u} = \vec{b}$

$\downarrow$

u_h aprox. řešení

2D: p.č. lin.

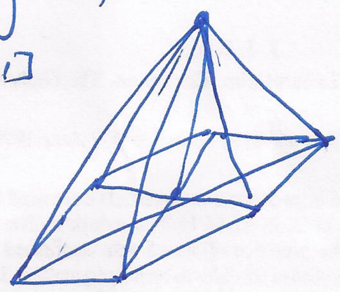


$p(x,y) = c_1 x + c_2 y + c_3$

(kvadr. $p(x,y) = c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x + c_5 y + c_6$)

aby u_h spoj., bereme jehlany s jedním v uzlu

$\Omega = [0,1]\times[0,1]$



P_I

$u_h(x) = \sum_I u_I P_I(x)$ $\qquad u|_{\partial\Omega} = \alpha(x,y)$

$\nabla u = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}$

$\Delta u = f(x,y)$ $\qquad u \in V$

$\nabla\cdot\nabla u = f$

$\int_\Omega \nabla\cdot\nabla u \, v \, dxdy = \int_\Omega f v \, dxdy$, $v \in V$, $v = 0$ na $\partial\Omega$

|| Green $\qquad$ vnější normála

$-\int_\Omega \nabla u\cdot\nabla v \, dxdy + \int_{\partial\Omega} \nabla u\cdot\vec{n} \underbrace{v}_{0 \text{ na } \partial\Omega} dxdy = \int_\Omega f v \, dxdy$

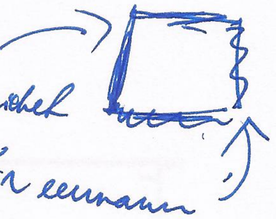
Hledej $u \in V$, $u = \alpha(x,y)$ na $\partial\Omega$: $\int_\Omega -\nabla u\cdot\nabla v \, dxdy = \int_\Omega f v \, dxdy$ $\forall v \in V$, $v=0$ na $\partial\Omega$

MKP u nahraď $u_h = \sum_J u_J P_J(x,y)$, $v \to v_h$, $f \to f_h$, $v_h := P_{I^0}$, I^0 není na hranici

$\sum_J u_J \underbrace{\int_\Omega -\nabla P_J \cdot \nabla P_{I^0} \, dxdy}_{A_{IJ}} = \sum_k f_k \int_\Omega P_k P_{I^0} \, dxdy$ $\qquad u_h$ hledá.

$\sum_{J^0} A_{I^0 J^0} u_{J^0} = \sum_k f_k \int_\Omega P_k P_{I^0} \, dxdy - \sum_{J^\partial} A_{I^0 J^\partial} \alpha_{J^\partial} = b_{I^0}$

• Obecnější okrajové podmínky:



$$\underset{\nabla\cdot\nabla u}{\Delta u \overset{}{=}} f, \qquad u = \alpha(x,y) \text{ na } \partial\Omega_D \text{ Dirichlet}$$

$$\nabla u\cdot\vec{n} = g(x,y) \text{ na } \partial\Omega_N \text{ Neumann}$$

$$\int_\Omega \nabla\cdot\nabla u \, v \, dxdy = \int_\Omega f v \, dxdy$$

$$-\int_\Omega \nabla u\cdot\nabla v \, dxdy + \int_{\partial\Omega_D} \nabla u\cdot\vec{n} \underset{0}{v} \, ds + \int_{\partial\Omega_N} \underset{g}{\nabla u\cdot\vec{n}} \, v \, ds = \int_\Omega f v \, dxdy$$

$v = 0$ na $\partial\Omega_D$ jako dříve

ale ~~...~~ hodnota u není zadaná na $\partial\Omega_N$

$\Rightarrow v \neq 0$ na $\partial\Omega_N$ – tam ale $\nabla u\cdot\vec{n} = g$

$\Rightarrow$ Hledej u t.ž.:

$$\int_\Omega -\nabla u\cdot\nabla v \, dxdy = \int_\Omega f v \, dxdy - \int_{\partial\Omega_N} g v \, ds$$

