Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Formulate theorem about continuous and compact embedding of $W^{1,p}(\Omega)$ into the proper Banach space(s).

Solution:

See lecture.

[100] 2. Let $\Omega \subset \mathbb{R}^d$ be Lipschitz set. Assume that $u \in L^2(0,T;W^{1,2}(\Omega))$ and $\mathbf{F} \in L^2(0,T;L^2(\Omega;\mathbb{R}^d))$. Assume that for all $\varphi \in \mathcal{C}^\infty((0,T)\times\Omega)$ fulfilling for all $x \in \Omega$ $\varphi(0,x) = \varphi(T,x) = 0$ there holds

$$\int_0^T \int_\Omega u\partial_t \varphi dx\, dt = \int_0^T \int_\Omega \mathbf{F}\cdot\nabla\varphi dx\, dt. \tag{E}$$

70% Prove that $u \in \mathcal{C}([0,T];L^2(\Omega))$.

30% Show that for all $t \in (0,T)$ there holds

$$\|u(t)\|_2^2 - \|u(0)\|_2^2 = -2\int_0^t \mathbf{F}\cdot\nabla u.$$

Solution:

Method I: Here, we rewrite the problem in the way that we can use directly the Theorem from the lecture. Hence, we recall that V, H, V^* forms a Gelfand triple if we set $V := W^{1,2}(\Omega)$ and $H := L^2(\Omega)$. First of all, we define $\mathcal{F} : (0,T) \to V^*$ by the formula

$$\langle \mathcal{F}(t), v\rangle := -\int_\Omega \mathbf{F}(t,x)\cdot\nabla v(x)\, dx \qquad \forall v \in V.$$

The fact that $\mathcal{F}(t) \in V^*$ for almost all $t \in (0,T)$ follows from the linearity of the integral, the Hölder inequality and the fact that $\mathbf{F} \in L^2(0,T;L^2)$. It also follows the definition that for almost all $t \in (0,T)$

$$\int_0^T \|\mathcal{F}(t)\|_{V^*}^2 \le \int_0^T\int_\Omega |\mathbf{F}|^2$$

and consequently, we see that $\mathcal{F} \in L^2(0,T;V^*)$. Next, we show that $\partial_t u = \mathcal{F}$. Consider $v \in V$ arbitrary and $\psi \in \mathcal{C}_0^\infty(0,T)$. Then we have, by using the properties of the Gelfand triple, the properties of the Bochner integral and by setting $\varphi := \psi v$ in (E)

$$\begin{aligned}
&\int_0^T \psi(t)\langle \mathcal{F}(t), v\rangle_V\, dt = -\int_0^T \psi(t)\int_\Omega \mathbf{F}(t,x)\cdot\nabla v(x) dx\, dt \\
&= -\int_0^T\int_\Omega u(t,x)\partial_t\psi(t)v(x)dx\, dt = -\int_\Omega\left(\int_0^T u(t,x)\partial_t\psi(t)\, dt\right)v(x)\, dx \\
&= -\left\langle \int_0^T u(t)\partial_t\psi(t)\, dt, v\right\rangle_V = \left\langle \int_0^T \partial_t u\psi(t)\, dt, v\right\rangle_V = \int_0^T \psi(t)\langle\partial_t u(t), v\rangle_V.
\end{aligned}$$

We used the definition of the weak derivative in Bochner spaces. Hence, we identified $\partial_t u$ with $\mathcal{F}$. Thus, we have $u \in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$ and we can use the Theorem from the lecture and get that $u \in \mathcal{C}([0,T];L^2(\Omega))$. Moreover, we can use the integration by parts formula together with (E) and the fact that $u \in L^2(0,T;V)$

$$\|u(t)\|_2^2 - \|u(0)\|_2^2 = 2\int_0^t \langle\partial_t, u\rangle_V = -2\int_0^t\int_\Omega \mathbf{F}\cdot\nabla u.$$

