Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Formulate the Lax–Milgram theorem.

Solution:
See lecture.

[100] 2. Consider the set
$$\Omega := \{(x,y) \in \mathbb{R}^2;\ x^2+y^2<1\}$$
and the function
$$f(x,y) := |x-y|^\alpha.$$

70% Prove that for $\alpha \geq 0$, we have $f \in W^{1,1}(\Omega)$. Derive rigorously $\partial_x f$ and $\partial_y f$ - the weak derivatives of f. Find the biggest $p>1$ for which $f \in W^{1,p}(\Omega)$.

30% Prove that for $\alpha < 0$, we have $f \notin W^{1,1}(\Omega)$.

Solution:

Let us define two open sets
$$\Omega_+ := \{(x,y) \in \Omega;\ x>y\},$$
$$\Omega_- := \{(x,y) \in \Omega;\ x<y\}.$$

Then f is smooth on Ω_+ and on Ω_- and we have that in $\Omega_+ \cup \Omega_-$ the following formulae for classical derivatives
$$\begin{aligned}\frac{\partial f}{\partial x} &= \alpha|x-y|^{\alpha-2}(x-y),\\ \frac{\partial f}{\partial y} &= -\alpha|x-y|^{\alpha-2}(x-y)\end{aligned} \tag{1}$$

Consequently, if $u \in W^{1,1}(\Omega)$ then for **weak derivatives** necessarily holds (1) almost everywhere in Ω. First, let us check the integrability of functions appearing in (1). In case $\alpha \geq 1$, we see that all quantities in (1) are bounded. Next, consider $\alpha < 1$ and $p \geq 1$ and compute
$$\int_\Omega \left(|x-y|^{\alpha-1}\right)^p \underset{x'=\frac{x-y}{\sqrt{2}},y'=\frac{x+y}{\sqrt{2}}}{=} 2^{\frac{(\alpha-1)p}{2}} \int_\Omega |x'|^{(\alpha-1)p}\,dx'dy' < \infty \qquad \Leftrightarrow \qquad p < \frac{1}{1-\alpha}$$

Consequently, if $\alpha < 0$ then p would be less than 1, hence, it would not be the Sobolev function. The case $\alpha = 0$ is trivial because then f is constant.

Hence, finally if we check that (1) is really the formula for the weak derivative then we get that $f \in W^{1,p}(\Omega)$ with $p = \infty$ if $\alpha \geq 1$ and for arbitrary $1 \leq p < \frac{1}{1-\alpha}$ if $\alpha \in (0,1)$. So let $\varphi \in \mathcal{C}_0^\infty(\Omega)$ be arbitrary and denote for $\varepsilon > 0$
$$\Omega_+^\varepsilon := \{(x,y) \in \Omega;\ x>y+\varepsilon\},$$
$$\Omega_-^\varepsilon := \{(x,y) \in \Omega;\ x<y-\varepsilon\},$$
$$\Omega^\varepsilon := \{(x,y) \in \Omega;\ y-\varepsilon \leq x \leq y+\varepsilon\}.$$

Note that on $\Omega_\pm^\varepsilon$ the function f is smooth and we can use the standard integration by parts. Thus, we have
$$\begin{aligned}\int_\Omega f\frac{\partial \varphi}{\partial x} &= \int_{\Omega_+^\varepsilon} f\frac{\partial \varphi}{\partial x} + \int_{\Omega_-^\varepsilon} f\frac{\partial \varphi}{\partial x} + \int_{\Omega^\varepsilon} f\frac{\partial \varphi}{\partial x}\\ &= -\int_{\Omega_+^\varepsilon} \varphi\frac{\partial f}{\partial x} - \int_{\Omega_-^\varepsilon} \varphi\frac{\partial f}{\partial x} + \int_{\Omega^\varepsilon} f\frac{\partial \varphi}{\partial x} + \int_{\partial\Omega_+^\varepsilon} f\varphi n_x + \int_{\partial\Omega_-^\varepsilon} f\varphi n_x\end{aligned} \tag{2}$$