MKP: $u_h, f_h, v_h = \sum P_{I^{0,N}}$ … vnitřní + na $\partial\Omega_N$

⋮

$$\sum_{I^{0,N}} A_{I^{0,N} J^{0,N}} u_{J^{0,N}} = b_{I^{0,N}} \Rightarrow u_h$$

• časové závislé problémy (vedení tepla)

$$\rho c_v \frac{\partial T}{\partial t} = -\nabla\cdot\left(-\lambda \nabla T\right), \quad \rho, c_v, \lambda \text{ const.}$$

$$\int_\Omega \frac{\partial T}{\partial t}\, v \,dxdy = \int_\Omega -\frac{\lambda}{\rho c_v}\nabla T\cdot\nabla v + \int_{\partial\Omega}\frac{\lambda}{\rho c_v}\nabla T\cdot\vec{n}\, v\, ds$$

kde $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\nabla T\cdot\vec{n} = 0$ na $\partial\Omega_N$

$v=0$ $\int_{\partial\Omega} = 0$

• časová diskretizace $T(k\cdot dt, x) = T_k(x)$

$t = 0, dt, 2dt, \dots$

$$\frac{\partial T}{\partial x} \approx \frac{T_h^{(k)} - T_h^{(k-1)}}{dt}$$

$$\int_\Omega T_h^{(k)} v \,dxdy + \int_\Omega \frac{\lambda\, dt}{\rho c_v}\nabla T_h^{(k)}\cdot\nabla v\,dxdy$$

$$= \int_\Omega T^{(k-1)} v\,dxdy$$

MKP: $T^{(k)} \to T_h^{(k)} = \sum_{J^0} T_J^{(k)} P_J(x,y)$, $v_h = P_{I^0}$

$\cancel{\sum_{J^0} A_{I^0 J^0} T_{J^0}^{(k)}}$

$$\sum_{J^0} T_{J^0}^{(k)} \underbrace{\left(\int_\Omega P_{J^0} P_{I^0}\,dxdy + \frac{\lambda\, dt}{\rho c_v}\int_\Omega \nabla P_{J^0}\cdot\nabla P_{I^0}\,dxdy\right)}_{A_{I^0 J^0}} = \underbrace{\sum_{k^0} T_{k^0}^{(k-1)} \int_\Omega P_{k^0} P_{I^0}}_{b_{I^0}^{(k-1)}}$$

~~Inicializace~~

$T^{(0)}$ poč. podm., $A\cdot\vec{T}^{(k)} = \vec{b}^{(k-1)}$, $T^{(0)} \to T^{(1)} \to T^{(2)}, \dots$

• nelineární problém

$$+\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+(\nabla u)^2}}\right) = f$$

minimalizace povrchu

$$\|\nabla u\| \ll 1 \Rightarrow \Delta u = f$$

$$\int_\Omega \nabla\cdot\left(\frac{\nabla u}{\sqrt{1+(\nabla u)^2}}\right) v \, dxdy = \iint_\Omega v \, dxdy$$

$$F(u,v) - \int_\Omega \frac{\nabla u}{\sqrt{1+(\nabla u)^2}}\cdot\nabla v + \int_{\partial\Omega}\frac{\nabla u}{\sqrt{1+(\nabla u)^2}}\cdot\vec{n}\, v \, ds - \int_\Omega f\cdot v \, dxdy = 0 \quad (*)$$

$\partial\Omega_N$ Neumann: $\frac{\nabla u}{\sqrt{1+(\nabla u)^2}}\cdot\vec{n} = g$