Method II: We modify the proof of integration by parts formula and in fact we do not use any Gelfand triple. Fix $h_0 > 0$. Assume that $\psi \in \mathcal{C}^\infty((0,T)\times\Omega)$ is arbitrary and consider $g \in \mathcal{C}^\infty(h_0, T-h_0)$. Consider $h \in (0,h_0)$ and define

$$(\psi(t,x)g(t))_{-h} := \frac{1}{h}\int_{t-h}^{t} \psi(\tau,x)g(\tau)\,d\tau, \qquad (\psi(t,x)g(t))_{h} := \frac{1}{h}\int_{t}^{t+h} \psi(\tau,x)g(\tau)\,d\tau.$$

Note that (see the lecture) it follows from the properties of the Bochner integral and weak derivatives that

$$\begin{aligned}
\partial_t(\psi(t,x)g(t))_{-h} &= \frac{\psi(t,x)g(t)-\psi(t-h,x)g(t-h)}{h},\\
\partial_t u_h(t,x) &= \frac{u(t+h,x)-u(t,x)}{h} \text{ weak derivative in } L^2(0,T;L^2(\Omega))
\end{aligned}$$

Next, we set $\varphi := (\psi g)_{-h}$ in (E). Using the fact that $h < h_0$ that g is compactly supported on $(h_0, T-h_0)$, we can write after using the Fubini theorem

$$\begin{aligned}
&-\int_0^T\int_\Omega \mathbf{F}_h(t,x)\cdot\nabla\psi(t,x)g(t)\,dx\,dt = -\frac{1}{h}\int_0^T\int_0^h\int_\Omega \mathbf{F}(t+\tau,x)\cdot\nabla\psi(t,x)g(t)\,dx\,d\tau\,dt\\
&= -\frac{1}{h}\int_0^T\int_0^h\int_\Omega \mathbf{F}(t,x)\cdot\nabla\psi(t-\tau,x)g(t-\tau)\,dx\,d\tau\,dt\\
&= -\int_0^T\int_\Omega \mathbf{F}\cdot\nabla(\psi g)_{-h}dx\,dt \overset{(E)}{=} -\int_0^T\int_\Omega u(t,x)\frac{\psi(t,x)g(t)-\psi(t-h,x)g(t-h)}{h}dx\,dt\\
&= \int_0^T\int_\Omega \psi(t,x)g(t)\frac{u(t+h,x)-u(t,x)}{h}dx\,dt = \int_0^T\int_\Omega \partial_t u_h(t,x)\psi(t,x)g(t)dx\,dt
\end{aligned} \tag{1}$$

Using the density argument, we see that (1) holds true even for all $\psi \in L^2(0,T;W^{1,2}(\Omega))$ and all $g \in L^\infty(0,T)$ compactly supported in $(h_0, T-h_0)$. Consider now $h_1, h_2 \in (0,h_0)$ and denote $w := u_{h_1} - u_{h_2}$. Then it follows from (1) that

$$\int_0^T\int_\Omega \partial_t w\psi g dx\,dt = \int_0^T\int_\Omega (\mathbf{F}_{h_2}-\mathbf{F}_{h_1})\cdot\nabla\psi g\,dx\,dt \tag{2}$$

Next, let $h_0 < t_1 < T/2 < \tau < T-h_0$ be arbitrary. We set $\psi := w$ and $g := \chi_{[t_1,\tau]}$ in (2). Note that such choice **is possible**. A simple integration and the Hölder inequality then gives

$$\begin{aligned}
\|u_{h_1}(t_1)-u_{h_2}(t_1)\|_2^2 &\le \|u_{h_1}(\tau)-u_{h_2}(\tau)\|_2^2\\
&\quad + 2\left(\int_{h_0}^{T-h_0}\|\mathbf{F}_{h_2}-\mathbf{F}_{h_1}\|_2^2\right)^{\frac12}\left(\int_{h_0}^{T-h_0}\|u_{h_1}-u_{h_2}\|_{W^{1,2}}^2\right)^{\frac12}
\end{aligned}$$

Integrating this result with respect to $\tau \in (T/2, T-h_0)$ and using the Young inequality, we have

$$\sup_{t_1\in[h_0,T/2]}\|u_{h_1}(t_1)-u_{h_2}(t_1)\|_2^2 \le 2T\int_{h_0}^{T-h_0}\|\mathbf{F}_{h_2}-\mathbf{F}_{h_1}\|_2^2 + \|u_{h_1}-u_{h_2}\|_{W^{1,2}}^2. \tag{3}$$

