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Positive Quadrant Dependence Tests for Copulas

Abstract

In this paper the interest is in testing the null hypothesis of positive quadrant dependence between two random variables. Such a testing problem is important since prior knowledge of positive quadrant dependence is a qualitative restriction that should be taken into account in further statistical analysis, e.g. when choosing an appropriate copula function to model the dependence structure. The key methodology of the proposed testing procedures consists of evaluating a 'distance' between a nonparametric estimator of a copula and the independence copula, which serves as a reference case in the whole set of copulas having the positive quadrant dependence property. We discuss choices of appropriate distances and nonparametric estimators of copula, and compare our methods with testing procedures based on bootstrap and multiplier techniques. The consistency of the testing procedures is established. In a simulation study we investigate the finite sample size and power performances of three types of test statistics, Kolmogorov-Smirnov, Cramér-von-Mises and Anderson-Darling statistics, together with several nonparametric estimators of a copula, including recently developed kernel type estimators. Finally we apply the testing procedures on some real data.

Keywords and phrases: Anderson-Darling test statistic; copula function; Cramér-von-Mises test; Kolmogorov-Smirnov distance; positive quadrant dependence; weak convergence.

1 Introduction

A copula C, associated with a random vector $\mathbf{X}$, is a function, which when given as arguments the marginal distribution functions of $\mathbf{X}$ results into the joint distribution function of all components of $\mathbf{X}$. It is uniquely defined on the set of continuous distribution functions.

In this paper, we focus on two dimensional random vectors $\mathbf{X} = (X, Y)$, with X and Y continuous random variables, where Sklar's theorem (see Sklar (1959)) gives that if $(X, Y) \sim H$ and $X \sim F$, $Y \sim G$, then

$$\forall x, y \qquad C(F(x), G(y)) = H(x, y)\ .$$

An important feature of the copula is that it is itself a distribution function on the unit square with the uniform marginals, as $F(X), G(Y) \sim U[0, 1]$ and

$$(F(X), G(Y)) \sim C.$$

Positive Quadrant Dependence (PQD) is a specific type of dependence between random variables (see Lehmann (1966)). Formally, we say that X and Y are positive quadrant dependent if

$$\forall x, y \qquad \mathbb{P}(X > x,\ Y > y) \geq \mathbb{P}(X > x)\mathbb{P}(Y > y). \tag{1}$$

This condition can be stated in terms of copulas as well

$$\forall u, v \in [0, 1] \qquad C(u, v) \geq uv. \tag{2}$$

Thus PQD can also be defined as a feature of a copula function and we shall refer both to PQD copulas and PQD random vectors.

A concept that is symmetric to PQD is the concept of Negative Quadrant Dependence (NQD), which swaps the inequality in the definition of PQD. The relation between both concepts can be seen in terms of monotonic transformations. Applying increasing functions to X and Y does not change the copula, thus neither their quadrant dependence. However, if an increasing function is applied to one random variable and a decreasing function to the other random variable, then the quadrant dependence of the transformed couple of random variables is changed.

The probabilistic approach in (1) gives an easy interpretation of PQD, as it says that the probability that random variables are jointly large is greater than when they are looked at separately. Separately here means independently, as independence between X and Y is characterized by the multiplicative copula $\Pi(u, v) = uv$.

Positive quadrant dependence might be a very realistic assumption in many situations. Think of for example life expectancies of men and women in various countries. One would expect that a higher life expectancy for men in one country goes along with a higher life expectancy for women in that country. Examples of positive quadrant dependence are ample in particular in insurance and finance. For a discussion about PQD in finance and actuarial sciences see Janic-Wróblewska et al. (2004) and Denuit and Scaillet (2004) and references therein. The knowledge about PQD or NQD of random variables is important for statistical inference. Indeed, if it is reasonable to assume, for example, positive quadrant dependence then such prior knowledge should be exploited in the statistical inference.

Janic-Wróblewska et al. (2004) and Denuit and Scaillet (2004) also investigate testing problems related to the type of dependence structure. In Janic-Wróblewska et al. (2004) rank tests are introduced for testing independence against positive quadrant dependence. Testing for independence against strict PQD was dealt with in Kochar and Gupta (1987). Denuit and Scaillet (2004) test for PQD against non-PQD and construct tests based on a distance concept considering the PQD definitions of both (1) and (2) using empirical cumulative distribution function estimators.

In this paper we are concerned with testing the null hypothesis of positive quadrant dependence versus not positive quadrant dependence, focusing as such on finding out whether a PQD assumption is justified. Starting from the PQD characteristic of a copula function given in (2), the basic idea of the testing procedures here is to investigate a distance between a nonparametric estimate of the unknown copula and the independence copula function. We consider various nonparametric estimators of a copula function along with three functional distances.

Testing for positive quadrant dependence was also studied in Scaillet (2005). In that paper the author constructs a Kolmogorov-Smirnov type of test based on the empirical copula estimator relying on the asymptotic distribution of the empirical process. Statistical inference is conducted by using a simulation-based multiplier method and a bootstrap method. The present paper contributes further on this testing problem in various aspects. Firstly, testing procedures based on other distance measures such as Cramér-von Mises and Anderson-Darling distance measures (see e.g. Anderson and Darling (1954)) should be studied, since they might reveal different power properties. See also Omelka et al. (2009). Secondly, in recent years other competitive and improved nonparametric estimators of a copula have been introduced and studied and it is worth to investigate how these estimators perform when used in testing procedures. In our study we consider the empirical copula estimator of Deheuvels (1979), kernel type estimators such as the integrated version of the density Mirror Reflection estimator (see Gijbels and Mielniczuk (1990)) and the Local Linear estimator (see Chen and Huang (2007)), as well as recent extensions (improvements) of these two kernel estimators introduced and studied in Omelka et al. (2009). Thirdly, relying on asymptotic theory is not always the best option, since the rate of convergence might require rather large samples before good finite sample behaviour is obtained. We therefore opt for a different approach here, and make use of the independence copula as a reference case included in the null hypothesis. Admittedly this approach also has drawbacks but, as will be seen, these are overruled by the advantages in power performance.

The paper is organized as follows. In Section 2, we briefly discuss the various nonparametric copula estimators and the different test statistics, and establish consistency of these testing procedures. Section 3 contains a simulation study illustrating the finite sample behaviour of the tests. In Section 4 we apply these procedures on real data examples. We conclude in Section 5 with some further discussions on the research topic.

2 Nonparametric copula estimation and test statistics

Copula estimation is closely related to the estimation of a cumulative distribution function with the main difference that no data from $(F(X), G(Y))$ are observed. Referring to the definition of a copula, an estimation procedure can be divided into two levels, estimation of the marginals and estimation of their joint distribution. If on both levels parametric assumptions are made, then maximum likelihood methods can be applied. However, it is common to make parametric assumptions on the joint level combined with nonparametric estimation of marginals, resulting in popular semi-parametric models. For this usage, there are many well described copula families differing in the number of parameters and characteristics (see e.g. Nelsen (2006)).

In this paper we are interested in a fully nonparametric approach, and in particular in recently developed estimation procedures described in Omelka et al. (2009). A basic idea behind this and previous estimation methods is to transform the observed data by a monotonic transformation, specifically by the empirical marginal distribution functions, and then to estimate the joint distribution function based on these pseudo-observations. As such we can unify random vectors, which have the same copula, regardless their marginal distributions.

Suppose we have a sample $(X_1, Y_1), \ldots, (X_n, Y_n) \sim_{iid} H = C(F, G)$. The pseudo-observations are defined as

$$\hat{U}_i = \frac{n}{n+1} F_n(X_i),\ \hat{V}_i = \frac{n}{n+1} G_n(Y_i),$$

where F_n and G_n are the empirical distributions. The modification $\frac{n}{n+1}$ to the empirical distribution simply pulls the pseudo-observations a bit more away from one (see Genest et al. (1995)). By doing so potential difficulties arising at boundaries can be reduced. The pseudo-observations are then treated as a sample from the random vector $(F(X), G(Y)) \sim C$ and the copula C can be estimated nonparametrically as a bivariate distribution on the unit square. However, because of the unit square domain there are boundary issues arising in the estimation task. Therefore, in our testing procedure, we investigate along with the empirical estimator, the kernel estimators of Chen and Huang (2007) and Gijbels and Mielniczuk (1990) together with their "shrunken" modifications proposed by Omelka et al. (2009) for better consistency results.

In summary our study involves the following copula estimators:

- Empirical copula estimator (Deheuvels (1979))

$$\hat{C}_n^{\mathrm{E}}(u,v) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}\{\hat{U}_i \le u, \hat{V}_i \le v\},$$

  where $\mathbb{I}\{A\}$ denotes the indicator function of A.

- Kernel Local Linear estimator (Chen and Huang (2007))

$$\hat{C}_n^{\mathrm{LL}}(u,v) = \frac{1}{n}\sum_{i=1}^{n} K_{u,h_n}\left(\frac{u-\hat{U}_i}{h_n}\right) K_{v,h_n}\left(\frac{v-\hat{V}_i}{h_n}\right),$$

  where h_n is a smoothing parameter and K_{u,h_n} is the integral of the modified kernel

$$k_{u,h}(x) = \frac{k(x)\left(a_2(u,h) - a_1(u,h)x\right)}{a_0(u,h)a_2(u,h) - a_1^2(u,h)} \mathbb{I}\{\frac{u-1}{h} < x < \frac{u}{h}\},$$

  where

$$a_\ell(u,h) = \int_{\frac{u-1}{h}}^{\frac{u}{h}} t^\ell k(t)dt \quad \text{for } \ell = 0,1,2$$

  and k is a symmetric, bounded on the unit interval, kernel function, for example the Epanechnikov kernel $k(x) = 0.75(1-x^2)\mathbb{I}\{|x| \le 1\}$.