Next, we let $\varepsilon \to 0_+$. We use the above computation and (1), so we have that for $\alpha > 0$ $\partial_x f \in \mathrm{Ł}^1(\Omega)$. Therefore

$$\lim_{\varepsilon\to 0_+} -\int_{\Omega_+^\varepsilon} \varphi \frac{\partial f}{\partial x} - \int_{\Omega_-^\varepsilon} \varphi \frac{\partial f}{\partial x} + \int_{\Omega^\varepsilon} f \frac{\partial \varphi}{\partial x} = -\alpha \int_\Omega (x-y)|x-y|^{\alpha-2}\varphi$$

Similarly, denoting

$$\Gamma_+^\varepsilon := \{(x,y) \in \Omega,\ x = y+\varepsilon\}, \quad \Gamma_-^\varepsilon := \{(x,y) \in \Omega,\ x = y-\varepsilon\}$$

and observing that the normal vector satisfies $n_x = 1/\sqrt{2}$ on Γ_+^ε and $n_x = -1/\sqrt{2}$ on Γ_-^ε, and using the fact that $\varphi = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\sqrt{2}\lim_{\varepsilon\to 0_+}\left(\int_{\partial\Omega_+^\varepsilon} f\varphi n_x + \int_{\partial\Omega_-^\varepsilon} f\varphi n_x\right) &= \lim_{\varepsilon\to 0_+}\int_{\Gamma_\varepsilon^+} f\varphi - \int_{\Gamma_\varepsilon^-} f\varphi \\ &= \lim_{\varepsilon\to 0_+}\varepsilon^\alpha\left(\int_{\Gamma_\varepsilon^+}\varphi - \int_{\Gamma_\varepsilon^-}\varphi\right) = 0\end{aligned}$$

Hence, letting $\varepsilon \to 0_+$ in (2) and using the above computation, we observe that (1) is indeed a formula for the weak derivative. For $\partial_y f$ we do the same computation.

[100] 3. Let $\Omega := B_1(0) \subset \mathbb{R}^2$. Consider the following problem: Find $u : (0,T)\times\Omega \to \mathbb{R}$ fulfilling

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + 2\frac{\partial^2 u}{\partial x_1 \partial x_2} &= 0 && \text{in } (0,T)\times\Omega,\\
\left(2\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2}\right)x_1 + \left(\frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_1}\right)x_2 &= 0 && \text{on } (0,T)\times\partial\Omega,\\
u(0) &= \sqrt{x_1^2 + x_2^2} && \text{in } \Omega,\\
\frac{\partial u}{\partial t}(0) &= 0 && \text{in } \Omega.
\end{aligned} \tag{P}$$

20% Define a notion of a weak solution u to (P). Check that it is meaningful.

60% Prove the existence and the uniqueness of a weak solution.

20% Is it true that $\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T;L^2(\Omega))$? Justify!

Solution:

First, it is important to notice that this is hyperbolic equation of second order with kind of Neumann data. (This is different to Dirichlet data, which was treated during lectures). Therefore, we **must not** fix the trace for test function.

First, we rewrite the (P) to the more standard form. First, we can observe that $\nu := (x_1, x_2)$ is the normal vector on $\partial\Omega$. Thus, if we define the matrix A as

$$A := \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

we can rewrite the problem (P) into the more familiar form

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \operatorname{div} A\nabla u &= 0 && \text{in } (0,T)\times\Omega,\\
A\nabla u \cdot \nu &= 0 && \text{on } (0,T)\times\partial\Omega,\\
u(0) &= \sqrt{x_1^2 + x_2^2} := u_0(x) && \text{in } \Omega,\\
\frac{\partial u}{\partial t}(0) &= 0 && \text{in } \Omega.
\end{aligned} \tag{P2}$$

