Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Formulate the continuous and compact embedding theorem of the Sobolev space $W^{1,p}(\Omega)$ into the space of Hölder continuous functions.

Solution:

See lecture.

[100] 2. Formulate and prove the Lax–Milgram theorem.

Solution:

See lecture.

[100] 3. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz set and $z_1, z_2 \in L^2(\Omega)$ be given and define the set

$$S := \left\{ f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega) : \int_\Omega \sum_{i=1}^2 (f_i - z_i) \frac{\partial \varphi}{\partial x_i} = 0 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega) \right\}.$$

Consider the functional $\mathcal{F} : S \to \mathbb{R}$ given as

$$\mathcal{F}(f) := \int_\Omega f_1^2 + Bf_2^2 + 2Cf_1 f_2 - 2f_1 + 2x_2 f_2,$$

where $C, B \in \mathbb{R}$ are given numbers. Finally, consider the minimization problem of the form: Find $f = (f_1, f_2) \in S$ such that for all $h = (h_1, h_2) \in S$ there holds

$$\mathcal{F}(f) \leq \mathcal{F}(h) \tag{P}$$

20 % Write the Euler–Lagrange equation corresponding to (P). Prove that any solution f to (P) satisfies the Euler–Lagrange equations.

30 % Show that if $B > 0$ and $C^2 < B$ then the problem (P) has a unique solution (minimizer). (Hint: show that infimum of the functional $\mathcal{F}$ is finite, take the minimizing sequence, show that it is bounded in a **reflexive space** L^2 and show that it converges weakly to minimum.)

50% Prove that there exists a matrix $A \in \mathbb{R}^{2\times 2}$ and $u \in W^{1,2}(\Omega)$, which is unique up to a constant, such that u is a weak solution to

$$-\mathrm{div} A\nabla u = -\mathrm{div} z \quad \text{in } \Omega$$

and in addition fulfils $A\nabla u = f$ almost everywhere in Ω. What is the trace of u?

Solution:

Euler–Lagrange equations: First, define

$$S_0 := \left\{ f = (f_1, f_2) \in L^2(\Omega) \times L^2(\Omega) : \int_\Omega \sum_{i=1}^2 f_i \frac{\partial \varphi}{\partial x_i} = 0 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega) \right\}.$$

Let f be a hypothetical solution to (P), $g \in S_0$ and $\varepsilon > 0$ be arbitrary. If we set $h := f + \varepsilon g$ then evidently $h \in S$ and therefore it can be used in (P) as a competitor. Thus, it follows from (P) that

$$\begin{aligned}&\int_\Omega f_1^2 + Bf_2^2 + 2Cf_1 f_2 - 2f_1 + 2x_2 f_2 = \mathcal{F}(f) \leq \mathcal{F}(h) \\ &= \int_\Omega (f_1 + \varepsilon g_1)^2 + B(f_2 + \varepsilon g_2)^2 + 2C(f_1 + \varepsilon g_1)(f_2 + \varepsilon g_2) - 2(f_1 + \varepsilon g_1) + 2x_2(f_2 + \varepsilon g_2).\end{aligned}$$

After a simple algebraic manipulation we deduce that (note that all integrals are well defined and finite thanks to the Hölder inequality)

$$0 \leq \int_\Omega 2f_1 \varepsilon g_1 + \varepsilon^2 g_1^2 + 2Bf_2 \varepsilon g_2 + B\varepsilon^2 g_2^2 + 2C(f_1 \varepsilon g_2 + f_2 \varepsilon g_1 + \varepsilon^2 g_1 g_2) - 2\varepsilon g_1 + 2x_2 \varepsilon g_2.$$

Finally, dividing by 2ε and letting $\varepsilon \to 0_+$ we get

$$0 \le \int_\Omega f_1 g_1 + B f_2 g_2 + C(f_1 g_2 + f_2 g_1) - g_1 + x_2 g_2.$$

Since $g \in S_0$ is arbitrary and $-g \in S_0$ as well, it follows that there holds

$$0 = \int_\Omega f_1 g_1 + B f_2 g_2 + C(f_1 g_2 + f_2 g_1) - g_1 + x_2 g_2 \qquad \text{for all } g \in S_0, \tag{E-L}$$

which is nothing else than the **Euler–Lagrange equation**. Moreover, we have just proven rigorously that if f solves (P) then it solves (E-L) as well.