$\partial\Omega_D$ Dir.: $u = \alpha$, $v = 0$

Hledáme u: $\forall v \in V$ platí $(*)$, tj. $F(u,v) = 0 \ \forall v$

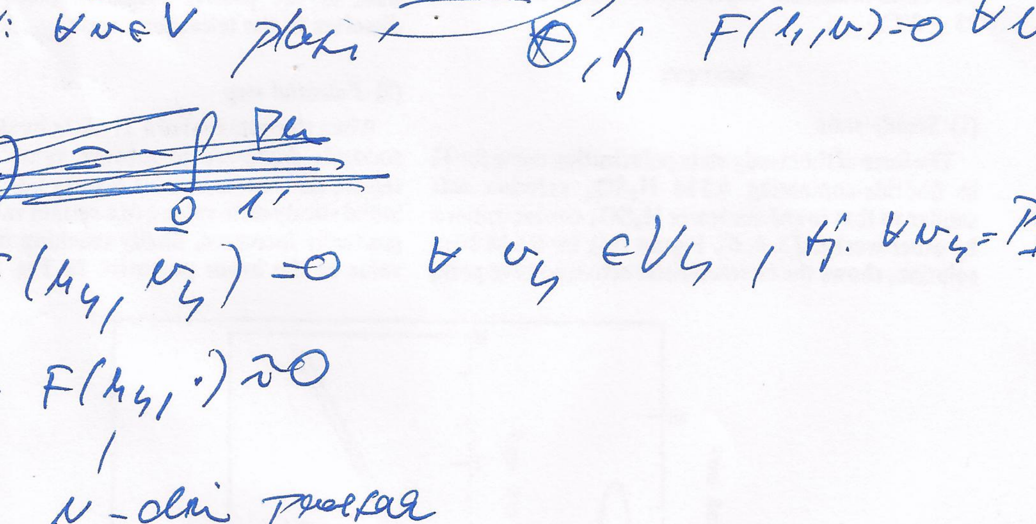
~~$F(u,v) = -\int_\Omega \frac{\nabla u}{1'}$~~

• MKP $F(u_h, v_h) = 0 \quad \forall v_h \in V_h$, tj. $\forall v_h = \varphi_I^{\Omega h}$

tj. $F(u_h, \cdot) = 0$

N-dim. soustava

v 1D: bych $\varphi(x) = 0$ nelin. algebr. rce



Newtonova metoda

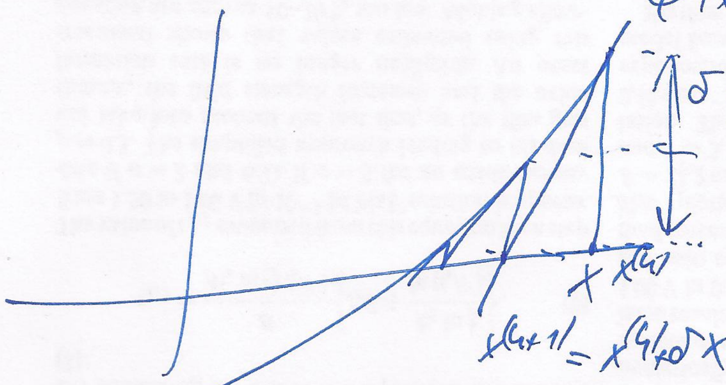
$$\frac{d\varphi}{dx}\Big|_{x^{(k-1)}} \cdot \left(x^{(k-1)} - x^{(k)}\right) = \varphi(x^{(k-1)})$$

~~$\frac{d\varphi}{dx}\Big|_{x^{(k)}} x^{(k)} = -\varphi(x^{k-1}) + \frac{d\varphi}{dx}\Big|_{x^{k-1}} x^{k-1}$~~

$$\frac{\partial F}{\partial u_h}\Big|_{u_h^{(k-1)}} u_h^{(k)} = -F(u_h^{k-1})$$

Newtonova metoda

$\varphi(x) \qquad \varphi(x)=0$



$$\varphi(x^{(k)}) = \left.\frac{d\varphi}{dx}\right|_{x^{(k)}} \cdot \underbrace{(x^{(k)} - x^{(k+1)})}_{-\delta x} = -\delta\varphi$$

$$F(u_h^{(k)}, v_h) = -\int_\Omega \left.\frac{\delta F}{\delta u}\right|_{u_h^{(k)}} \delta u_h \, d\Omega \qquad \forall v_h \in V_h$$

$$u_h^{(k+1)} := u_h^{(k)} + \delta u_h$$

$$F(u + \delta u) = F(u) + \underbrace{\delta F}_{\int_\Omega \frac{\delta F}{\delta u} \delta u} + O(\delta u)^2$$

Fenics má implementováno automaticky