- Kernel Local Linear Shrunken estimator (Omelka et al. (2009))

$$\hat{C}_n^{\mathrm{LLS}}(u,v) = \frac{1}{n}\sum_{i=1}^{n} K_{u,h_n}\left(\frac{u-\hat{U}_i}{b(u)h_n}\right) K_{v,h_n}\left(\frac{v-\hat{V}_i}{b(v)h_n}\right),$$

  where $b(w) = \sqrt{\min(w, 1-w)}$.

- Kernel Mirror-Reflection estimator (Gijbels and Mielniczuk (1990))

$$\begin{aligned} \hat{C}_n^{\mathrm{MR}}(u,v) \quad &= \quad \frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{9}\left[K\left(\frac{u-\hat{U}_i^{(\ell)}}{h_n}\right) - K\left(\frac{-\hat{U}_i^{(\ell)}}{h_n}\right)\right] \\ &\qquad \times \left[K\left(\frac{v-\hat{V}_i^{(\ell)}}{h_n}\right) - K\left(\frac{-\hat{V}_i^{(\ell)}}{h_n}\right)\right], \end{aligned}$$

  where $\{(\hat{U}_i^{(\ell)}, \hat{V}_i^{(l)}),\ i = 1,\ldots,n,\ \ell = 1,\ldots,9\}$
  $= \{(\pm\hat{U}_i, \pm\hat{V}_i), (\pm\hat{U}_i, 2-\hat{V}_i), (2-\hat{U}_i, \pm\hat{V}_i), (2-\hat{U}_i, 2-\hat{V}_i),\ i = 1,\ldots,n\},$

  and K is the integral of the considered kernel k.

- Kernel Mirror-Reflection Shrunken estimator (Omelka et al. (2009))

$$\hat{C}_n^{\mathrm{MRS}}(u,v) = \frac{1}{n}\sum_{i=1}^{n}\sum_{\ell=1}^{9}\left[K\left(\frac{u-\hat{U}_i^{(\ell)}}{b(u)h_n}\right)-K\left(\frac{-\hat{U}_i^{(\ell)}}{b(u)h_n}\right)\right] \times\left[K\left(\frac{v-\hat{V}_i^{(\ell)}}{b(v)h_n}\right)-K\left(\frac{-\hat{V}_i^{(\ell)}}{b(v)h_n}\right)\right].$$

It should be mentioned that in Chen and Huang (2007) the pseudo-observations are obtained via kernel methods. The authors showed however that a strong undersmoothing is needed in this step, and hence we here decided directly for a rank estimation, which coincides with the limiting case that the smoothing parameter tends to zero.

The test statistics for testing for positive quadrant dependence are based on distances between the estimated copula and the independence copula. The distances measure the violation part of the copula estimator with the positive quadrant dependence hypothesis under the null. We focus on measures based on L_∞ and L_2 distances. Denote by $\hat{C}_n$ the estimated copula distribution function. We then consider the following statistics

- Kolmogorov-Smirnov

$$S_n^{\mathrm{KS}} = \sqrt{n}\sup_{u,v\in[0,1]}(uv-\hat{C}_n(u,v))_+, \qquad \text{where } (\cdot)_+ = \max(\cdot,0)$$

- Cramér-von Mises

$$S_n^{\mathrm{CvM}} = n\int_{\mathbb{I}^2}(uv-\hat{C}_n(u,v))_+^2 d\hat{C}_n(u,v) \qquad \text{with } \mathbb{I}^2 = [0,1]\times[0,1]\ .$$

- Anderson-Darling

$$S_n^{\mathrm{AD}} = n\int_{\mathbb{I}^2}\frac{(uv-\hat{C}_n(u,v))_+^2}{uv(1-u)(1-v)} d\hat{C}_n(u,v)\ .$$

Note that the correction factor $(uv(1-u)(1-v))^{-1}$ in the Anderson-Darling distance puts more attention to the borders of a copula. This weight factor is in fact the asymptotic variance of the empirical copula estimator (based on pseudo-observations) when the true underlying copula is the independence copula. Such a weighting factor is also appealing from an intuitive point of view since copulas often have very similar values near the borders, and the closer one comes to them the smaller the absolute differences between the copulas are. In addition crucial differences between copulas (and therefore between dependency structures) are often hidden close to the borders.

With a specified copula estimator and a functional distance, and an i.i.d. sample $(X_1,Y_1),\ldots,(X_n,Y_n)$ from a joint distribution with underlying copula C, we can build a test statistic S_n to test the null hypothesis of positive quadrant dependence

$$H_0:\ \forall u,v\in[0,1] \qquad C(u,v)\geq uv$$

against the negation of this

$$H_1:\ \exists u,v\in[0,1] \qquad C(u,v)< uv.$$

The distribution of S_n under the null hypothesis is unknown and there are various options to tackle this problem. A first option is to rely on asymptotic results for the copula estimator at hand. For all copula estimators mentioned above weak convergence results are available. Possible drawbacks of this approach are that the asymptotics might kick in only for rather large sample sizes, that the test statistics are non-trivial functionals of the copula estimator, and that it typically requires estimation of partial derivatives of the copula function, resulting in a rather complex estimation procedure. A second approach is to use resampling methods to mimic the distribution of S_n under the null hypothesis. The multiplier and bootstrap methods of Scaillet (2005) follow these approaches. A potential problem with these two methods is that the resampling should in fact be done under the null hypothesis, which is not the case.

The approach we follow here consists of making reference to the specific case of the independence copula that is included in the null hypothesis of positive quadrant dependence. To approximate the distribution of S_n under H_0 we draw samples from the independence copula $\Pi(u, v) = uv$, and use these drawings in a Monte Carlo setting to approximate the critical values of the test. Admittedly, this is just selecting one specific element out of the families of all copulas under the null, but the selection makes sense given the importance of the independent case in general.

More precisely, the test works as follows. For a sample of size n with true (unknown) underlying copula C we

$$\text{reject } H_0 \text{ if } S_n > c^{\Pi}_{\alpha,n}, \tag{3}$$

where $c^{\Pi}_{\alpha,n}$ is the quantile of the test statistic S_n under the independence copula Π. By using the independence copula to obtain the critical values, we expect to reach the upper bound of the type I error of the test in our composite null hypothesis testing problem. It is also expected that the deeper we are in the null hypothesis (i.e. the larger the discrepancy is between $C(u, v)$ and $\Pi(u, v) = uv$), the smaller the actual significance level of the test will be. This issue is investigated in the finite sample study in Section 3.3.

An alternative in hypothesis testing is to calculate the P-value, the probability (under the null) that the considered test statistic exceeds its observed value. In practice this leads to a rejection rule based on an estimated P-value denoted by p_n:

$$\text{reject } H_0 \text{ if } p_n < \alpha.$$

In a bootstrap or multiplier method an estimator for the p-value is

$$p_{n,m} = \frac{1}{m} \sum_{i=1}^{m} \mathbb{I}\{S_{n,m} > S_n\}, \tag{4}$$

where $S_{n,m}$ is obtained either from a bootstrap $S^{(\mathrm{B})}_{n,m}$ or a multiplier $S^{(\mathrm{M})}_{n,m}$ method. See Scaillet (2005).

To guarantee the consistency of the proposed testing procedure (3) some regularity assumptions are needed. These differ for the different estimators (see Table 1).

(C1) The first order partial derivatives of C with respect to u and v, denoted by C_u and C_v respectively, are continuous on the set $[0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

(C2) The second order partial derivatives of C denoted by C_{uu}, C_{uv} and C_{vv}, satisfy

$$C_{uu}(u, v) = O\left(\frac{1}{u(1-u)}\right), \; C_{vv}(u, v) = O\left(\frac{1}{v(1-v)}\right), \; C_{uv}(u, v) = O\left(\frac{1}{\sqrt{uv(1-u)(1-v)}}\right).$$

(C3) The second order partial derivatives C_{uu}, C_{uv} and C_{vv} are bounded on $[0,1]^2$.

(C4) Let μ_C be the measure associated with a copula C, λ_2 the Lebesgue measure on $[0,1]^2$, $I_0 = \{(u,v) \in [0,1]^2 : \; C(u,v) = uv\}$ and ∂I_0 the boundary of the set I_0. Then $\mu_C(\partial I_0) = \lambda_2(\partial I_0) = 0$.

(Bw) The bandwidth h_n satisfies: $h_n = O(n^{-1/3})$.

The following table lists the assumptions which are needed for the weak convergence of the process $\sqrt{n}(\hat{C}_n - C)$ in the space of the bounded functions $\ell^\infty([0,1]^2)$ to a centered Gaussian process G_C (see Theorems 1 and 2 in Omelka et al. (2009)).

Estimator	Assumptions
$\hat{C}_n^{\mathrm{E}}$	**C1**
$\hat{C}_n^{\mathrm{LL}}$, $\hat{C}_n^{\mathrm{MR}}$	**C3**, **Bw**
$\hat{C}_n^{\mathrm{LLS}}$, $\hat{C}_n^{\mathrm{MRS}}$	**C1**, **C2**, **Bw**

Table 1: Assumptions for the various estimators.