Moreover, it is also evident that the matrix A is elliptic, i.e., for all $\xi \in \mathbb{R}^2$ we have

$$A\xi \cdot \xi \geq c_1|\xi|^2$$

for some $c_1 > 0$. Hence, now it is evident how we define a notion of a weak solution. We set $V := W^{1,2}(\Omega)$ and say that $u : (0,T)\times\Omega$ is a weak solution to (P) if $u \in L^2(0,T;V)$, $\partial_t u \in L^2(0,T;L^2(\Omega))$ and $\partial_{tt}^2 u \in L^2(0,T;V^*)$ and for almost all $t \in (0,T)$ and all $\varphi \in V$ satisfies

$$\langle \partial_{tt} u, \varphi \rangle + \int_\Omega A\nabla u \cdot \nabla\varphi = 0. \tag{3}$$

In addition, we require that $u(0) = u_0$ and $\partial_t u(0) = 0$. Note that thanks to the assumptions on $\partial_t u$ and $\partial_{tt}^2 u$, we know that $u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_t u \in \mathcal{C}([0,T];V^*)$

and therefore we can talk about the point-wise values of $u(t)$ and $\partial_t u(t)$, in particular for $t = 0$.

Next, we show that there exists a unique weak solution. Moreover, we show that u in addition belongs to $L^\infty(0,T;V)$ and $\partial_t u \in L^\infty(0,T;L^2(\Omega))$.

We start with the existence part. We can find $\{w_i\}_{i=1}^\infty$ a basis of V which is orthonormal in $L^2(\Omega)$. Then we look for u^n given by

$$u^n(t,x) := \sum_{i=1}^n \alpha_i^n(t) w_i$$

solving

$$\begin{aligned} &\int_\Omega \partial_{tt} u^n w_i + \int_\Omega A\nabla u^n \cdot \nabla w_i = 0 \qquad i = 1,\dots,n, \\ &u^n(0) = P^n u_0, \qquad \partial_t u^n(0) = 0, \end{aligned} \tag{4}$$

where $P^n u_0 := \sum_{i=1}^n w_i \int_\Omega u_0 w_i$.

Next, we rewrite (4) as the ordinary equations of secdond order for α^n. Using orthonormality of the basis we have

$$\begin{aligned} &(\alpha_i^n)''(t) + \sum_{j=1}^n \alpha_j^n(t) \int_\Omega A\nabla w_j \cdot \nabla w_i = 0 \qquad i = 1,\dots,n, \\ &\alpha_i^n(0) = \int_\Omega u_0 w_i, \qquad (\alpha_i^n)'(0) = 0. \end{aligned} \tag{5}$$

The system (5) is the system of ordinary differential equations of second order with constant coefficients and has always a solution.

Next, we focus on a priori (n-independnet) estimates. We also recal the property of P^n, namely $\|P^n v\|_2 \le \|v\|_2$ and $\|P^n v\|_V \le C\|v\|_V$ for all $v \in V$. Also Notice that since $\nabla u_0 = \frac{x}{|x|}$, we see that $u_0 \in V$ and $\|P^n u_0\|_V \le C$.

We multiply the i-th equation in (4) by $(\alpha_i^n)'(t)$ and sum the result over $i = 1,\dots,n$ to obtain

$$\int_\Omega \partial_{tt} u^n \partial_t u^n + \int_\Omega A\nabla u^n \cdot \nabla \partial_t u^n = 0.$$

Hence using the fact the A is symmetric, we get

$$\frac{d}{dt}\left(\|\partial_t u^n(t)\|_2^2 + \int_\Omega A\nabla u^n(t) \cdot \nabla u^n(t) \right) = 0.$$

Integration with respect to time then leads to

$$\begin{aligned} &\|\partial_t u^n(t)\|_2^2 + \int_\Omega A\nabla u^n(t) \cdot \nabla u^n(t) = \|\partial_t u^n(0)\|_2^2 + \int_\Omega A\nabla u^n(0) \cdot \nabla u^n(0) \\ &= \int_\Omega A\nabla P^n u_0 \cdot \nabla P^n u_0 \le C\|P^n u_0\|_V^2 \le C. \end{aligned} \tag{6}$$