Thus, since $\mathbf{F}_h \to \mathbf{F}$ strongly in $L^2(0,T;L^2)$ and $u_h \to u$ strongly in $L^2(0,T;W^{1,2})$, it follows from (3) that u_h is a Cauchy sequence in $\mathcal{C}([h_0,T/2];L^2(\Omega))$ and since it is a Banach space, it has the limit $u \in \mathcal{C}([h_0,T/2];L^2(\Omega))$. Similarly we can prove the result also on the time interval $[T/2, T-h_0]$ and since h_0 was arbitrary, we can extend it onto the whole time interval $(0,T)$, i.e., for all $h_0 > 0$ we have

$$u_h \to u \qquad \text{strongly in } \mathcal{C}([h_0, T-h_0]; L^2(\Omega)). \tag{4}$$

It just remains to cover the interval $[0,T]$ and to prove also the energy equality. We start with the following observation. Using (1) with $\psi := u_h$ and $g = \chi_{[t_1,t_2]}$ with $0 < t_1 < t_2$ we have for sufficiently small $h > 0$ that

$$-2\int_{t_1}^{t_2}\int_\Omega \mathbf{F}_h(t,x)\cdot\nabla u_h(t,x)\,dx\,dt = \|u_h(t_2)\|_2^2 - \|u_h(t_1)\|_2^2.$$

Finally, using (4), we deduce

$$-2\int_{t_1}^{t_2}\int_\Omega \mathbf{F}(t,x)\cdot\nabla u(t,x)\,dx\,dt = \|u(t_2)\|_2^2 - \|u(t_1)\|_2^2. \tag{5}$$

Hence, to finish the proof, it remains to show that

$$u(t) \to u(0) \text{ strongly in } L^2(\Omega) \text{ for } t \to 0_+.$$

And also similarly for $t \to T_-$. However, now we can use the whole procedure, where we just replace u by

$$w(t,x) := u(t+\delta,x) - u(t,x).$$

Then, (5) gives for $\delta > 0$ small

$$\|u(t_1+\delta)-u(t_1)\|_2^2 = \|u(t_2+\delta)-u(t_2)\|_2^2 + 2\int_{t_1}^{t_2}\int_\Omega (\mathbf{F}(t+\delta,x)-\mathbf{F}(t,x))\cdot\nabla(u(t+\delta,x)-u(t,x))\,dx\,dt. \tag{6}$$

Thus, we see that the sequence $u(t)$ is Cauchy in $L^2(\Omega)$ as $t \to 0_-$. Hence, we have

$$u(t) \to u(0).$$

[100] 3. Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz set and $d \in \mathbb{N}$. Assume that $\mathbb{B} \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ is given elliptic matrix. In addition, let $\partial\Omega_1$ and $\partial\Omega_2$ be two smooth disjoint parts of the boundary such that $\overline{\partial\Omega_1 \cup \partial\Omega_2} = \partial\Omega$ and $|\partial\Omega_1|_{d-1} > 0$. Denote $V := \{\varphi \in W^{1,2}(\Omega);\ \varphi = 0 \text{ on } \partial\Omega_1\}$. Define a set

$$K := \left\{\mathbf{f} = (f_1, \ldots, f_n) \in L^2(\Omega; \mathbb{R}^d);\ \forall\varphi \in V\ \int_\Omega \mathbf{f}\cdot\nabla\varphi = \int_\Omega \varphi - \int_{\partial\Omega_2} x_1 n_1 \varphi\right\},$$

where the outer normal vector of $\partial\Omega$ is denoted as $n = (n_1, \ldots, n_d)$, and a functional

$$J(\mathbf{f}) := \int_\Omega \mathbb{B}\mathbf{f}\cdot\mathbf{f} - x_2 f_1 - x_1 f_2 + 2x_d f_d.$$

30% Show that there exists unique $\mathbf{f} \in K$ such that (Present the complete proof.)

$$J(\mathbf{f}) \le J(\tilde{\mathbf{f}}) \qquad \text{for all } \tilde{\mathbf{f}} \in K. \tag{7}$$

20% Derive the Euler–Lagrange equations to (7) and show that the unique minimizer satisfies such equations.