The limiting Gaussian process G_C has on $[0,1]^2 \setminus \{(0,0),(0,1),(1,0),(1,1)\}$ the representation

$$G_C(u,v) = B_C(u,v) - C_u(u,v)\, B_C(u,1) - C_v(u,v)\, B_C(1,v) \;, \tag{5}$$

where B_C is a two-dimensional pinned C-Brownian sheet on $[0,1]^2$, i.e. it is a centered Gaussian process with covariance function

$$\mathrm{E}\left[B_C(u,v) B_C(u',v')\right] = C(u \wedge u', v \wedge v') - C(u,v)C(u',v') \;, \tag{6}$$

and G_C is defined to be zero in the 'corner points' $\{(0,0),(0,1),(1,0),(1,1)\}$.

Remark 1. *Note that from Condition **(C1)** (or also from Condition **(C3)**) and the properties of a copula it follows that the limiting process G_C is equal to zero on the borders of the unit square $[0,1]^2$. This issue will be used in the proofs.*

Theorem 1 below establishes the consistency results for testing procedure (3), and is similar to results provided in Scaillet (2005) for the tests therein. The proof of Theorem 1 is given in the Appendix.

Theorem 1. *If $\sqrt{n}(\hat{C}_n - C)$ converges weakly to a zero mean Gaussian process and* **(C4)** *holds, we have that*

i) if H_0 is true, then

$$\lim_{n\to\infty} \mathbb{P}(\textit{reject } H_0) = \lim_{n\to\infty} \mathbb{P}\left(S_n > c^{\Pi}_{\alpha,n}\right) \leq \alpha,$$

ii) if H_0 is false, then

$$\lim_{n\to\infty} \mathbb{P}(\textit{reject } H_0) = 1.$$

It finally should be mentioned that we rely on Monte Carlo simulations to approximate the quantile $c^{\Pi}_{\alpha,n}$. This does not affect the above consistency results, as the Monte Carlo simulation can be carried out with high precision.

Remark 2.

1. *Condition **(C4)** is in fact not necessary for the limiting results on the Kolmogorov-Smirnov test statistic, as can be seen from the proof in the Appendix.*

2. *Condition **(C4)** can in fact be weakened a bit, by assuming that $\mu_C(B) = \lambda_2(B)$ for all measurable subsets B of I_0. Both conditions, **(C4)** and this weaker version, are very mild conditions, and we were not able to construct examples for which these are violated.*

3 Simulation study

In this section we investigate the finite sample behaviour of the testing procedure based on the various nonparametric estimators and the different distance measures. The simulation study mainly focuses on evaluations of powers of the tests in Sections 3.1 and 3.2, but also studies on the actual size of the tests are provided in Section 3.3. The simulation study also includes a comparison with the testing procedures of Scaillet (2005) and moreover extends this paper with results on bootstrap and multiplier based testing methods for other distance measures.

Since the tests are based on nonparametrically estimating the marginal distributions using ranks, they are unaffected by monotonic transformations. Therefore, for the purpose of the study, a direct sampling from the concerned copula distribution was done. For the multiplier method the samples were additionally transformed to have exponential marginal distributions with parameter 1. For all computations the R software (see R Development Core Team (2008)) was used with in particular the copula package of Yan (2007). See also Yan and Kojadinovic (2009).

Throughout the simulation study the significance level is 0.05 and the sample size is $n = 200$. Computation of the critical values was based on 10000 samples from the independence copula and the power performance was based on 1000 samples. For the Kolmogorov-Smirnov based test statistics, the supremum is searched for on an equally spaced grid of points with distance 0.05 between two consecutive grid points. For the bootstrap and the multiplier methods, we used $m = 1000$ repetitions for approximating the p-value, according to (4).

Several copula models are considered in the simulation study. These also include models studied in Scaillet (2005), namely Frank, Gaussian and Farlie-Gumbel-Morgenstern (FGM) copula families for Kendall's tau equal to -0.11 and -0.16. In order to have some more challenging testing problems we also discuss results for two families of mixtures of copulas. A mixture of Frank copulas introduces different dependency structure than these of the frequently used copula families. A second family of mixtures of copulas is the modified Mardia family, which is a convex mixture of the Frechét-Hoeffding boundary copulas and the independence copula.

3.1 Classical copula families

Simulations were done for five classical copula families: Frank, Clayton, Gumbel, Gaussian and FGM copulas; and this for two different values of Kendall's tau: -0.11 and -0.16. The Clayton and Gumbel copula families are often used for modeling heavy dependencies in right tails. However, all members of the Gumbel family have the positive quadrant dependence property. Therefore, this family cannot be directly included in the power study, as there are no members violating the PQD condition. It is however interesting to investigate the power of the tests for copulas for which there is a heavy negative quadrant dependence. Transferring the heavy tails from the upper-right corner to the bottom-right corner can easily be obtained by considering $(U, 1-V)$, where $(U,V) \sim C_{\text{Gumbel}}$. This construction preserves the absolute magnitude of Kendall's tau, but changes the sign, so in the above sense we refer to this Gumbel copula as a Gumbel copula with a negative tau.

Table 2 presents the simulation results on the power study for the five classical copula families (with significance level 0.05). The entries "M" and "B" in the tables are the results obtained by the Multiplier and Bootstrap methods for the empirical copula introduced in Scaillet (2005). It is clear that with the decrease of Kendall's tau the overall power increases and also the differences between the various tests decrease. Further it is to be noted that among the five nonparametric copula estimators and the three distance measures, the worst results are almost always for the testing procedure using the empirical copula estimator and the Kolmogorov-Smirnov distance. The bootstrap method of Scaillet (2005) (and its extension to other distances) works worse than the empirical method (short for the method using the empirical copula with resampling from the independence copula) in case of the Kolmogorov-Smirnov distance, but it works slightly better in cases of the Cramér-von Mises and Anderson-Darling distances. The multiplier method of Scaillet (2005) (and its extensions) works similar to the empirical method for the Kolmogorov-Smirnov distance, but much worse in case of the Cramér-von Mises and Anderson-Darling distances.

	Frank			Gaussian			FGM			Gumbel			Clayton		
	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
	$\tau = -0.11$														
E	.613	.729	.719	.526	.673	.699	.605	.726	.749	.553	.710	.773	.573	.704	.770
LL	.718	.742	.543	.670	.721	.622	.727	.762	.536	.710	.765	.791	.737	.768	.773
MR	.744	.743	.745	.680	.683	.688	.741	.741	.742	.681	.698	.693	.700	.690	.697
LLS	.675	.747	.696	.638	.704	.715	.689	.745	.719	.650	.736	.803	.678	.735	.815
MRS	.675	.745	.738	.638	.701	.723	.689	.745	.765	.650	.728	.783	.676	.730	.787
B	.451	.752	.746	.354	.698	.721	.433	.751	.765	.386	.715	.780	.388	.714	.778
M	.615	.573	.469	.529	.521	.471	.605	.569	.460	.557	.540	.522	.576	.560	.537
	$\tau = -0.16$														
E	.896	.947	.950	.855	.946	.961	.909	.961	.960	.849	.934	.958	.875	.948	.969
LL	.945	.958	.873	.939	.969	.937	.966	.974	.869	.937	.961	.957	.962	.967	.966
MR	.951	.956	.956	.949	.957	.956	.970	.973	.973	.929	.932	.933	.941	.933	.934
LLS	.930	.960	.949	.915	.961	.972	.945	.972	.962	.907	.948	.967	.946	.959	.979
MRS	.930	.959	.955	.915	.960	.970	.945	.971	.970	.906	.946	.965	.944	.957	.970
B	.794	.952	.960	.728	.953	.967	.790	.972	.971	.718	.934	.960	.752	.951	.972
M	.896	.892	.845	.859	.886	.868	.911	.917	.864	.845	.864	.862	.871	.893	.897

Table 2: *Power study results for Frank, Gaussian and FGM copulas with Kendall's tau equal to* -0.11 *and* -0.16.

The performance results for the Cramér-von Mises and the Anderson-Darling based statistics are more comparable. It seems that for "well behaving" copulas like Frank, Gaussian and FGM the Cramér-von Mises based statistics seem to be working better, but for "heavier-tailed" copulas like Gumbel and Clayton the Anderson-Darling based test statistics perform best.

As for the cross-estimators analysis (comparing the performances of the 5 estimators), the only visible pattern is in the Gumbel and Clayton case, where the mirror type of estimator performs not very good regardless the distance. Also the local linear estimator combined with the Anderson-Darling distance seems to perform worse in case of the "well behaving" copulas.

In conclusion, overall it seems recommendable to use the mirror reflection shrunken or local linear shrunken estimators combined with the Anderson-Darling or Cramér-von-Mises distance measure.

3.2 Mixed copulas examples

The copulas in the previous section violated the PQD condition in a simple manner by simply being negative quadrant dependent. In particular this implies that the whole copula function is below the independence copula, and hence the violation is on the whole unit square. However PQDness is a global feature, and it is interesting to see how the tests work on examples where the PQD condition is violated only locally. Therefore we consider in this section two examples of copulas which are neither PQD, nor negative quadrant dependent (NQD), in contrast to these in the previous analysis.

When it comes to copulas which are only locally NQD, then Kendall's tau is no longer an appropriate measure for the difficulty of a testing problem, especially when a considered copula is on average symmetric around the independence copula. In order to have some guidance regarding the difficulty of a testing problem, there is a need for a different appropriate measure of departure from PQDness.

