Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Define the notion of **Bochner measurability** and **Bochner integral**.

Solution:

See lecture.

[100] 2. Let $\Omega \subset \mathbb{R}^2$ be a ball of radius $\frac{1}{2}$ centered at zero, $0 < T \leq \frac{1}{2}$ unit ball and let $u \in L^2(0,T; W_0^{1,2}(\Omega))$ and $f \in L^\infty((0,T)\times\Omega; \mathbb{R}^2)$. Assume in addition that for all $\varphi \in \mathcal{C}_0^\infty((0,T)\times\Omega)$ there holds

$$\int_0^T \int_\Omega u(t,x) \frac{\partial \varphi(t,x)}{\partial t} dx\, dt = \int_0^T \int_\Omega \frac{f \cdot \nabla \varphi(t,x)}{(t^2+|x|^2)^{\frac{3}{4}} \ln(t^2+|x|^2)} dx dt. \tag{B}$$

Prove **in details** that $u \in \mathcal{C}([0,T]; L^2(\Omega))$.

Solution:

Method I: Here we recall the lemma from the lecture: Let V, H, V^* be a Gelfand triple and $u \in L^2(0,T;V)$ and $\partial_t u \in L^2(0,T;V^*)$. Then $u \in \mathcal{C}([0,T];H)$. The proof is a part of the exam - see lecture.

Hence, we want to apply this result for our u and the setting $V := W_0^{1,2}(\Omega 0$ and $H := L^2(\Omega)$. Since we assume that $u \in L^2(0,T;W_0^{1,2}(\Omega))$ it remains to check that $\partial_t u \in L^2(0,T;(W_0^{1,2}(\Omega))^*)$.

For that purpose, let us define for almost all $t \in (0,T)$, $F(t) \in (W_0^{1,2}(\Omega))^*$ by

$$\langle F(t), v\rangle := -\int_\Omega \frac{f \cdot \nabla v(x)}{(t^2+|x|^2)^{\frac{3}{4}} \ln(t^2+|x|^2)} dx \qquad \text{for all } v \in W_0^{1,2}(\Omega). \tag{D}$$

Evidently, for any $t \in (0,T)$ the integral is well defined and also linear with respect to v and so $F(t) \in V^*$. In addition, due to the assumptions on f, we see that $F : t \mapsto F(t)$ is Bochner measurable. Finally, we show that $F \in L^2(0,T;V^*)$. To do so, for any fix $t \in (0,T)$ we have (by the Hölder inequality)

$$\begin{aligned}
\|F(t)\|_{V^*} &:= \sup_{v \in V; \|v\|_V \leq 1} \langle F(t), v\rangle \overset{(D)}{=} \sup_{v \in V; \|v\|_V \leq 1} -\int_\Omega \frac{f \cdot \nabla v(x)}{(t^2+|x|^2)^{\frac{3}{4}} \ln(t^2+|x|^2)} dx \\
&\leq \|f\|_{L^\infty(0,T)\times\Omega} \sup_{v \in V; \|v\|_V \leq 1} \|\nabla v\|_2 \left(\int_\Omega \frac{1}{(t^2+|x|^2)^{\frac{3}{2}} \ln^2(t^2+|x|^2)} dx \right)^{\frac{1}{2}} \\
&\leq C \left(\int_\Omega \frac{1}{(t^2+|x|^2)^{\frac{3}{2}} \ln^2(t^2+|x|^2)} dx \right)^{\frac{1}{2}},
\end{aligned}$$

where C is independent of t. Hence applying the second power to the above inequality and integrating with respect to $t \in (0,T)$, using also the fact that $|x|^2 + t^2 \leq \frac{1}{2}$, we see that (here, $B := B_{\frac{\sqrt{2}}{2}} \subset \mathbb{R}^3$)

$$\int_0^T \|F(t)\|_{V^*}^2 \leq C \int_0^T \int_\Omega \frac{1}{(t^2+|x|^2)^{\frac{3}{2}} \ln^2(t^2+|x|^2)} dx\, dt \leq C \int_B \frac{1}{|z|^3 \ln^2(|z|)}\, dz < \infty.$$