Existence of minimizer: First, we show that $\mathcal{F}$ is bounded from below. To do so, we use the assumption that $C^2 < B$. For arbitrary $\delta_1, \delta_2 \in (0,1)$, we have (we use the Young inequality)

$$\begin{aligned}
&\int_\Omega f_1^2 + B f_2^2 + 2C f_1 f_2 - 2f_1 + 2x_2 f_2 \\
&\quad \ge \int_\Omega f_1^2 + B f_2^2 - 2\left(\sqrt{(1-\delta_1)}|f_1|\right)\left(\frac{C|f_2|}{\sqrt{1-\delta_1}}\right) - 2\frac{\sqrt{1+|x|^2}}{\sqrt{\delta_2}}(\sqrt{\delta_2}|f|) \\
&\quad \ge \int_\Omega (\delta_1 - \delta_2) f_1^2 + \left(B - \frac{C^2}{1-\delta_1} - \delta_2\right) f_2^2 - \frac{1+|x|^2}{\delta_2}.
\end{aligned}$$

Hence, since $B > C^2$ and Ω is bounded (because it is Lipschitz), we can choose $\delta_1, \delta_2 > 0$ sufficiently small (depending on B and C) and find positive constants C_1 and C_2 such that

$$\mathcal{F}(f) \ge C_1(B,C)\|f\|_2^2 - C_2(B,C,\Omega) \qquad \text{for all } f \in L^2(\Omega) \times L^2(\Omega). \tag{A-E}$$

Next, let us denote

$$I := \inf_{f \in S} \mathcal{F}(f).$$

Due to (A-E), we see that $I > -\infty$. From the definition of infima, it follows that there exists a sequence $\{f^n\}_{n=1}^\infty \subset S$ such that

$$I = \lim_{n \to \infty} \mathcal{F}(f^n). \tag{I}$$

Setting $h := z$ in (P) (it is a possible choice since $z \in S$), it also follows that

$$I \le \mathcal{F}(z) \le C(1 + \|z\|_2^2),$$

where the second inequality follows from the Hölder inequality. Consequently, using also (A-E), we see that there exists n_0 such that for all $n \ge n_0$ we have

$$C_1(B,C)\|f^n\|_2^2 - C_2(B,C,\Omega) \le \mathcal{F}(f^n) \le I + 1 \le C(2 + \|z\|_2^2)$$

and therefore

$$\|f^n\|_2^2 \le C.$$

Consequently, f^n is a bounded sequence in $L^2(\Omega) \times L^2(\Omega)$ which is a reflexive space and therefore there exists a subsequence f^{n_k} and f such that

$$f^{n_k} \rightharpoonup f \text{ weakly in } L^2(\Omega) \times L^2(\Omega).$$

Note that the above convergence means that for any $g_1, g_2 \in L^2(\Omega)$ there holds

$$\lim_{n_k \to \infty} \int_\Omega f_1^{n_k} g_1 + f_2^{n_k} g_2 = \int_\Omega f_1 g_1 + f_2 g_2. \tag{w-c}$$

Using the definition of S, the fact that $f^n \in S$ and (w-c) we can deduce that $f \in S$ as well. Finally, we show that f is really a minimizer, i.e., it solves (P). Indeed, using the computation above (A-E), we have that (using the fact that $B > C^2$)

$$0 \le (f_1^{n_k} - f_1)^2 + B(f_2^{n_k} - f_2)^2 + 2C(f_1^{n_k} - f_1)(f_2^{n_k} - f_2)$$

Consequently,

$$\begin{aligned} 0 \le& \int_\Omega (f_1^{n_k})^2 + f_1^2 - 2f_1^{n_k} f_1 + B(f_2^{n_k})^2 + Bf_2^2 - 2Bf_2^{n_k} f_2 \\ &+ 2Cf_1^{n_k} f_2^{n_k} - 2Cf_1^{n_k} f_2 - 2Cf_1 f_2^{n_k} + 2Cf_1 f_2 \\ =& \mathcal{F}(f^{n_k}) + \mathcal{F}(f) + \int_\Omega f_1^{n_k}(2 - 2f_1 - 2Cf_2) + f_2^{n_k}(-2x_2 - 2Bf_2 - 2Cf_1) + 2f_1 - 2x_2 f_2 \end{aligned}$$

Next we use (w-c) with $g_1 := 2 - 2f_1 - 2Cf_2$ and $g_2 := -2x_2 - 2Bf_2 - 2Cf_1$, the relation (I) and let $n_k \to \infty$ in the above inequality to conclude

$$\begin{aligned} 0 \le& \lim_{n_k \to \infty} \Big(\mathcal{F}(f^{n_k}) + \mathcal{F}(f) + \int_\Omega f_1^{n_k}(2 - 2f_1 - 2Cf_2) \\ &+ f_2^{n_k}(-2x_2 - 2Bf_2 - 2Cf_1) + 2f_1 - 2x_2 f_2 \Big) \\ =& I + \mathcal{F}(f) + \int_\Omega f_1(2 - 2f_1 - 2Cf_2) + f_2(-2x_2 - 2Bf_2 - 2Cf_1) + 2f_1 - 2x_2 f_2 \\ =& I - \mathcal{F}(f). \end{aligned}$$

But since I is infimum, it necessarily follows from the above inequality that f solves (P). (Recall, that we know that $f \in S$)

Finally, we show that there is a unique minimizer. Let f and $\tilde{f}$ be two solutions to (P) and denote $w := f - \tilde{f}$. Since both functions solve (P) then they also satisfy the Euler-Lagrange equation (E-L). Hence, subtracting (E-L) for $\tilde{f}$ from the identity (E-L) for f we obtain

$$0 = \int_\Omega w_1 g_1 + Bw_2 g_2 + C(w_1 g_2 + w_2 g_1) \qquad \text{for all } g \in S_0. \tag{Uniq}$$

Then, because $f, \tilde{f} \in S$, we get that $w \in S_0$ and so we can set $g := w$ in (Uniq). Doing so, we obtain

$$0 = \int_\Omega w_1^2 + Bw_2^2 + 2Cw_1 w_2 \ge C_1 \|w\|_2^2, \tag{Uniq2}$$

where for the second inequality, we used the fact that $B > C^2$ (see the very similar computation above (A-E)). Hence, $w = 0$ and so $f = \tilde{f}$.