Two such straightforward measures are a maximum violation measure and a mean of violation measure defined as

$$a_C = \max_{u,v} \left(uv - C(u,v)\right)_+ \qquad \text{and} \qquad b_C = \int \left(uv - C(u,v)\right)_+ dudv$$

respectively. These measures are similar in spirit to some dependence measures discussed in Nelsen (2006). We calculated the measures a_C and b_C for the copulas considered in Section 3.1. For all these copulas the values of a and b for specific values of Kendall's tau turned out to be very similar. Table 3 presents approximate average values with respect to tau

τ	a	b
-0.11	0.029	0.014
-0.16	0.043	0.020

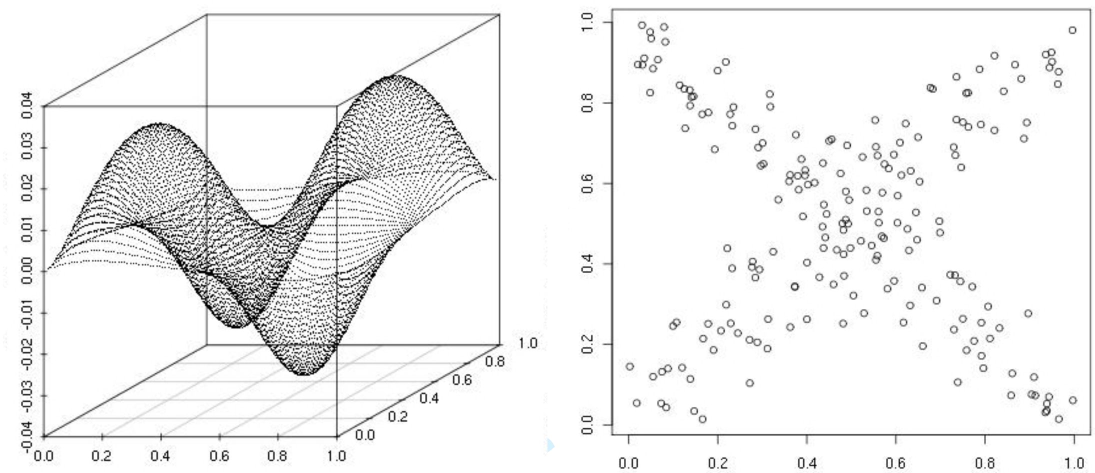
Table 3: *Approximate average values of a and b for Frank, Gaussian, FGM, Gumbel and Clayton copulas with different taus.*

We now consider copulas that are convex mixtures of other copulas and violate the PQDness condition only locally instead of globally.

A first mixture that we study is a symmetric mixture of Frank copulas

$$(U, V) \sim 0.5 \cdot C_{\text{Frank}(\theta)} + 0.5 \cdot C_{\text{Frank}(-\theta)}. \tag{7}$$

To give some insight in such a mixture of copulas, we present in Figure 1 (a) the difference between such a mixture copula (with $\theta = 17.5$) and the independence copula. Figure 1 (b) depicts a sample from such a copula. Note that for increasing θ one moves further away from the independence case. The larger θ the more concentrated the observations are along both diagonals. Table 4 provides the values of a and b for a set of parameters theta.



(a) (b)

Figure 1: *(a). Difference between the convex mixture of Frank copulas given by (7) and the independence copula; (b) a sample from this convex mixture distribution.*

θ	a	b
1.47	0.002	0.0003
9.5	0.029	0.0060
17.5	0.043	0.0084

Table 4: *Approximate values of a and b for mixture of Frank copula with different thetas.*

The parameter values $\theta = 9.5$ and $\theta = 17.5$ were chosen such that the values of a are approximately equal to these from the previous models in Section 3.1. The parameter value $\theta = 1.47$ serves as a reference to the case of a single Frank copula with the same parameter (when τ was equal to -0.16). Note from Table 3 that the corresponding b-values are much smaller than these for the 'classical' copulas. As will be seen from the simulation results, these mixture copulas present a situation of local violation of the PQD condition that is more difficult to detect. As such we believe that the 'b' measure of PQD-"badness" of a copula is more appropriate than the 'a'-measure.

The simulation results for the mixture of Frank copulas are given in Table 5. The power results for $\theta = 1.47$ are close to the significance levels, which suggests that the mixture in this case is hard to distinguish from the independence case. More informative are the simulation results for $\theta = 9.5$ and $\theta = 17.5$. Although there is much more variability in the power values than in the previous study, we again notice that the mirror type of estimator does not perform well regardless the distance, which was also the case for the Gumbel and Clayton copulas. Regarding the cross-distance analysis it can be concluded, also from these simulation results, that the tests based on the Anderson-Darling distance seem to perform best. We can also see that, both the bootstrap and the multiplier methods work worse in all of the cases, when compared to the empirical method, with exception of the multiplier method combined with the Kolmogorov-Smirnov distance.

	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
	θ =1.47			θ=9.5			θ =17.5		
E	.043	.050	.054	.377	.389	.617	.759	.777	.926
LL	.048	.044	.064	.494	.261	.802	.840	.520	.969
MR	.050	.050	.049	.094	.108	.106	.152	.153	.163
LLS	.049	.050	.052	.494	.276	.571	.809	.578	.880
MRS	.049	.050	.048	.491	.269	.527	.810	.561	.853
B	.017	.055	.055	.230	.291	.496	.605	.639	.872
M	.041	.016	.007	.429	.143	.194	.828	.455	.594

Table 5: *Power results for the mixture of Frank copulas given by* (7) *with parameters* $\theta = 1.47$*,* 9.5*,* 17.5.

The second example of models is a mixture of the Fréchet-Hoeffding boundary copulas and the independence copula, which is closely related to the Mardia family. More precisely, consider a copula given by

$$C = \frac{\theta^2(1-\theta)}{2}\gamma \cdot W + (1-\gamma\theta^2)\cdot \Pi + \frac{\theta^2(1+\theta)}{2}\gamma \cdot M, \tag{8}$$

where $\theta \in [-1;1]$, $\gamma \leq 1/\theta^2$ and where $W(u,v) = \max(u+v-1,0)$ and $M(u,v) = \min(u,v)$ are the Fréchet-Hoeffding lower and upper bounds respectively (see Nelsen (2006)). Re-arranging (8) gives

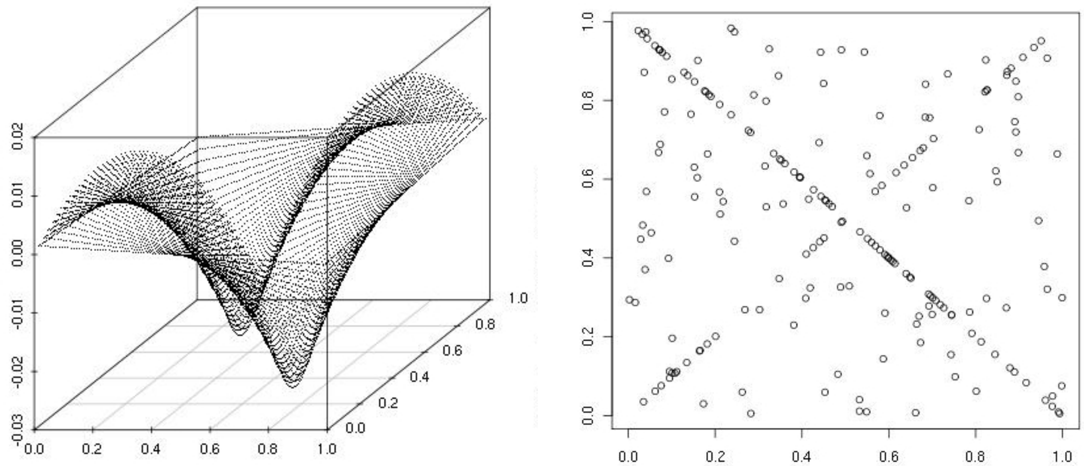
$$C = \Pi + \gamma \cdot (C_{\text{Mardia}} - \Pi), \tag{9}$$

where

$$C_{\text{Mardia}} = \frac{\theta^2(1-\theta)}{2}\cdot W + (1-\theta^2)\cdot \Pi + \frac{\theta^2(1+\theta)}{2}\cdot M,$$

is the original Mardia copula family. Expression (9) reveals the motivation behind this mixture copula – to scale the differences between the Mardia copula and the independence copula. Scaling of this difference does not change the area in the unit square where it is negative or positive. For θ equal to -0.5, -0.2, 0.2, 0.5 respectively there is 0.125, 0.32, 0.68, 0.875 percentage of the area above the independence copula function. For comparison the symmetric mixture of Frank copulas displayed in Figure 1 (a) is for half of the area above the independence copula.

To give some idea of the form of such copulas, we depict a copula from this family in Figure 2 (a) together with a sample generated from it in Figure 2 (b). The parameter θ in this copula is responsible for the concentration of the observations on the diagonals. If θ is positive then it is more probable to have observations from the M copula, which concentrates on the $[(0,0),(1,1)]$ diagonal. If θ is negative then more observations come from the W copula, which concentrates on the $[(0,1),(1,0)]$ diagonal. The parameter γ



(a) (b)

Figure 2: *Difference between the mixture of Mardia and independence copulas given by* (8) *and the independence copula function with* (θ, γ) *equal to* $(-0.2, 11.9444)$ *(in (a)) together with a samples from this distribution in (b).*

additionally decreases ($\gamma > 1$) or increases ($\gamma < 1$) the probability of having independent observations in the sample.