Hence, $F \in L^2(0,T;V^*)$. Finally, we show that $\partial_t u = F$. Let $\eta \in \mathcal{C}_0^\infty(0,T)$ and $v \in \mathcal{C}_0^\infty(\Omega)$ be arbitrary. Using (B) with $\varphi(t,x) := \eta(t)v(x)$, we see that (using the properties of the Gelfand triple and the Bochner integral)

$$\begin{aligned}
&\left\langle \int_0^T \partial_t \eta(t) u(t)\, dt, v \right\rangle_V = \int_\Omega \left(\int_0^T \partial_t \eta(t) u(t,x)\, dt \right) v(x)\, dx \\
&= \int_0^T \partial_t \eta(t) \int_\Omega u(t,x) v(x) dx\, dt = \int_0^T \eta(t) \int_\Omega \frac{f \cdot \nabla v(x)(t,x)}{(t^2+|x|^2)^{\frac{3}{4}} \ln(t^2+|x|^2)} dx dt \\
&= -\int_0^T \eta(t) \langle F(t), v\rangle\, dt = -\left\langle \int_0^T \eta(t) F(t)\, dt, v \right\rangle.
\end{aligned}$$

Since $\mathcal{C}_0^\infty(\Omega)$ is dense in V, we see that the above identity holds for all $v \in V$ and consequently we have

$$\int_0^T \partial_t \eta(t) u(t)\, dt = -\int_0^T \eta(t) F(t)\, dt \quad \text{ in } V^*.$$

But this is nothing else than the definition of $\partial_t u = F$.

Method II: Here we closely follows the theorem about integration by parts from the lecture (with no need for definition of the Gelfand triple and the corresponding duality pairing).

We prove that $u \in \mathcal{C}([0,T/2]; L^2(\Omega))$. The proof for the interval $(T/2, T]$ is similar. For any $n \in \mathbb{N}$, $n > 4$ we define

$$u^n(t) := n \int_t^{t+\frac{1}{n}} u(\tau)\, d\tau \qquad \text{Bochner integral on the space } W_0^{1,2}$$

Evidently, we also have for almost all $t \in (0, T/2)$

$$\partial_t u^n(t) = n(u(t+n^{-1}) - u(t))$$

and then also $\partial_t u^n \in L^2(0,T;V)$ (**not uniformly** with respect to n). Let $\varphi \in \mathcal{C}_0^\infty((0,T/2)\times\Omega)$ be arbitrary and extend it by zero. Then

$$\varphi^\tau(t,x) := \varphi(t-\tau, x)$$

satisfies $\varphi^\tau \in \mathcal{C}_0^\infty((0,T)\times\Omega)$ whenever $\tau < T/2$. Consequently, φ^τ can be used in (B) to get

$$\int_0^T \int_\Omega u(t,x) \frac{\partial \varphi(t-\tau,x)}{\partial t} dx\, dt = \int_0^T \int_\Omega \frac{f \cdot \nabla \varphi(t-\tau,x)}{(t^2+|x|^2)^{\frac{3}{4}} \ln(t^2+|x|^2)} dx dt.$$

which gives

$$\int_0^T \int_\Omega u(t+\tau,x) \frac{\partial \varphi(t,x)}{\partial t} dx\, dt = \int_0^T \int_\Omega \frac{f(t+\tau,x) \cdot \nabla \varphi(t,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)} dx dt.$$

Integration with respect to $\tau \in (0, n^{-1})$ thus leads to

$$\begin{aligned}
&\int_0^T \int_\Omega u^n(t,x) \frac{\partial \varphi(t,x)}{\partial t} dx\, dt \\
&= \int_0^T \int_\Omega \left(n \int_0^{n^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \right) \cdot \nabla \varphi(t,x) dx dt.
\end{aligned}$$

Finally, using the fact that $\partial_t u^n \in L^2(0,T;V) \subset L^2(0,T;L^2(\Omega))$, we can use the integration by parts to conclude

$$\int_0^T \int_\Omega \partial_t u^n(t,x) \varphi(t,x) dx\, dt = -\int_0^T \int_\Omega \left(n \int_0^{n^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \right) \cdot \nabla \varphi(t,x) dx dt.$$

Since $\varphi \in \mathcal{C}_0^\infty((0,T/2)\times\Omega)$ is arbitrary, the above identity implies that for almost all $t \in (0,T/2)$ and all $v \in W_0^{1,2}(\Omega)$

$$\int_\Omega \partial_t u^n(t) v = -\int_\Omega \left(n \int_0^{n^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \right) \cdot \nabla v(x)\, dx. \tag{B2}$$

Finally, let us denote $w^{n,m} := u^n - u^m$ and then it follows from (B2) that

$$\begin{aligned}\int_\Omega \partial_t w^{n,m}(t) v = -\int_\Omega \Bigg(n \int_0^{n^{-1}} &\frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \\ &-m \int_0^{m^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \Bigg) \cdot \nabla v(x)\, dx.\end{aligned} \tag{B3}$$

