PDE 1, WS 2018-2019 Written exam January 29, 2019

Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Introduce the notion of Gelfand triple and give the proper meaning to $\langle \ldots, \ldots \rangle_V$ for properly chosen space V.

Solution:

See lecture.

[100] 2. 30% Define the notion of a set Ω, which is $\mathcal{C}^{0,\alpha}$.

35% Consider the domain Ω given as

$$\Omega := \left\{(x,y) \in \mathbb{R}^2;\ |x| + |y| < 1\right\}.$$

Show from the definition that Ω is Lipschitz.

35% Show from the definition that Ω is **not** $\mathcal{C}^1$.

Solution:

Only a sketch:

Definition: We say that $\Omega \in \mathcal{C}^{0,\alpha}$, if it is open and there exist positive numbers α, β, M orthogonal transformations T_r and M $\mathcal{C}^{0,\alpha}$ functions a_r such that if we define

$$\begin{aligned}
V_r^+ &:= \{(x_r', x_{r_d}) \in \mathbb{R}^d;\ |x_{r_i}'| < \alpha \text{ for } i = 1, \dots d-1,\ a_r(x_r') < x_{r_d} < a_r(x_r') + \beta\} \\
V_r^- &:= \{(x_r', x_{r_d}) \in \mathbb{R}^d;\ |x_{r_i}'| < \alpha \text{ for } i = 1, \dots d-1,\ a_r(x_r') - \beta < x_{r_d} < a_r(x_r')\} \\
\Lambda_r &:= \{(x_r', x_{r_d}) \in \mathbb{R}^d;\ |x_{r_i}'| < \alpha \text{ for } i = 1, \dots d-1,\ a_r(x_r') = x_{r_d}\}
\end{aligned}$$

Then

$$T_r(V_r^+) \subset \Omega,\ T_r(V_r^-) \subset \mathbb{R}^d \setminus \Omega,\ \partial\Omega = \Cap_{r=1}^M T_r(\Lambda_r).$$

Task 2: We set $M = 4$ and define $a_r(x_r^1) := |x_r^1|$ for all $r = 1, \dots, 4$. We also set $\alpha = 3/4$ and $\beta := 1/4$ and define the set V_r (it is the same set for each r). Finally, we find four proper orthogonal transformations as

$$\begin{aligned}
T_1(x_1, x_2) &:= (x_1, x_2 - 1), \\
T_2(x_1, x_2) &:= (x_1, 1 - x_2), \\
T_3(x_1, x_2) &:= (x_2 - 1, x_1), \\
T_4(x_1, x_2) &:= (1 - x_2, x_1).
\end{aligned}$$

Finally, it is easy to check that such a setting gives the desired property. (Note that T_1 covers neighborhood of $(0, -1)$, T_2 represents $(0, 1)$, T_3 stands for $(-1, 0)$ and T_4 for $(1, 0)$.

Task 3 Assume for a contradiction that Ω is $\mathcal{C}^1$. Let us consider a point $(0, -1) \in \partial\Omega$. Then we can find an orthogonal transformation T, a $\mathcal{C}^1$ function a such that for all $|x_1| \leq \alpha$

$$T(x_1, a(x_1)) \subset \partial\Omega$$

in addition there is $\tilde{x}$, $|\tilde{x}| < \alpha$ such that $T(\tilde{x}, a(\tilde{x})) = (0, -1)$ Since any orthogonal transformation in 2D can be written as (for some t, c_1, c_2)

$$y_1 := x_1 \cos t + x_2 \sin t + c_1, \qquad y_2 := -x_1 \sin t + x_2 \cos t + c_2$$

We obtain from the constrain that for all $|x_1| \leq \alpha$

$$|x_1 \cos t + a(x_1) \sin t + c_1| + |-x_1 \sin t + a(x_1) \cos t + c_2| = 1. \tag{1}$$

In addition we know that here exists $\tilde{x}$ such that

$$f_1(\tilde{x}) := \tilde{x} \cos t + a(\tilde{x}) \sin t + c_1 = 0, \qquad f_2(\tilde{x}) := -\tilde{x} \sin t + a(\tilde{x}) \cos t + c_2 = -1.$$

Consequently, it follows from (1) that on some neighborhood of $\tilde{x}$, we have

$$-1 \leq -\tilde{x}\sin t + a(\tilde{x})\cos t + c_2 < 0.$$

Thus, f_2 has a minimum at $\tilde{x}$, so

$$f_2'(\tilde{x}) = -\sin t + a'(\tilde{x})\cos t = 0. \tag{2}$$

Let us also assume for a moment that

$$f_1'(\tilde{x}) := \cos t + a'(\tilde{x})\sin t = 0.$$

However, combining this with (2) we see that it cannot happen (multiply f_2' by $-\sin t$, f_1' by $\cos t$ and sum the resulting identities). Hence, we have $f_1'(\tilde{x}) \neq 0$, and since $f_1 \in \mathcal{C}^1$ and we have $f_1(\tilde{x}) = 0$ then the function $|f_1(x_1)|$ cannot have derivative at $\tilde{x}$. Since (1) is equivalent to

$$|x_1 \cos t + a(x_1)\sin t + c_1| = 1 + f_2(x_1) \tag{3}$$

we see that the right hand side has derivative at $\tilde{x}$ while the left hand side does not. Which is a contradiction.