Table 6 gives values for the a and b measures for the Mardia copula with different parameter values theta. Note that there is no monotonic relation between a and θ.

θ	a	b
-0.5	0.035	0.0103
-0.2	0.004	0.0008
0.2	0.002	0.0002
0.5	0.004	0.0001

Table 6: *Approximate values of a and b for the Mardia copula for different theta values.*

Table 7 summarizes the obtained simulation results for some Mardia copulas. Note that the cases with $\theta = -0.2$, 0.2 and 0.5 really represent difficult to very difficult testing problems. Despite the 0.125 area violation of PQDness, the deviation from the independence copula is simply too small for the Mardia copula with parameter 0.5 to be caught by the tests. For $\theta = 0.2$, with the area of violation being 0.32, the powers are mostly below the significance level. Yet, these results are not surprising, when compared to the ones for parameter -0.2. It occurs that the Mardia copula with $\theta = -0.2$ has similar values of a and b as the mixture of Frank copulas with parameter 1.47, which was hard to be distinguished from the independence case. See Tables 4 and 6. The area of violation for the Mardia copula with $\theta = -0.2$ is 0.68, which is more than the 0.5 area of violation for the Frank mixture with parameter 1.47. Thus, the results are expected to be slightly better for the former case. Comparing Tables 5 and 7 this indeed seems to be the case. Our tests are thus also more sensitive to the case of asymmetric copulas (around the independence copula). The results for the Mardia copula with parameter $\theta = -0.5$ can be interpreted easier after the next simulation part, which incorporates also different values for the γ parameter in (8) and (9). We can already notice however that in general the bootstrap and multiplier methods work worse than the empirical one, again with exception of the multiplier method combined with the Kolmogorov-Smirnov distance.

	$\theta = -0.5$			$\theta = -0.2$			$\theta = 0.2$			$\theta = 0.5$		
	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
E	.592	.746	.805	.067	.081	.092	.037	.041	.040	.001	.001	.007
LL	.582	.635	.836	.081	.082	.130	.037	.036	.076	.004	.001	.146
MR	.517	.564	.555	.073	.071	.072	.035	.039	.038	0	0	0
LLS	.610	.644	.770	.082	.075	.101	.039	.040	.051	.001	.001	.041
MRS	.610	.638	.716	.082	.075	.092	.039	.039	.044	.001	0	.002
B	.413	.673	.755	.029	.084	.097	.018	.043	.046	0	0	.003
M	.598	.522	.508	.066	.038	.020	.036	.014	.008	.001	0	0

Table 7: *Power results for the Mardia copula with parameters* $\theta = -0.5, -0.2, 0.2, 0.5$.

Overall the results are better for tests based on the Anderson-Darling distance and the Cramér-von-Mises distances, combined with the shrunken type of kernel estimators.

Thanks to the construction of the modified Mardia copula via the scaling factor γ in (9), it is possible to "adjust" γ in order to get a testing model for which the values of a and b are close to these for the previous analysis. By doing so we will be able to compare results for these type of mixture copulas, with local violations of the PQD condition, with copulas for which there is a global violation of this condition. Table 8 lists possible values of γ, which give either the same a or b values as for the copulas considered in Section 3.1. It is not possible to obtain all values of a and b for all theta, because of the constraint $\gamma\theta^2 < 1$.

γ	a	b
	$\theta = -0.5$	
0.825	0.029	0.009
1.223	0.043	0.013
1.358	0.048	0.014
1.940	0.068	0.020

γ	a	b
	$\theta = -0.2$	
8.056	0.029	0.006
11.944	0.043	0.010
17.560	0.063	0.014
	$\theta = 0.2$	
18.125	0.029	0.003

Table 8: *Approximate values of a and b for mixtures of Mardia and independence copulas given by* (8) *for different values of gamma and theta.*

	$\gamma = 0.825$			$\gamma = 1.223$			$\gamma = 1.358$			$\gamma = 1.940$		
	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
E	.461	.597	.670	.781	.887	.939	.863	.952	.977	.990	.999	.999
LS	.461	.509	.700	.746	.785	.907	.825	.848	.959	.985	.981	.995
MR	.428	.457	.449	.691	.719	.714	.759	.800	.793	.955	.968	.967
LLS	.472	.520	.624	.776	.803	.897	.848	.870	.925	.987	.990	.997
MRS	.472	.515	.564	.775	.800	.856	.849	.870	.917	.987	.990	.998
B	.299	.542	.605	.634	.838	.899	.729	.906	.949	.973	.997	.998
M	.462	.385	.348	.784	.742	.705	.865	.843	.813	.989	.989	.992

Table 9: *Power results for the mixtures of Mardia and independence copulas given by* (8) *for* $\theta = -0.5$ *and different gamma values.*

Table 9 contains the simulation results for the modified Mardia copula in (8) with $\theta = -0.5$ and various values of γ. In terms of a (b) the results for γ equal to 0.825 (1.358) can be compared with the results from the study in Section 3.1 in case $\tau = -0.11$. The conclusions in terms of possible best choices for distance measures and nonparametric estimators remain the same.

	$\gamma = 8.056$			$\gamma = 11.944$			$\gamma = 17.560$		
	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
E	.396	.542	.710	.742	.897	.965	.978	.999	1
LL	.408	.387	.791	.721	.664	.963	.953	.913	.996
MR	.256	.285	.281	.440	.491	.487	.711	.755	.751
LLS	.427	.396	.657	.745	.712	.903	.965	.946	.992
MRS	.425	.394	.565	.744	.702	.849	.963	.945	.989
B	.241	.449	.632	.549	.796	.913	.920	.986	.996
M	.396	.287	.293	.743	.678	.677	.978	.964	.965

Table 10: *Power results for the mixtures of Mardia and independence copulas given by* (8) *for* $\theta = -0.2$ *and different gamma values.*

The same analysis has been done for rescaled Mardia copulas with $\theta = -0.2$ and various values for the γ parameter. We can see from the results presented in Table 10 that there is the same gradation in test performances as in case of $\theta = -0.5$ and that similar conclusions hold.

3.3 Size simulation study for Frank copula

In this section the aim is to investigate the actual size properties of the proposed tests and to compare these also with the actual size results for the testing procedures of Scaillet (2005). We therefore focus on the Frank copula, since this model served for the simulation study of this kind in Scaillet (2005). In particular we are interested to see how the actual size is influenced when having a true copula equal to the independence copula or a true copula that has stronger PQD characteristics.

	$\tau = -0.07$			$\tau = -0.03$			$\tau = 0$			$\tau = 0.03$			$\tau = 0.07$		
	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD	KS	CvM	AD
E	.337	.405	.408	.117	.135	.135	.056	.059	.061	.008	.011	.010	.003	0	0
LL	.396	.420	.263	.134	.139	.084	.055	.060	.049	.017	.011	.031	0	.001	.008
MR	.413	.410	.411	.131	.128	.130	.059	.061	.060	.012	.008	.008	.001	.001	.001
LLS	.377	.412	.385	.134	.134	.119	.055	.061	.058	.011	.010	.013	.001	.001	0
MRS	.377	.408	.422	.134	.134	.136	.055	.061	.063	.011	.010	.010	.001	.001	0
B	.199	.430	.430	.054	.142	.146	.025	.062	.066	.002	.011	.012	0	0	0
M	.333	.263	.176	.113	.065	.043	.057	.029	.014	.006	.002	.001	.003	0	0

Table 11: *Simulation results for power and actual size for Frank copula with different values of Kendall's tau.*

Results from Table 11 suggest that the actual sizes of the tests are quite good and that the significance level holds when the copula does not have too strong PQD-characteristics. Note that for the independence based methods in the case of a true independence copula (the case $\tau = 0$) the differences between the significance level and the actual size are just to be explained by the Monte Carlo simulation error. The general tendency when going deeper into the null hypothesis (i.e. τ taking on positive values) is clearly visible, since the actual size is decreasing. When getting more and more into violation of the null hypothesis (i.e. τ taking on bigger negative values) the power increases. Again here the test based on the local linear estimator combined with the Anderson-Darling distance measure has the worst performance. This was also noticed from previous simulation results, and hence the use of this specific test statistic should be avoided. One can also see that the bootstrap method combined with Kolmogorov-Smirnov distance and the multiplier method combined with Cramér-von Mises or Anderson-Darling do not hold the level. Moreover, neither of the considered tests holds the level for $C \in H_0$, except for independence based tests in the border (independence; $\tau = 0$) case.

4 Applications

We illustrate the usefulness of the testing procedures on three data examples. A first example concerns the well-known data set from the insurance market introduced by Frees and Valdez (1998) and described in detail in Denuit and Scaillet (2004). The second example is on a data set on life expectancy at birth. This example is similar to the one in Scaillet (2005) but we use an updated data set available from the World Factbook of the Central Intelligence Agency (2008). Finally, in the last example we investigate the dependence structure between the Belgian stock index BEL20 and the exchange rate between the currencies Euro and American Dollar.

4.1 Insurance claim data

This data set consists of 1466 uncensored claims (losses) and claims' costs (ALAE) which are presented together with the corresponding pseudo-observations in Figure 3.

The empirical value of Kendall's tau for this sample is 0.31, which is relatively high when thinking of our testing procedures framework of Section 3. In addition the sample size is much higher than the sample size ($n = 200$) considered in the simulation study.