Setting $v := w^{n,m}$ and integrating the result over (t_1,t_2) we get

$$\begin{aligned}\|w^{n,m}(t_1)\|_2^2 &\le \|w^{n,m}(t_2)\|_2^2 - 2\int_{t_1}^{t_2}\int_\Omega \Bigg(n \int_0^{n^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \\ &\qquad -m \int_0^{m^{-1}} \frac{f(t+\tau,x)}{((t+\tau)^2+|x|^2)^{\frac{3}{4}} \ln((t+\tau)^2+|x|^2)}\, d\tau \Bigg) \cdot \nabla w^{n,m}\, dx\, dt \\ &\le \|w^{n,m}(t_2)\|_2^2 + C\left(\int_0^{T/2} \|\nabla w^{n,m}\|_2^2\right)^{\frac{1}{2}} \|f\|_\infty \left(\int_0^T \int_\Omega \frac{1}{(t^2+|x|^2)^{\frac{3}{2}} \ln(t^2+|x|^2)}\, dx\, dt \right)^{\frac{1}{2}} \\ &\le \|w^{n,m}(t_2)\|_2^2 + C\left(\int_0^{T/2} \|\nabla w^{n,m}\|_2^2\right)^{\frac{1}{2}}.\end{aligned}$$

Due tu the properties of u^n, we can find $t_2 \in (0,T/2)$ such that $u^n(t_2) \to u(t_2)$ in $L^2(\Omega)$ and consequently the sequence $w^{n,m}(t_2)$ is Cauchy in L^2. In addition, we also know that $\nabla u^n \to \nabla u$ in $L^2(0,T/2;L^2(\Omega)$ and consequently the sequence $\nabla w^{n,m}$ is also Cauchy in $L^2(0,T/2;L^2)$. Therefore taking a supremum with respect to $t_1 \in (0,T/2)$ in the above inequality, we get that $w^{n,m}$ is Cauchy in $\mathcal{C}([0,T/2];L^2(\Omega))$ and consequently the limit $u \in \mathcal{C}([0,T];L^2(\Omega))$.

[100] 3. Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz set and define

$$K := \left\{ f = (f_1, f_2, f_3) \in L^2(\Omega; \mathbb{R}^3) : \ \forall \varphi \in \mathcal{C}_0^\infty(\Omega) \ \int_\Omega f \cdot \nabla \varphi = 0 \right\}$$

and the functional

$$\mathcal{F}(f) := \int_\Omega f_1^2 + 2f_2^2 + 3f_3^2 + 2f_1 f_2 - x \cdot f \, dx.$$

Consider the following problem:

$$\text{Find } f \in K \text{ such that } \mathcal{F}(f) \le \mathcal{F}(g) \text{ for all } g \in K. \tag{F}$$

20% Derive the Euler–Lagrange equations to (F).

30% Prove that to solve (F) is equivalent to solve the Euler–Lagrange equations.

20% Show the existence and the uniqueness of the solution to (F) and to its Euler–Lagrange equations. (Hint: apply the Lax–Milgram theorem on Euler–Lagrange equations)

30% The problem (F) is the dual formulation of some elliptic problem. Find the elliptic problem and show that a solution u to that problem fulfills $\nabla u = \mathbb{B} f$ for some matrix $\mathbb{B}$.

Solution:

Let us define a matrix $\mathbb{A}$ as

$$\mathbb{A} := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then $\mathcal{F}$ can be equivalently written as

$$\mathcal{F}(f) = \int_\Omega \mathbb{A} f \cdot f - x \cdot f \, dx.$$

Euler-Lagrange equations: Since K is linear space then for any $t \in \mathbb{R}_+$ and any $g \in K$, we have that $f + tg \in K$. Consequently, if f solves (F) then we have

$$\int_\Omega \mathbb{A} f \cdot f - x \cdot f \, dx \le \int_\Omega \mathbb{A}(f + tg) \cdot (f + tg) - x \cdot (f + tg) \, dx.$$

By simple manipulation it leads to

$$0 \le \int_\Omega t^2 \mathbb{A} g \cdot g + 2t \mathbb{A} f \cdot g - t x \cdot g \, dx.$$

Division by $t > 0$ and limit $t \to 0_+$ then gives

$$0 \le \int_\Omega 2 \mathbb{A} f \cdot g - x \cdot g \, dx.$$

Since $-g \in K$ as well we see that the above inequality must hold with equality sign, i.e.

$$\int_\Omega \mathbb{A} f \cdot z \, dx = \frac{1}{2} \int_\Omega x \cdot z \, dx \qquad \text{for all } z \in K. \tag{E-L}$$