[100] 3. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set and $n \in \mathbb{N}$. Assume that $f_i \in L^2(\Omega)$ for $i = 1, \dots, n$. Consider the problem: Find n functions $u_i : \Omega \to \mathbb{R}$ (here $i = 1, \dots, n$) solving

$$-\triangle u_i + \sum_{j=1}^{n} a_{ij} u_j = f_i \quad \text{in } \Omega, \qquad i = 1, \dots, n,$$

$$u_i = 0 \quad \text{on } \partial\Omega, \qquad i = 1, \dots, n,$$

where $a_{ij} \in \mathbb{R}$ are given.

20% Find a proper definition of a weak solution. Check that for given data such a definition is meaningful.

30% Use the Lax-Milgram theorem and show that if the matrix $A = \{a_{ij}\}_{i,j=1}^n$ is positively semidefinite, then there exists a unique weak solution. (Hint, find a proper bilinear form and a proper function space (a subspace of $W^{1,2} \times \cdots \times W^{1,2}$), for which you can use the Lax-Milgram theorem.)

30% In case that A is not positively definite, find the **sharp** relation between the spectrum of A and the spectrum of the operator $-\Delta u$ that guarantees the existence of a weak solution for every $f_i \in L^2(\Omega)$, with $i = 1, \dots, n$. (Hint, consider a proper basis $\{w_i\}_{i=1}^\infty$ of $W_0^{1,2}$ and consider a solution of the form $u_i = \sum_{j=1}^\infty b_{ij} w_j$.)

20% Consider $\Omega := (0, \pi)^2$, $n = 2$ and $f_1 = f_2 = \sin x_1 \sin x_2$. Find the **sharp** assumption on the matrix A for which you can find a solution. Is it unique? (Hint: I hope you remember the homework about eigen-functions and eigen-vectors for the Laplace operator on the square.)

Solution:

Weak formulation: We deal with homogeneous Dirichlet problem. Hence, we say that u_i with $i = 1, \dots, n$ is a weak solution if $u_i \in W_0^{1,2}(\Omega)$ for all $i = 1, \dots, n$ and for all $\varphi \in W_0^{1,2}(\Omega)$ and all $i = 1, \dots, n$ there holds

$$\int_\Omega \nabla u_i \cdot \nabla \varphi + \int_\Omega \sum_{j=1}^n a_{ij} u_j \varphi = \int_\Omega f_i \varphi. \tag{4}$$

Thanks to the Hölder inequality all integrals are well defined. Equivalently, we can set

$$V := \underbrace{W_0^{1,2}(\Omega) \times \cdots \times W_0^{1,2}(\Omega)}_{n-\text{times}}$$

and look for $u = (u_1, \dots, u_n) \in V$ such that for all $\varphi \in V$ there holds

$$\int_\Omega \nabla u : \nabla \varphi + Au \cdot \varphi = \int_\Omega f \cdot \varphi. \tag{5}$$

Here, we set $f := (f_1, \dots, f_n)$, the matrix $\{A\}_{ij} := a_{ij}$, the symbol "$\cdot$"denotes the scalar product in $\mathbb{R}^n$ and the symbol ":"denotes the scalar product in $\mathbb{R}^n \times \mathbb{R}^d$.

Existence of solution for positively semidefinite A: For our purposes, we define the bilinear B form on V as

$$B(u, v) := \int_\Omega \nabla u : \nabla v + Au \cdot v = \int_\Omega \sum_{i=1}^n \sum_{j=1}^d \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} + \sum_{i,j=1}^n a_{ij} u_j v_i$$

and $F \in V^*$ as

$$\langle F, v\rangle_V := \int_\Omega f \cdot v = \int_\Omega \sum_{i=1}^n f_i v_i, \qquad \text{for all } v \in V.$$

Clearly, the problem then reduces to: find $u \in V$ such that for all $v \in V$ there holds

$$B(u,v) = \langle F, v\rangle_V. \tag{6}$$

The existence and uniqueness will be proven by the Lax-Milgram theorem and the fact that A is assumed to be positively semidefinite. Indeed, V is a Hilbert space. The form B is evidently also bilinear and V-bounded, which follows from the Hölder inequality. Thus, it just remains to prove coercivity. However, using the fact that A is positively semidefinite, we get (we use that u has zero trace)

$$B(u,u) = \sum_{i=1}^n \|\nabla u_i\|_2^2 + \int_\Omega Au \cdot u \geq \sum_{i=1}^n \|\nabla u_i\|_2^2 \overset{\text{Poincaré}}{\geq} c_1 \sum_{i=1}^n \|u_i\|_{1,2}^2 = c_1 \|u\|_V^2.$$

Hence, B satisfies all assumptions of the Lax–Milgram theorem. Thus, existence and uniqueness is proven.

