

Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: _____

Question	1	2	3	Score
Maximum points	0	100	100	200
Points				

- [0] 1. Define the Sobolev space $W^{1,p}(\Omega)$. For which p 's is it separable, reflexive or Hilbert?

Solution:

See lecture.

- [100] 2. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz and $\mathbb{A} : \Omega \rightarrow \mathbb{R}^{d \times d}$ be bounded measurable **symmetric** matrix-valued mapping that is for almost all $x \in \Omega$ elliptic matrix. Consider the problem: find $\lambda_i \in \mathbb{R}$ and u_i such that

$$\begin{aligned} -\operatorname{div}(\mathbb{A} \nabla u_i) &= \lambda_i u_i && \text{in } \Omega, \\ u_i &= 0 && \text{on } \partial\Omega \end{aligned}$$

fulfilling $\int_{\Omega} (u_i)^2 = 1$.

- How many weak solutions (λ_i, u_i) does the above problem have? (none, finite, infinite, countable, uncountable, etc...)
- What can you say about the set of all solutions to $\{u_i\}$ to the above problem?

Solution:

The problem has infinitely many but countable number of solutions, λ_i can be ordered such that $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\lim_{j \rightarrow \infty} \lambda_j = \infty$. The set $\{u_i\}_{i=1}^{\infty}$ forms a basis of $W_0^{1,2}(\Omega)$ and also $L^2(\Omega)$. Moreover, for $i \neq j$ we have the identities

$$\int_{\Omega} u_i u_j = 0, \quad \int_{\Omega} \mathbb{A} \nabla u_i \cdot \nabla u_j = 0.$$

For the proof see the lecture and the theorem about the spectrum of symmetric elliptic operator.

- [100] 3. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz set and $n \in \mathbb{N}$. Assume that $f_i \in L^2(\Omega)$ for $i = 1, \dots, n$. Consider the problem: Find n functions $u_i : \Omega \rightarrow \mathbb{R}$ (here $i = 1, \dots, n$) solving

$$-(4 + \sin x_2) \frac{\partial^2 u}{\partial x_1^2} - (4 + \sin x_1) \frac{\partial^2 u}{\partial x_2^2} + 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + \sum_{j=1}^n a_{ij} u_j = f_i \quad \text{in } \Omega, \quad i = 1, \dots, n,$$

$$u_i = 0 \quad \text{on } \partial\Omega, \quad i = 1, \dots, n,$$

where $a_{ij} \in \mathbb{R}$ are given.

- 20% Find a proper definition of a weak solution. Check that for given data such a definition is meaningful.
- 40% Use the Lax–Milgram theorem and show that if the matrix $A = \{a_{ij}\}_{i,j=1}^n$ is positively semidefinite, then there exists a unique weak solution. (Hint, find a proper bilinear form and a proper function space (a subspace of $W^{1,2} \times \dots \times W^{1,2}$), for which you can use the Lax–Milgram theorem.)
- 40% In case that A is not positively definite, find the **sharp** relation between the spectrum of A and the spectrum of the proper elliptic operator that guarantees the existence of a weak solution for every $f_i \in L^2(\Omega)$, with $i = 1, \dots, n$. (Hint, consider a proper basis $\{w_i\}_{i=1}^\infty$ of $W_0^{1,2}$ and consider a solution of the form $u_i = \sum_{j=1}^\infty b_{ij} w_j$.)

Solution:

Weak formulation: We deal with homogeneous Dirichlet problem. Hence, we say that u_i with $i = 1, \dots, n$ is a weak solution if $u_i \in W_0^{1,2}(\Omega)$ for all $i = 1, \dots, n$ and for all $\varphi \in W_0^{1,2}(\Omega)$ and all $i = 1, \dots, n$ there holds

$$\int_{\Omega} \nabla u_i \cdot \nabla \varphi + \int_{\Omega} \sum_{j=1}^n a_{ij} u_j \varphi = \int_{\Omega} f_i \varphi. \quad (1)$$

Thanks to the Hölder inequality all integrals are well defined. Equivalently, we can set

$$V := \underbrace{W_0^{1,2}(\Omega) \times \dots \times W_0^{1,2}(\Omega)}_{n\text{-times}}$$

and look for $u = (u_1, \dots, u_n) \in V$ such that for all $\varphi \in V$ there holds

$$\int_{\Omega} \nabla u : \nabla \varphi + Au \cdot \varphi = \int_{\Omega} f \cdot \varphi. \quad (2)$$

Here, we set $f := (f_1, \dots, f_n)$, the matrix $\{A\}_{ij} := a_{ij}$, the symbol “ \cdot ” denotes the scalar product in \mathbb{R}^n and the symbol “ $:$ ” denotes the scalar product in $\mathbb{R}^n \times \mathbb{R}^d$.

Existence of solution for positively semidefinite A : For our purposes, we define the bilinear B form on V as

$$B(u, v) := \int_{\Omega} \nabla u : \nabla v + Au \cdot v = \int_{\Omega} \sum_{i=1}^n \sum_{j=1}^d \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} + \sum_{i,j=1}^n a_{ij} u_j v_i$$

and $F \in V^*$ as

$$\langle F, v \rangle_V := \int_{\Omega} f \cdot v = \int_{\Omega} \sum_{i=1}^n f_i v_i, \quad \text{for all } v \in V.$$

Clearly, the problem then reduces to: find $u \in V$ such that for all $v \in V$ there holds

$$B(u, v) = \langle F, v \rangle_V. \quad (3)$$

The existence and uniqueness will be proven by the Lax-Milgram theorem and the fact that A is assumed to be positively semidefinite. Indeed, V is a Hilbert space. The form B is evidently also bilinear and V -bounded, which follows from the Hölder inequality. Thus, it just remains to prove coercivity. However, using the fact that A is positively semidefinite, we get (we use that u has zero trace)

$$B(u, u) = \sum_{i=1}^n \|\nabla u_i\|_2^2 + \int_{\Omega} Au \cdot u \geq \sum_{i=1}^n \|\nabla u_i\|_2^2 \stackrel{\text{Poincaré}}{\geq} c_1 \sum_{i=1}^n \|u_i\|_{1,2}^2 = c_1 \|u\|_V^2.$$

Hence, B satisfies all assumptions of the Lax-Milgram theorem. Thus, existence and uniqueness is proven.