Equivalence: In previous step we showed that (F) $\Longrightarrow$ (E-L). Here, we focus on the opposite implication. Let $f, g \in K$ such that f fulfills (E-L). Then, we have

$$\begin{aligned}
0 &\leq \int_\Omega (f_1 - g_1 + f_2 - g_2)^2 + (f_2 - g_2)^2 + 3(f_3 - g_3)^2 = \int_\Omega \mathbb{A}(f-g)\cdot(f-g) \\
&= \int_\Omega \mathbb{A}g\cdot g + \mathbb{A}f\cdot f - 2\mathbb{A}g\cdot f = \int_\Omega \mathbb{A}g\cdot g - \mathbb{A}f\cdot f + 2\mathbb{A}f\cdot(f-g) \\
&= \mathcal{F}(g) - \mathcal{F}(f) + \int_\Omega 2\mathbb{A}f\cdot(f-g) - x\cdot(f-g) = \mathcal{F}(g) - \mathcal{F}(f),
\end{aligned}$$

where for the last equality we used (E-L) with $z := f - g$. Hence, we see that f solves (F) and the equivalence is proven.

Existence and uniqueness: We know that we just need to show the unique solvability to the problem (E-L). For that purpose, it is direct to observe that K is a Hilbert space equipped with the standard L^2-norm. Next, we define a bilinear form

$$\mathcal{B}(f,g) := \int_\Omega \mathbb{A}f\cdot g \qquad \text{for all } f, g \in K$$

and $F \in K^*$ by

$$\langle F, g\rangle_K := \frac{1}{2}\int_\Omega x\cdot g \qquad \text{for all } g \in K.$$

Thanks to the Hölder inequality both definitions are meaningfull and $\mathcal{B}$ is also bounded. In addition, since $\mathbb{A}$ is evidently an elliptic matrix, we also have that there exists $C_1 > 0$ such that for all $f \in K$

$$\mathcal{B}(f,f) \geq C_1\|f\|_K^2 = C_1\int_\Omega f_1^2 + f_2^2 + f_3^2.$$

Thus, according to Lax-Milgram lemma there exists a unique $f \in K$ solving the problem

$$\mathcal{B}(f,z) = \langle F, z\rangle,$$

which is however equivalent to (E-L).

Dual problem: Let us define

$$u_0 := \frac{|x|^2}{4}.$$

Then $u_0 \in W^{1,2}(\Omega)$ and $\nabla u_0 = \frac{x}{2}$. With this notation, we can rewrite (E-L) as

$$\int_\Omega \mathbb{A}f\cdot z\,dx = \int_\Omega \nabla u_0\cdot z\,dx \qquad \text{for all } z \in K. \tag{E-L2}$$

Then, since $\mathbb{A}$ is positively definite, it has an inverse $\mathbb{B} := \mathbb{A}^{-1}$, in fact we have

$$\mathbb{B} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Consider the problem

$$\operatorname{div}(\mathbb{B}\nabla u) = 0 \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega.$$

From the lecture, we know it has exactly one solution $u \in W^{1,2}(\Omega)$, fulfilling $(u - u_0) \in W_0^{1,2}(\Omega)$ and for all $\varphi \in W_0^{1,2}(\Omega)$

$$\int_\Omega \mathbb{B}\nabla u \cdot \nabla\varphi = 0. \tag{WF}$$

In addition, since $\mathcal{C}_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, we see that K also satisfies

$$K = \left\{ f = (f_1, f_2, f_3) \in L^2(\Omega; \mathbb{R}^3) : \ \forall\varphi \in W_0^{1,2}(\Omega) \ \int_\Omega f \cdot \nabla\varphi = 0 \right\} \tag{K}$$

and therefore also

$$\mathbb{B}\nabla u \in K. \tag{1}$$

Finally, since $\mathbb{A}$ is elliptic, we have ($\mathbb{A} = \mathbb{B}^{-1}$)

$$\begin{aligned} C_1 \int_\Omega |\mathbb{B}\nabla u - f|^2 &\le \int_\Omega \mathbb{A}(\mathbb{B}\nabla u - f) \cdot (\mathbb{B}\nabla u - f) = \int_\Omega (\nabla u - \mathbb{A}f) \cdot (\mathbb{B}\nabla u - f) \\ &= \int_\Omega \underbrace{(\nabla u - \nabla u_0)}_{\in W_0^{1,2}} \cdot \underbrace{(\mathbb{B}\nabla u - f)}_{\in K} + \int_\Omega (\nabla u_0 - \mathbb{A}f) \cdot \underbrace{(\mathbb{B}\nabla u - f)}_{\in K} = 0, \end{aligned}$$

where the first integral vanishes thanks to (K) and the second integral vanishes thanks to (E-L2).