30% The problem (7) is a dual formulation of some primary problem(s). Derive the classical formulation of the primary problem to (7) and show its "equivalence"to dual prolem.

20% Under which assumptions there hold $\mathbf{f} \in W^{1,2}_{loc}(\Omega; \mathbb{R}^d)$ and/or $\mathbf{f} \in W^{1,2}(\Omega; \mathbb{R}^d)$?

Solution:

Existence and uniqueness: Note that we can assume that $\mathbb{B}$ is symmetric (since the antisymmetric part of $\mathbb{B}$ is not visible for J). First, we show that K is nonempty. Using integration by parts, it is not difficult to show that $(-x_1, 0, \ldots, 0) \in K$. Next, using the ellipticity of $\mathbb{B}$, the fact that Ω is bounded and the Hölder inequality, we also have that for all $\mathbf{f} \in L^2(\Omega; \mathbb{R}^d)$

$$J(\mathbf{f}) \ge C_1\|\mathbf{f}\|_2^2 - C_2\|\mathbf{f}\|_2 \ge \frac{C_1}{2}\|\mathbf{f}\|_2^2 - C, \tag{8}$$

where C is a constant independent of $\mathbf{f}$ and C_1 is the ellipticity constant of $\mathbb{B}$. Hence, since K is not empty and J is bounded from below, we have

$$I := \inf_{\mathbf{f}\in K} J(\mathbf{f}) > -\infty.$$

Hence, from the definition of infima, we have that there exists a sequence $\mathbf{f}^n \in K$ such that

$$I = \lim_{n\to\infty} J(\mathbf{f}^n).$$

Since, $J((-x_1, 0, \ldots, 0)) \le C_3$, we see that there exists n_0 such that for all $n \ge n_0$ we have

$$\|\mathbf{f}^n\|_2^2 - C \le J(\mathbf{f}^n) \le J((-x_1, 0, \ldots, 0)) + 1 \le C_3 + 1.$$

Consequently, we have that

$$\|\mathbf{f}^n\|_2 \le \tilde{C}. \tag{9}$$

Using the reflexivity of L^2, we have that there exists $\mathbf{f} \in L^2(\Omega;\mathbb{R}^d)$ such that for a subsequence that we do not relabel

$$\mathbf{f}^n \rightharpoonup \mathbf{f} \qquad \text{weakly in } L^2(\Omega;\mathbb{R}^d).$$

The above weak convergence means that for all $\psi \in L^2(\Omega)$ and all $i = 1,\dots,d$, there holds

$$\lim_{n\to\infty} \int_\Omega f_i^n \psi = \int_\Omega f_i \psi. \tag{10}$$

First, we show that the limit $\mathbf{f} \in K$. Using (10) and the fact that $\mathbf{f}^n \in K$ and $\nabla\varphi \in L^2$, we get

$$\int_\Omega \mathbf{f}\cdot\nabla\varphi = \lim_{n\to\infty}\int_\Omega \mathbf{f}^n\cdot\nabla\varphi = \int_\Omega \varphi - \int_{\partial\Omega_2} x_1 n_1 \varphi,$$

which means $\mathbf{f} \in K$. Next, using the ellipticity and the symmetry of $\mathbb{B}$ and the weak convergence (10), we deduce

$$\begin{aligned}
I &= \lim_{n\to\infty} J(\mathbf{f}^n) = \liminf_{n\to\infty} \int_\Omega \mathbb{B}\mathbf{f}^n\cdot\mathbf{f}^n - x_2 f_1^n - x_1 f_2^n + 2x_d f_d^n \\
&= \liminf_{n\to\infty}\int_\Omega \mathbb{B}(\mathbf{f}^n-\mathbf{f})\cdot(\mathbf{f}^n-\mathbf{f}) + \mathbb{B}\mathbf{f}\cdot(\mathbf{f}^n-\mathbf{f}) + \mathbb{B}\mathbf{f}\cdot\mathbf{f}^n - x_2 f_1^n - x_1 f_2^n + 2x_d f_d^n \\
&\geq \lim_{n\to\infty}\int_\Omega \underbrace{\mathbb{B}\mathbf{f}}_{\in L^2}\cdot\underbrace{(\mathbf{f}^n-\mathbf{f})}_{\rightharpoonup 0 \text{ in } L^2} + \underbrace{\mathbb{B}\mathbf{f}}_{\in L^2}\cdot\underbrace{\mathbf{f}^n}_{\rightharpoonup \mathbf{f} \text{ in } L^2} - \underbrace{(x_2, x_1, 0, \dots, 0, -2x_d)}_{\in L^2}\cdot\underbrace{\mathbf{f}}_{\rightharpoonup \mathbf{f} \text{ in } L^2} \\
&= \int_\Omega \mathbb{B}\mathbf{f}\cdot\mathbf{f} - x_2 f_1 - x_1 f_2 + 2x_d f_d = J(\mathbf{f}).
\end{aligned}$$