Characterization via spectrum: First, we know that there exists a basis $\{w_j\}_{j=1}^{\infty}$ of $W_0^{1,2}$, which is orthogonal in $W_0^{1,2}$ and orthonormal in L^2 , which consists of eigen functions and eigen vectors of Laplace operator, i.e.,

$$\begin{aligned} \int_{\Omega} w_i w_k &= \delta_{ik}, \\ \int_{\Omega} \nabla w_i \cdot \nabla \varphi &= \lambda_i \int_{\Omega} w_i \varphi, \quad \text{for all } \varphi \in W_0^{1,2}(\Omega). \end{aligned}$$

Note, that we also have $\lambda_i > 0$ for all i . Since, it is a basis, every u_i and f_i can be written as

$$u_i = \sum_{j=1}^{\infty} b_{ij} w_j, \quad f_i = \sum_{j=1}^n F_{ij} w_j \text{ with } F_{ij} := \int_{\Omega} f_i w_j. \quad (4)$$

Due to the property of basis, we can equivalently rewrite (1) as

$$\int_{\Omega} \nabla u_i \cdot \nabla w_k + \int_{\Omega} \sum_{j=1}^n a_{ij} u_j w_k = \int_{\Omega} f_i w_k \quad \text{for all } i = 1, \dots, n \text{ and } k \in \mathbb{N}.$$

Using, the definition (4) and the orthogonality and orthonormality of the basis, it reduces to: Find $b_{ik} \in \mathbb{R}$ such that

$$b_{ik} \lambda_k + \sum_{j=1}^n a_{ij} b_{jk} = F_{ik} \quad \text{for all } i = 1, \dots, n \text{ and } k \in \mathbb{N}. \quad (5)$$

Hence, if we denote by $\mathbf{b}_k, \mathbf{F}_k \in \mathbb{R}^n$ as

$$\mathbf{b}_k := (b_{1k}, \dots, b_{nk}), \quad \mathbf{F}_k := (F_{1k}, \dots, F_{nk})$$

Then the relation (5) can be reformulated as: for every $k \in \mathbb{N}$ find $\mathbf{b}_k \in \mathbb{R}^n$ such that

$$(\lambda_k \mathbb{I} + A)\mathbf{b}_k = \mathbf{F}_k, \quad (6)$$

where \mathbb{I} is the identity matrix in $\mathbb{R}^{n \times n}$. Next, (6) is just linear algebraic equation. Therefore, if we require that there exists unique solution to (6) for arbitrary \mathbf{F}_k , then necessarily $\lambda_k \mathbb{I} + A$ must be a regular matrix, which means nothing else that

$$\lambda_k \notin \text{spt}(-A) \text{ for all } k \in \mathbb{N}. \quad (7)$$

Solution for special choice of f_1 and f_2 : Here, we recall the homework, where you proved that the eigen-functions and eigen-values for the Laplace operator with zero trace on the square are of the form (we do not normalize them to have L^2 norm equal to 1)

$$w_{lk} = \sin(lx_1) \sin(kx_2), \quad \lambda_{lk} = l^2 + k^2, \quad l, k \in \mathbb{N}.$$

Hence, it follows from (7) that if for all $l, k \in \mathbb{N}$

$$-l^2 - k^2 \notin \text{spt}(A)$$

then we have a unique solution.

Now, we can follow the preceding step and look for u of the form

$$u_i = \sum_{l,k=1}^{\infty} b_{ilk} w_{lk}, \quad f_1 = w_{21}, f_2 = w_{11}. \quad (8)$$

Repeating step by step the previous procedure, we end up with the system of equations

$$\begin{aligned} b_{1kl} \lambda_{kl} + a_{11} b_{1kl} + a_{12} b_{2kl} &= \begin{cases} 1 & \text{if } k = 1 \text{ \& } l = 1, \\ 0 & \text{otherwise} \end{cases} \\ b_{2kl} \lambda_{kl} + a_{21} b_{1kl} + a_{22} b_{2kl} &= \begin{cases} 1 & \text{if } k = 1 \text{ \& } l = 1, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (9)$$

Hence, if we consider $l > 1$ or $k > 1$ then we can simply set $b_{ikl} = 0$ to fulfill (9). However, if spectrum of $-A$ contains $(l^2 + k^2)$ we can choose b_{ikl} as a corresponding eigen-vector to A , so we have non-uniqueness.

Finally, if the spectrum of A does not contain -2 , the system (9) always have a (non-unique) solution. On the other, if the spectrum of $-A$ contains 2 then we can first rewrite (9) for $k = l = 1$. Then it reduces (denoting $(B = b_{111}, b_{211})$) to find B such that

$$(2\mathbb{I} + A)B = (1, 1).$$

Since -2 is in spectrum of A the above equation has a (nonunique) solution if and only if

$$0 = w_1 + w_2 = (1, 1) \cdot (w_1, w_2) \quad \text{for any solution } w \text{ to } (2\mathbb{I} + A^T)w = 0.$$

Or in other words, the vector $(1, 1)$ must be orthogonal to eigen-vectors of matrix A^T corresponding to eigen-value -2 .