Characterization via spectrum: First, we know that there exists a basis $\{w_j\}_{j=1}^\infty$ of $W_0^{1,2}$, which is orthogonal in $W_0^{1,2}$ and orthonormal in L^2, which consists of eigen functions and eigen vectors of Laplace operator, i.e.,

$$\int_\Omega w_i w_k = \delta_{ik},$$
$$\int_\Omega \nabla w_i \cdot \nabla \varphi = \lambda_i \int_\Omega w_i \varphi, \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega).$$

Note, that we also have $\lambda_i > 0$ for all i. Since, it is a basis, every u_i and f_i can be written as

$$u_i = \sum_{j=1}^\infty b_{ij} w_j, \qquad f_i = \sum_{j=1}^n F_{ij} w_j \text{ with } F_{ij} := \int_\Omega f_i w_j. \tag{7}$$

Due to the property of basis, we can equivalently rewrite (4) as

$$\int_\Omega \nabla u_i \cdot \nabla w_k + \int_\Omega \sum_{j=1}^n a_{ij} u_j w_k = \int_\Omega f_i w_k \qquad \text{for all } i = 1, \ldots, n \text{ and } k \in \mathbb{N}.$$

Using, the definition (7) and the orthogonality and orthonormality of the basis, it reduces to: Find $b_{ik} \in \mathbb{R}$ such that

$$b_{ik}\lambda_k + \sum_{j=1}^n a_{ij} b_{jk} = F_{ik} \qquad \text{for all } i = 1, \ldots, n \text{ and } k \in \mathbb{N}. \tag{8}$$

Hence, if we denote by $\mathbf{b}_k, \mathbf{F}_k \in \mathbb{R}^n$ as

$$\mathbf{b}_k := (b_{1k}, \ldots, b_{nk}), \qquad \mathbf{F}_k := (F_{1,k}, \ldots, F_{nk})$$

Then the relation (8) can be reformulated as: for every $k \in \mathbb{N}$ find $\mathbf{b}_k \in \mathbb{R}^n$ such that

$$(\lambda_k \mathbb{I} + A)\mathbf{b}_k = \mathbf{F}_k, \tag{9}$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R}^{n\times n}$. Next, (9) is just linear algebraic equation. Therefore, if we require that there exists unique solution to (9) for arbitrary $\mathbf{F}_k$, then necessarily $\lambda_k \mathbb{I} + A)$ must be a regular matrix, whin means nothing else that

$$\lambda_k \notin \text{spt } (-A) \text{ for all } k \in \mathbb{N}. \tag{10}$$

Solution for special choice of f_1 and f_2: Here, we recall the homework, where you proved that the eigen-functions and eigen-values for the Laplace operator with zero trace on the square are of the form (we do not normalize them to have L^2 norm equal to 1)

$$w_{lk} = \sin(lx_1)\sin(kx_2), \qquad \lambda_{lk} = l^2 + k^2, \qquad l, k \in \mathbb{N}.$$

Hence, it follows from (10) that if for all $l, k \in \mathbb{N}$

$$-l^2 - k^2 \notin \text{spt } (A)$$

then we have a unique solution.

Now, we can follow the preceding step and look for u of the form

$$u_i = \sum_{lk=1}^{\infty} b_{ilk} w_{lk}, \qquad f_1 = w_{21}, f_2 = w_{11}. \tag{11}$$

Repeating step by step the previous procedure, we end up with the system of equations

$$\begin{aligned} b_{1kl}\lambda_{kl} + a_{11}b_{1kl} + a_{12}b_{2kl} &= \begin{cases} 1 & \text{if } k = 1 \;\&\; l = 1, \\ 0 & \text{otherwise} \end{cases} \\ b_{2kl}\lambda_{kl} + a_{21}b_{1kl} + a_{22}b_{2kl} &= \begin{cases} 1 & \text{if } k = 1 \;\&\; l = 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{12}$$

Hence, if we consider $l > 1$ or $k > 1$ then we can simply set $b_{ikl} = 0$ to fulfill (12). However, if spectrum of $-A$ contains $(l^2 + k^2)$ we can chose b_{ikl} as a corresponding eigen-vector to A, so we have non-uniqueness.

Finally, if the spectrum of A does not contain -2, the system (12) always have a (non-unique) solution. On the other, if the spectrum of $-A$ contains 2 then we can first rewrite (12) for $k = l = 1$. Then it reduces (denoting $(B = b_{111}, b_{211}))$ to find B such that

$$(2\mathbb{I} + A)B = (1, 1).$$

Since -2 is in spectrum of A the above equation has a (nonunique) solution if and only if

$$0 = w_1 + w_2 = (1, 1) \cdot (w_1, w_2) \qquad \text{for any solution } w \text{ to } (2\mathbb{I} + A^T)w = 0.$$

Or in other words, the vector $(1, 1)$ must be orthogonal to eigen-vectors of matrix A^T corresponding to eigen-value -2.