The approximated P-values provided in Table 12 confirm that there is no proof for rejecting the positive quadrant dependency structure between the data in this example.

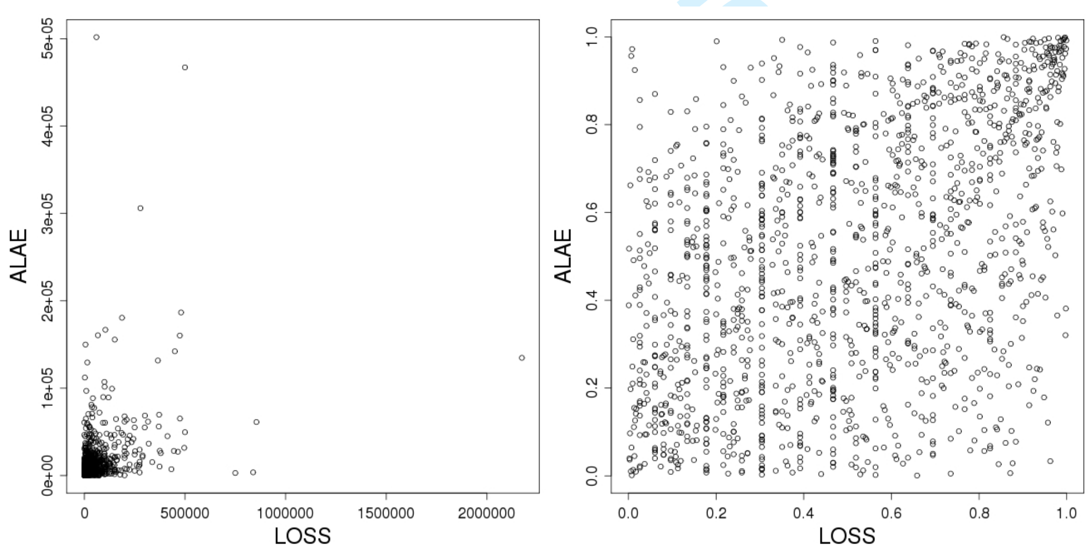


Figure 3: *Scatterplot of the insurance claim data and plot of the corresponding pseudo-observations.*

	KS	CvM	AD
E	.825	1	1
LL	.989	.990	.975
MR	.812	1	1
LLS	.996	1	1
MRS	.992	.991	.971
B	.989	1	1
M	.813	1	1

Table 12: *Approximated P-values for the data set of losses and losses' costs.*

4.2 Life expectancy at birth for men and women

This data set consists of estimated life expectancy at birth for men and women in 223 countries. In Figure 4 we can see a high concentration of pseudo-observations around the positive diagonal, which suggests that there is strong positive dependence structure. This is also supported by the empirical value of Kendall's tau which equals 0.86. From the approximated P-values in Table 13 we can indeed see that there is a very strong evidence for not rejecting the positive quadrant dependence. The evidence is even much stronger here than in the previous example, although here the sample size is more than six times smaller.

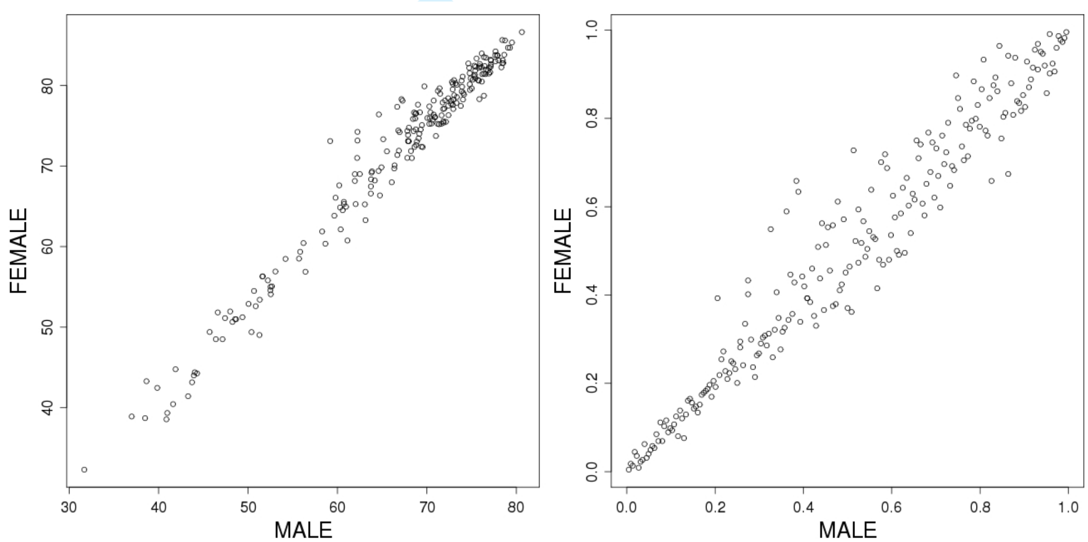


Figure 4: *Scatterplot of the data concerning life expectancy at birth of male vs. female and plot of the corresponding pseudo-observations.*

4.3 The BEL20 index and the EUR/DOL exchange rate

This data set consists of observations on the Belgian stock index BEL20 and the currency exchange rate EUR/DOL from the period January 2, 2008 till January 8, 2009, resulting into 259 common observations. Figure 5 depicts the log-returns of the index and the currency exchange rate together with a plot of their pseudo-observations. In contrast to the previous examples, it is less clear from the plots whether the data are positive quadrant dependent or not.

	KS	CvM	AD
E	1	1	1
LL	.995	.999	.999
MR	.875	.994	.994
LLS	.999	1	1
MRS	.999	1	1
B	1	1	1
M	1	1	1

Table 13: *Approximated P-values for the data set of life expectancy at birth for male vs. female.*

The empirical Kendall's tau value equals -0.06. This suggests that there is no positive dependence. However, when we look at Table 14 then, with 5% significance level, there is no evidence to reject the null hypothesis, except in the case of one test. Since this one single case is the case of local linear estimation with the Anderson-Darling distance measure, it makes sense to conclude that there is no evidence against positive dependence.

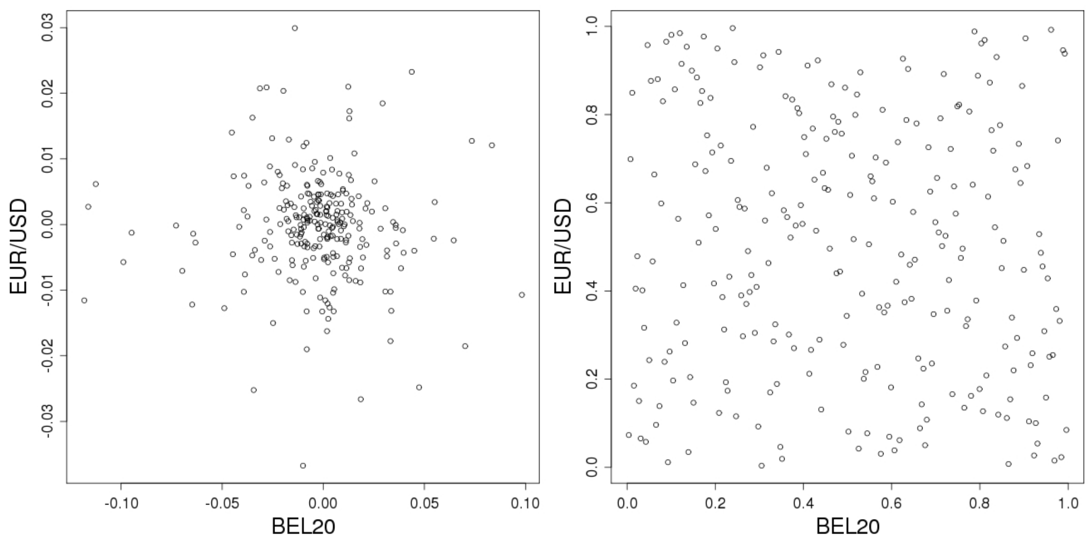


Figure 5: *Scatterplot of the data of log-returns of BEL20 vs. EUR/DOL and plot of the corresponding pseudo-observations.*

	KS	CvM	AD
E	.255	.084	.064
LL	.115	.074	.049
MR	.099	.086	.087
LLS	.144	.081	.084
MRS	.143	.081	.061
B	.549	.113	.088
M	.247	.166	.217

Table 14: *Approximated P-values for the data set of log-returns of BEL20 vs. EUR/DOL.*

The first two data sets were examples of clear positive quadrant dependence, with non rejected null hypothesis with very high P-values. In the first case it was a matter of strong positive quadrant dependence, but large sample size, and in the second case of very strong positive quadrant dependence and a moderate sample size.

The third data set presents a less obvious case. In most of the tests we are close to the significance level of 5%. Additional tests for independence do not reject that the data come from the Π copula.

5 Conclusion and Further discussion

The main goal of this paper was to propose well performing testing procedures for testing the null hypothesis of positive quadrant dependence. We relied on recently developed copula estimators and on different functional distances. We proved the consistency of the proposed tests and provided a simulation study to illustrate the finite sample performances on a diverse set of testing problems, including quite challenging problems. The testing procedures were illustrated on real data applications.