Thus, we have $I \leq J(\mathbf{f}) \leq I$ and therefore $\mathbf{f}$ is a minimizer.

Concerning the uniqueness. Assume that $\tilde{\mathbf{f}} \neq \mathbf{f}$ is another minimizer, i.e, $J(\tilde{\mathbf{f}}) = I$ and $\int_\Omega |\mathbf{f}-\tilde{\mathbf{f}}|^2 > 0$. Then $(\tilde{\mathbf{f}}+\mathbf{f})/2 \in K$ and from definition of I we deduce

$$\begin{aligned}
I \leq J((\tilde{\mathbf{f}}+\mathbf{f})/2) &= \frac{1}{4}\int_\Omega \mathbb{B}(\mathbf{f}+\tilde{\mathbf{f}})\cdot(\mathbf{f}+\tilde{\mathbf{f}}) - 2x_2(f_1+\tilde{f}_1) - 2x_1(f_2+\tilde{f}_2) + 4x_d(f_d+\tilde{f}_d) \\
&= \frac{1}{2}J(\mathbf{f}) + \frac{1}{2}J(\tilde{\mathbf{f}}) - \frac{1}{4}\int_\Omega \mathbb{B}(\mathbf{f}-\tilde{\mathbf{f}})\cdot(\mathbf{f}-\tilde{\mathbf{f}}) \leq I - \frac{C_1}{4}\|\mathbf{f}-\tilde{\mathbf{f}}\|_2^2 < I,
\end{aligned}$$

which is a contradiction.

Euler–Lagrange equations: Let $\mathbf{f}$ be a minimizer to (7). Denote

$$K_0 := \left\{\mathbf{f} = (f_1,\dots,f_n) \in L^2(\Omega;\mathbb{R}^d);\ \forall\varphi\in V \ \int_\Omega \mathbf{f}\cdot\nabla\varphi = 0\right\},$$

. Next, let $\mathbf{g} \in K_0$ be arbitrary. Then for all $\varepsilon > 0$ we have $(\mathbf{f}\pm\varepsilon\mathbf{g}) \in K$ and from the definition of minima, it follows

$$\begin{aligned}
&\int_\Omega \mathbb{B}\mathbf{f}\cdot\mathbf{f} - x_2 f_1 - x_1 f_2 + 2x_d f_d = J(\mathbf{f}) \leq J(\mathbf{f}\pm\varepsilon\mathbf{g}) \\
&\qquad = \int_\Omega \mathbb{B}(\mathbf{f}\pm\varepsilon\mathbf{g})\cdot(\mathbf{f}\pm\varepsilon\mathbf{g}) - x_2(f_1\pm\varepsilon g_1) - x_1(f_2\pm\varepsilon g_2) + 2x_d(f_d\pm\varepsilon g_d)
\end{aligned}$$

and consequently, a simple algebraic manipulation gives

$$0 \le \varepsilon^2 \int_\Omega \mathbb{B}\mathbf{g}\cdot\mathbf{g} \pm \varepsilon \int_\Omega 2\mathbb{B}\mathbf{f}\cdot\mathbf{g} - x_2 g_1 - x_1 g_2 + 2x_d g_d$$

Hence, dividing by $\varepsilon > 0$ and letting $\varepsilon \to 0_+$, we get

$$\int_\Omega 2\mathbb{B}\mathbf{f}\cdot\mathbf{g} - x_2 g_1 - x_1 g_2 + 2x_d g_d = 0 \qquad \forall \mathbf{g} \in K_0, \tag{11}$$

which is the Euler–Lagrange equation for (7).