Elliptic problem: In fact, here we will show that the minimization problem is a dual problem to certain elliptic equation. Define the matrix

$$Q := \begin{pmatrix} 1 & C \\ C & B \end{pmatrix}$$

Since matrix Q is regular and positively definite (because $B > C^2$), we can find its inverse,

$$A := Q^{-1}$$

and A is positively definite as well, i.e., it satisfies for all $\xi \in \mathbb{R}^2$

$$\sum_{i,j=1}^{2} A_{ij}\xi_i\xi_j \geq c_1|\xi|^2 \tag{1}$$

with some $c_1 > 0$. Finally, we set u_0

$$u_0 := x_1 - \frac{x_2^2}{2}.$$

The purpose of this setting is that we can now equivalently rewrite the Euler–Lagrange equation (E-L) as

$$0 = \int_\Omega (A^{-1}f - \nabla u_0) \cdot g \qquad \text{for all } g \in S_0. \tag{E-L–II}$$

Next, we define an elliptic problem

$$-\text{div } A\nabla u = -\text{div} z \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega. \tag{El}$$

Since A is elliptic matrix, we can use the Lax-Milgram theorem and find a unique $u \in W^{1,2}(\Omega)$ being the weak solution, i.e., $u - u_0 \in W_0^{1,2}(\Omega)$ and for all $\varphi \in W_0^{1,2}(\Omega)$ there holds

$$\int_\Omega (A\nabla u - z) \cdot \nabla\varphi = 0. \tag{w-f}$$

Finally, we show that $A\nabla u = f$. First, it follows from the fact that $f \in S$ and (w-f) that $(A\nabla u - f) \in S_0$. Next, we can compute (using the ellipticity of A)

$$\begin{aligned}
c_1\|\nabla u - A^{-1}f\|_2^2 &= c_1 \int_\Omega |\nabla u - A^{-1}f|^2 \leq \int_\Omega A(\nabla u - A^{-1}f) \cdot (\nabla u - A^{-1}f) \\
&= \int_\Omega (A\nabla u - f) \cdot (\nabla u - A^{-1}f) \\
&= \int_\Omega \underbrace{(A\nabla u - f)}_{\in S_0} \cdot \nabla(\underbrace{u - u_0}_{\in W_0^{1,2}}) + \int_\Omega \underbrace{(A\nabla u - f)}_{\in S_0} \cdot (\nabla u_0 - A^{-1}f) = 0,
\end{aligned}$$

where the first integral is equal to zero just because of definition of S_0 and the second integral vanishes thanks to (E-L–II).

Alternative proof of the existence of minimizer: Here, we quickly present a proof for the existence of minimizer f, which is just based on the use of Lax-Milgram theorem. First, we consider the problem (El). By using the Lax-Milgram theorem we get the unique u a weak solution. Next, we define $f := A\nabla u$ and show that it solves (P). First, since u is a weak solution, we have

$$0 = \int_\Omega (A\nabla u - z)\cdot\nabla\varphi = \int_\Omega (f - z)\cdot\nabla\varphi,$$

for all $\varphi \in W_0^{1,2}(\Omega)$. Hence, $f \in S$. Next, we show that f satisfies the Euler–Lagrange equations (E-L). Note that (E-L) can be equivalently rewritten as (E-L–II). Hence, for arbitrary $g \in S_0$ we can compute

$$\int_\Omega (A^{-1}f - \nabla u_0)\cdot g = \int_\Omega \nabla\underbrace{(u - u_0)}_{\in W_0^{1,2}(\Omega)}\cdot\underbrace{g}_{\in S_0} = 0,$$

where the last equality follows from the definition of S_0. Therefore we obtained that f solves (E-L). Finally, we show that any solution to (E-L) is also a solution to (P). Using the definition of A, the fact that it is elliptic and the definition of u_0 again we have that

$$\begin{aligned}\mathcal{F}(\tilde f) - \mathcal{F}(f) &= \int_\Omega (A^{-1}\tilde f - 2\nabla u_0)\cdot\tilde f - (A^{-1}f - 2\nabla u_0)\cdot f \\ &= \int_\Omega 2(A^{-1}f - \nabla u_0)\cdot\underbrace{(\tilde f - f)}_{\in S_0} + \underbrace{A^{-1}(\tilde f - f)\cdot(\tilde f - f)}_{\geq 0} \\ &\geq 0,\end{aligned}$$

where the second inequality follows from the fact that f solves (E-L–II). Hence, f is a minimizer.