Other discrepancy measures than the three considered so far can be thought of. From our extensive study, we find it worth to report on the following modifications of the Cramér-von Mises (CvM) and the Anderson-Darling (AD) distance measures:

$$S_n^{\mathrm{CvM2}} = n \int_{\mathbb{I}^2} (uv - \hat{C}_n(u,v))_+^2 dudv \quad \text{and} \quad S_n^{\mathrm{AD2}} = n \int_{\mathbb{I}^2} \frac{(uv - \hat{C}_n(u,v))_+^2}{uv(1-u)(1-v)} dudv \,. \tag{10}$$

A selection of simulation results for these alternative distances are provided in Tables 15—18, which present parts of Tables 2, 5, 7 and 9 extended with the simulation results for the distance measures in (10). Note the considerable improved power for the tests based on these distances for the classical copulas, as well as the 'switch' in performance between the bootstrap and multiplier based tests. The multiplier method gives for CvM2 and AD2 based test statictics comparable powers to the empirical method for classical copulas, but still has far lower powers for the mixture copulas.

	Frank				FGM				Gumbel			
	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2
	$\tau = -0.11$											
E	.729	.741	.719	.739	.726	.745	.749	.757	.710	.705	.773	.755
LL	.742	.747	.543	.727	.762	.761	.536	.747	.765	.757	.791	.791
MR	.743	.750	.745	.751	.741	.751	.742	.753	.698	.693	.693	.693
LLS	.747	.752	.696	.747	.745	.753	.719	.765	.736	.735	.803	.776
MRS	.745	.755	.738	.748	.745	.752	.765	.764	.728	.731	.783	.764
B	.752	.694	.746	.663	.751	.703	.765	.676	.715	.638	.780	.674
M	.573	.740	.469	.737	.569	.746	.460	.749	.540	.703	.522	.744

Table 15: *Simulation results on classical copulas: extension of part of Table 2.*

Our main findings from all simulations can be summarized as follows:

- The use of copula kernel type estimators increases the power of the testing procedures in particular for Kolmogorov-Smirnov based tests. For the other test statistics they only lead to slightly higher powers for 'classical' copulas, but can possibly lead to a power loss for 'mixture' alternatives.

- When using the empirical copula estimate (E), the Cramér-von Mises (CvM) and the Anderson-Darling (AD) based statistics give consistently higher power than

	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2
	θ =1.47				θ=9.5				θ =17.5			
E	.050	.051	.054	.049	.389	.315	.617	.512	.777	.656	.926	.855
LL	.044	.048	.064	.052	.261	.218	.802	.518	.520	.398	.969	.880
MR	.050	.050	.049	.048	.108	.098	.106	.103	.153	.111	.163	.133
LLS	.050	.049	.052	.051	.276	.266	.571	.468	.578	.513	.880	.814
MRS	.050	.051	.048	.050	.269	.264	.527	.453	.561	.507	.853	.789
B	.055	.037	.055	.031	.291	.171	.496	.250	.639	.378	.872	.541
M	.016	.052	.007	.048	.143	.227	.194	.318	.455	.470	.594	.643

Table 16: *Simulation results on mixtures on Frank copulas: extension of part of Table 5.*

	$\theta = -0.5$				$\theta = -0.2$				$\theta = 0.2$			
	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2
E	.746	.595	.805	.647	.081	.076	.092	.084	.041	.039	.040	.041
LL	.635	.574	.836	.667	.082	.078	.130	.093	.036	.037	.076	.045
MR	.564	.531	.555	.535	.071	.071	.072	.073	.039	.040	.038	.039
LLS	.644	.584	.770	.656	.075	.074	.101	.088	.040	.037	.051	.041
MRS	.638	.582	.716	.638	.075	.074	.092	.087	.039	.037	.044	.040
B	.673	.505	.755	.495	.084	.056	.097	.053	.043	.026	.046	.025
M	.522	.557	.508	.570	.038	.071	.020	.073	.014	.037	.008	.039

Table 17: *Simulation results on Mardia copulas: extension of part of Table 7.*

	$\gamma = 8.056$				$\gamma = 11.944$				$\gamma = 17.560$			
	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2	CvM	CvM2	AD	AD2
E	.542	.351	.710	.449	.897	.661	.965	.760	.999	.930	1	.972
LL	.387	.324	.791	.504	.664	.568	.963	.762	.913	.824	.996	.960
MR	.285	.252	.281	.263	.491	.429	.487	.445	.755	.683	.751	.707
LLS	.396	.335	.657	.449	.712	.601	.903	.735	.946	.870	.992	.947
MRS	.394	.333	.565	.430	.702	.596	.849	.721	.945	.871	.989	.942
B	.449	.253	.632	.286	.796	.496	.913	.550	.986	.835	.996	.868
M	.287	.306	.293	.353	.678	.566	.677	.618	.964	.885	.965	.904

Table 18: *Simulation results on mixtures of Mardia copulas: extension of part of Table 10.*

the Kolmogorov-Smirnov (KS) based tests in all considered situations. Note that modified distance versions of these tests lead to considerably higher powers.

- The proposed resampling from an independence copula method seems to work comparatively with a bootstrap (B) method for classical families of copulas when using CvM and AD and the empirical copula. For CVM2 and AD2 our method works slightly better. For mixture copulas our method gives significantly higher power than bootstrap or multiplier methods for CvM and AD. The method of resampling from the independent copula consistently increases the power of KS based tests.

In the simulation study, we obtained much better results for the 'classical' copula families than for the more challenging 'mixtures' of copulas. An issue that we did not discuss so far is the choice of the smoothing parameter h_n for the kernel estimators. We applied the data-driven method proposed by Omelka et al. (2009), where the bandwidth is selected as a minimizer of the integrated asymptotic mean squared error of the copula estimator using Frank copula as a reference copula. This reference copula is fitted via the empirical Kendall's tau from the data. In the case of non-PQD examples which are oscillating around the independence copula, the empirical Kendall's tau is close to zero, and as a consequence the chosen data-driven bandwidth is very likely large and not very appropriate. This problem deserves further research and other methods of bandwidth selection, tailored also for testing purposes, should be developed.

The testing procedures described in this paper use Monte Carlo simulation with the independence case as a reference case to obtain the critical values. Overall these tests outperform tests based on bootstrap or multiplier techniques. Another possible approach would be to resample from a nonparametric copula estimate that satisfies the null hypothesis. This could possibly lead to increased powers. This approach is the subject of current research.

Appendix: Proof of Theorem 1

We now prove the two parts of Theorem 1.

For part i) we need to prove that if H_0 is true, then $\lim_n \mathbb{P}(\text{reject } H_0) \leq \alpha$.

Let $\epsilon > 0$ be fixed and consider $\delta > 0$ and $\eta > 0$, to be specified later on. Denote

$$\begin{aligned} I_0 &= \{(u,v):\ C(u,v) = uv\} \\ I_1^\delta &= \{(u,v):\ uv < C(u,v) \leq uv + \delta\} \\ I_2^\delta &= \{(u,v):\ uv + \delta < C(u,v)\}. \end{aligned}$$

Further, let S_n^A denote the test statistic restricted to the subset $A \subset \mathbb{I}^2 = [0,1] \times [0,1]$, e.g. $S_n^{A,\mathrm{KS}} = \sqrt{n}\sup_{(u,v)\in A}(uv - \hat{C}_n(u,v))_+$.

We further distinguish between the cases of the Cramér-von-Mises and the Anderson-Darling distance measures on the one hand, and the Kolmogorov-Smirnov distance measure on the other hand.

- For the Cramér-von Mises and the Anderson-Darling distance measures we proceed as follows. For brevity of presentation we only give details for the Cramér-von Mises distance, the one for the Anderson-Darling distance being along the same lines.

  First note that

$$\begin{aligned} \mathbb{P}\left(S_n > c^\Pi_{\alpha,n}\right) &= \mathbb{P}\left(S_n^{I_0} + S_n^{I_1^\delta} + S_n^{I_2^\delta} > c^\Pi_{\alpha,n} - \eta + \eta\right) \\ &\leq \mathbb{P}\left(S_n^{I_0} > c^\Pi_{\alpha,n} - \eta\right) + \mathbb{P}\left(S_n^{I_1^\delta} > \eta\right) + \mathbb{P}\left(S_n^{I_2^\delta} > 0\right) . \end{aligned} \tag{A.1}$$

  Recall that G_C denotes the limiting Gaussian process of $\sqrt{n}(\hat{C}_n - C)$ and put $c^\Pi_\alpha := \lim_n c^\Pi_{\alpha,n}$. On I_0, $C(u,v) = \Pi(u,v)$ and from (5) and (6) it can be deduced that the covariance functions of the processes G_C and G_Π coincide on the interior of I_0 denoted by $\mathrm{int}(I_0)$ (because of assumption **(C1)**), which imply that the processes of $\{G_C, (u,v) \in \mathrm{int}(I_0)$ and $\{G_\Pi, (u,v) \in \mathrm{int}(I_0)\}$ have the same distribution. The assumption on ∂I_0 (see **(C4)**) ensures that zero probability is given to this set. These considerations justify the passage from integrals involving G_C and dC to integrals involving G_Π and $d\Pi$ in the bound of the first component in (A.1) below. Therefore, with the help of the weak convergence of the process $\sqrt{n}(\hat{C}_n - C)$ and assumption **(C4)** one obtains, for all sufficiently large n, the bound

$$\begin{aligned}
\mathbb{P}\left(S_n^{I_0} > c^{\Pi}_{\alpha,n} - \eta\right) &= \mathbb{P}\left(n \iint_{I_0} \left(C(u,v) - \hat{C}_n(u,v)\right)_+^2 d\hat{C}_n(u,v) > c^{\Pi}_{\alpha,n} - \eta\right) \\
&\leq \mathbb{P}\left(\iint_{I_0} (-G_C(u,v))_+^2 \, dC(u,v) > c^{\Pi}_{\alpha} - 2\eta\right) + \epsilon \\
&= \mathbb{P}\left(\iint_{I_0} (-G_\Pi(u,v))_+^2 \, d\Pi(u,v) > c^{\Pi}_{\alpha} - 2\eta\right) + \epsilon \\
&\leq \mathbb{P}\left(\iint_{\mathbb{I}^2} (-G_\Pi(u,v))_+^2 \, d\Pi(u,v) > c^{\Pi}_{\alpha}\right) + 2\epsilon \\
&= \alpha + 2\epsilon \, .
\end{aligned}$$