Primary formulation: Let $\alpha \in \mathbb{R}$ be fixed but arbitrary. Denote $u_0 := \frac{x_1x_2 - x_d^2}{2} + \alpha$. Then $u_0 \in W^{1,2}(\Omega)$ and $\nabla u_0 = \frac{1}{2}(x_2, x_1, 0, \dots, 0, -2x_d)$. Hence, we can rewrite

$$J(\mathbf{f}) = 2\int_\Omega \frac{\mathbb{B}\mathbf{f}\cdot\mathbf{f}}{2} - \nabla u_0\cdot\mathbf{f}$$

and (11) has the form

$$\int_\Omega \mathbb{B}\mathbf{f}\cdot\mathbf{g} - \nabla u_0\cdot\mathbf{g} = 0 \qquad \forall \mathbf{g} \in K_0. \tag{12}$$

Finally, we introduce the problem (which is the **primary problem** to (7))

$$\begin{aligned} -\operatorname{div}(\mathbb{B}^{-1}\nabla u) &= 1 && \text{in } \Omega,\\ u &= u_0 && \text{on } \partial\Omega_1,\\ \mathbb{B}^{-1}\nabla u \mathbf{n} &= -x_1 n_1 && \text{on } \partial\Omega_2. \end{aligned} \tag{13}$$

Note that the matrix $\mathbb{B}^{-1}$ exists since $\mathbb{B}$ is symmetric and elliptic. The weak formulation of (13) is the following

$$\int_\Omega \mathbb{B}^{-1}\nabla u\cdot\nabla\varphi = \int_\Omega \varphi - \int_{\partial\Omega_2} x_1 n_1 \varphi \quad \forall \varphi \in V. \tag{14}$$

The existence and uniqueness of a weak solution $u \in W^{1,2}$, such that $(u - u_0) \in V$ follows by the Lax–Milgram theorem (see the lecture and use the fact that $|\partial\Omega_1|_{(d-1)} > 0$). Finally we show that (therefore, we called it the primary formulation)

$$\boxed{\nabla u = \mathbb{B}\mathbf{f}}. \tag{15}$$

To do so, we compute

$$\begin{aligned} C\int_\Omega |\mathbb{B}\mathbf{f} - \nabla u|^2 &\le \int_\Omega \mathbb{B}^{-1}(\mathbb{B}\mathbf{f} - \nabla u)\cdot(\mathbb{B}\mathbf{f} - \nabla u) = \int_\Omega (\mathbb{B}\mathbf{f} - \nabla u)\cdot(\mathbf{f} - \mathbb{B}^{-1}\nabla u)\\ &= \int_\Omega (\mathbb{B}\mathbf{f} - \nabla u_0)\cdot\underbrace{(\mathbf{f} - \mathbb{B}^{-1}\nabla u)}_{\in K_0} + \int_\Omega \nabla(\underbrace{u_0 - u}_{\in V})\cdot(\mathbf{f} - \mathbb{B}^{-1}\nabla u)\\ &\overset{(12),(14)}{=} 0, \end{aligned}$$

where we also used the fact that $\mathbf{f} \in K$. Thus, (15) holds true.

Regularity: We use the elliptic regularity(from the lecture). We assume that $\mathbb{B}$ is in addition Lipschitz (and consequently also $\mathbb{B}^{-1}$). Then we use the elliptic regularity for (13). Hence, we have $u \in W^{2,2}_{loc}(\Omega)$. In addition, to get that $u \in W^{2,2}(\Omega)$ we must assume that $\Omega \in C^{1,1}$ and also that $\overline{\partial\Omega_1} \cap \overline{\partial\Omega_2} = \emptyset$. Thus using (15), we deduce $\mathbf{f} \in W^{1,2}_{loc}$ provided that $\mathbb{B}$ is Lipschitz and $\mathbf{f} \in W^{1,2}$ provided that $\Omega \in C^{1,1}$ and also that $\overline{\partial\Omega_1} \cap \overline{\partial\Omega_2} = \emptyset$.