The second component in (A.1) can be bounded for all n sufficiently large and for small enough δ

$$\begin{aligned}
\mathbb{P}\left(S_n^{I_1^\delta} > \eta\right) &= \mathbb{P}\left(n \iint_{I_1^\delta} \left(C(u,v) - \hat{C}_n(u,v)\right)_+^2 d\hat{C}_n(u,v) > \eta\right) \\
&\leq \mathbb{P}\left(\iint_{I_1^\delta} (-G_C(u,v))_+^2 \, dC(u,v) > \eta\right) + \epsilon \leq 2\,\epsilon. \qquad \text{(A.2)}
\end{aligned}$$

The last inequality in (A.2) may be justified as follows. As the limiting process G_C is centered and Gaussian, for each $\epsilon > 0$ there exists $K < \infty$ such that

$$\mathbb{P}\left(\sup_{\mathbb{I}^2} |G_C(u,v)| > K\right) < \epsilon. \qquad \text{(A.3)}$$

Thus with probability greater than $1 - \epsilon$ it holds that

$$\iint_{I_1^\delta} (-G_C(u,v))_+^2 \, dC(u,v) \leq K^2 \mu_C(I_1^\delta), \qquad \text{(A.4)}$$

where μ_C is the measure associated with the copula C. Let $\{\delta_k,\, k \in \mathbb{N}\}$ be a sequence of positive numbers decreasing to zero. The definition of I_1^δ implies that $\bigcap_{k=1}^{\infty} I_1^{\delta_k} = \emptyset$. Now, the continuity of a measure (see e.g. Lemma 1.14 of Kallenberg (1997)) yields

$$\lim_{k\to\infty} \mu_C(I_1^{\delta_k}) = 0,$$

which implies that $\mu_C(I_1^\delta)$ can be made arbitrary small, and hence the right-hand side of (A.4) can be made smaller than η by taking δ small enough. This together with (A.3) yields (A.2).

Finally, the third component in (A.1) can be bounded by ϵ for all n sufficiently large

$$\begin{aligned}
\mathbb{P}\left(S_n^{I_2^\delta} > 0\right) &\leq \mathbb{P}\left(\sup_{I_2^\delta} \left(uv - \hat{C}_n(u,v)\right) > 0\right) \\
&\leq \mathbb{P}\left(\sup_{I_2^\delta} \left(C(u,v) - \hat{C}_n(u,v)\right) > \delta\right) \\
&\leq \epsilon. \qquad \text{(A.5)}
\end{aligned}$$

- For the Kolmogorov-Smirnov distance measure we proceed as follows:

$$\mathbb{P}\left(S_n > c^{\Pi}_{\alpha,n}\right) \le \mathbb{P}\left(S_n^{I_0 \cup I_1^\delta} > c^{\Pi}_{\alpha,n}\right) + \mathbb{P}\left(S_n^{I_2^\delta} > 0\right) . \tag{A.6}$$

The first term on the right-hand side of (A.6) may be bounded as

$$\begin{aligned}\mathbb{P}\left(S_n^{I_0 \cup I_1^\delta} > c^{\Pi}_{\alpha,n}\right) &= \mathbb{P}\left(\sqrt{n} \sup_{I_0 \cup I_1^\delta} \left(uv - \hat{C}_n(u,v)\right)_+ > c^{\Pi}_{\alpha,n}\right) \\ &\le \mathbb{P}\left(\sqrt{n} \sup_{I_0 \cup I_1^\delta} \left(C(u,v) - \hat{C}_n(u,v)\right)_+ > c^{\Pi}_{\alpha,n}\right) \\ &\le \mathbb{P}\left(\sup_{I_0 \cup I_1^\delta} (-G_C(u,v))_+ > c^{\Pi}_{\alpha} - \eta\right) + \epsilon \end{aligned} \tag{A.7}$$

Let $\{\delta_k, k \in \mathbb{N}\}$ be again a decreasing sequence of positive numbers going to zero and for $k \in \mathbb{N}$ put

$$A_k = \left[\sup_{I_0 \cup I_1^{\delta_k}} (-G_C(u,v))_+ > c^{\Pi}_{\alpha} - \eta\right].$$

Note that $A_k \supset A_{k+1}$ and by the almost sure continuity of the paths of the process G_C (see e.g. Addendum 1.5.8 of van der Vaart and Wellner (1996)) we have

$$\bigcap_{k=1}^{\infty} A_k = \left[\sup_{I_0} (-G_C(u,v))_+ \ge c^{\Pi}_{\alpha} - \eta\right] \cup N, \quad \text{where } \mathbb{P}(N) = 0.$$

Once more using the continuity of the probability measure from above we get

$$\lim_{k\to\infty} \mathbb{P}\left(\sup_{I_0 \cup I_1^{\delta_k}} (-G_C(u,v))_+ > c^{\Pi}_{\alpha} - \eta\right) = \mathbb{P}\left(\sup_{I_0} (-G_C(u,v))_+ \ge c^{\Pi}_{\alpha} - \eta\right).$$

Thus the probability on the right-hand side of (A.7) may be for sufficiently small δ and η bounded as

$$\begin{aligned}&\mathbb{P}\left(\sup_{I_0 \cup I_1^\delta} (-G_C(u,v))_+ > c^{\Pi}_{\alpha} - \eta\right) \le \mathbb{P}\left(\sup_{I_0} (-G_C(u,v))_+ \ge c^{\Pi}_{\alpha} - \eta\right) + \epsilon \\ &\le \mathbb{P}\left(\sup_{I_0} (-G_C(u,v))_+ \ge c^{\Pi}_{\alpha}\right) + 2\epsilon = \mathbb{P}\left(\sup_{I_0} (-G_\Pi(u,v))_+ \ge c^{\Pi}_{\alpha}\right) + 2\epsilon \\ &\le \mathbb{P}\left(\sup_{\mathbb{I}^2} (-G_\Pi(u,v))_+ \ge c^{\Pi}_{\alpha}\right) + 2\epsilon = \alpha + 2\epsilon, \end{aligned} \tag{A.8}$$

which together with (A.7) gives for all sufficiently large n

$$\mathbb{P}\left(S_n^{I_0 \cup I_1^\delta} > c^{\Pi}_{\alpha,n}\right) \le \alpha + 3\epsilon.$$

The first equality in (A.8) follows from the fact that according to Remark 1, the limiting processes G_C and G_Π are zero on the boundary of $[0,1]^2$, and therefore it suffices to look at the supremum over the set $I_0 \cap \mathrm{int}([0,1]^2) = I_0 \cap]0,1[^2$. Below we show that for all points of this set it holds that

$$C_u(u,v) = v = \Pi_u(u,v) \qquad \text{and} \qquad C_v(u,v) = u = \Pi_v(u,v) \ , \tag{A.9}$$

which then implies that the limiting processes G_C and G_Π have the same distribution on $I_0 \cap]0,1[^2$. For proving statement (A.9) suppose there is a point $(u_0, v_0) \in I_0 \cap]0,1[^2$, such that for instance $C_u(u_0,v_0) < v_0$. Put $\varepsilon = v_0 - C_u(u_0,v_0)$. Then by Taylor expansion for a sufficiently small $\Delta > 0$, we have that

$$\begin{aligned} C(u_0+\Delta, v_0) &= C(u_0,v_0) + \Delta\, C_u(u_0,v_0) + o(\Delta) \\ &= u_0\, v_0 + \Delta\, C_u(u_0,v_0) + o(\Delta) \\ &< u_0\, v_0 + \Delta\, (v_0 - \tfrac{\varepsilon}{2}) \\ &< (u_0+\Delta)\, v_0 \\ &= \Pi(u_0+\Delta, v_0) \ , \end{aligned}$$

which contradicts the null hypothesis of PQD. A similar argument can be given if $C_u(u_0,v_0) > v_0$ or $C_v(u_0,v_0) \neq u_0$. This completes the proof of (A.9) and the bound for the first component in (A.6).

The second component in (A.6) can be bounded by ϵ for all n sufficiently large in an analogous manner as in (A.5).

For part two of Theorem 1 we need to prove: if H_0 is false, then $\lim_{n\to\infty} \mathbb{P}(\text{reject } H_0) = 1$. Since it holds that

$$\exists (u,v) \in \mathbb{I}^2 : \quad C(u,v) < uv,$$

and because of the continuity of the distances, the test statistic S_n converges to infinity in probability, so

$$\mathbb{P}(S_n^C > c_{\alpha,n}^{\Pi}) \longrightarrow 1.$$

□

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