Thus, using the fact that A is elliptic, we get

$$\sup_{t\in(0,T)} \|\partial_t u(t)\|_2^2 + \|\nabla u^n(t)\|_2^2 \le C \tag{7}$$

with C being independent of n. Note that in the above equation we do not control u^n in V, since we do not have the Poincaré inequality (Neumann problem). Thus, to add the remaining information, we use the properties of the Bochner integral and the fact we know initial condition. Indeed,

$$\|u^n(t)\|_2 = \left\|\int_0^t \partial_\tau u^n(\tau)d\tau + u^n(0)\right\|_2 \le \int_0^t \|\partial_t u(t)\|_2 + \|P^n u_0\|_2 \le \|u_0\| + T\sup_t \|\partial_t u^n(t)\|_2$$

Hence, using the first part in (7), we deduce the final estimate

$$\|\partial_t u^n\|_{L^\infty(0,T;L^2(\Omega))} + \|u^n\|_{L^\infty(0,T;V)} \le C(\|u_0\|_V, T). \tag{8}$$

Finally, we derive the estimate for $\partial_{tt}u^n$. Using the orthonormality of the basis, we have for all $v \in V$

$$\begin{aligned}\langle \partial_{tt}u^n(t), v\rangle &= \int_\Omega \partial_{tt}u^n(t)v \overset{Orthogonality}{=} \int_\Omega \partial_{tt}u^n(t)P^n v\\ &\overset{(4)}{=} -\int_\Omega A\nabla u^n(t)\cdot \nabla P^n v \le C\|\nabla u^n(t)\|_2 \|\nabla P^n v\|_2 \overset{(7)}{\le} C\|v\|_V.\end{aligned}$$

Consequently, using the definition of the norm, we have

$$\|\partial_{tt}u^n\|_{V^*} := \sup_{v\in V;\, \|v\|_V=1} \langle \partial_{tt}u^n(t), v\rangle \le C$$

and we have

$$\|\partial_{tt}u^n\|_{L^\infty(0,T;V)} \le C. \tag{9}$$

Hence, using (8) and (9), we can find u such that

$$u^n \rightharpoonup^* u \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V), \tag{10}$$
$$\partial_t u^n \rightharpoonup^* \partial_t u \qquad \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{11}$$
$$\partial_{tt}u^n \rightharpoonup^* \partial_{tt}u \qquad \text{weakly}^* \text{ in } L^\infty(0,T;V^*). \tag{12}$$

Thus, u is a good candidate for weak solution, it just remains to check whether it satisfies (3) and fulfils also the initial conditions. Since from this point, the proof is identical to the proof for hyperbolic equations with homogeneous boundary data, we refer to the lecture for the rest of the proof of the existence theorem.

To get uniqueness, we also closely follow the proof for Dirichlet problem. For arbitrary $s > 0$. Due to the linearity, it is enough to show that if u is a weak solution with zero initial data then it is identical zero. We define (Bochner integral)

$$\varphi(t) := \begin{cases} \int_t^s u(\tau)d\tau & \text{if } t \le s,\\ 0 & \text{if } t \in (s,T).\end{cases}$$

Since $\varphi \in L^2(0,T;V)$ we can use it in (3) to get for almost all t the identity

$$\langle \partial_{tt} u(t), \varphi(t)\rangle + \int_\Omega A\nabla u(t)\cdot \nabla\varphi(t) = 0.$$

Integrating the result over $t \in (0,T)$ and using integration by parts, we have

$$\begin{aligned}
0 &= \int_0^T \langle \partial_{tt} u, \varphi\rangle + \int_\Omega A\nabla u \cdot \nabla \varphi \overset{\varphi(T)=0, \partial_t u(0)=0}{=} -\int_0^T \langle \partial_t u, \partial_t \varphi\rangle + \int_0^T \int_\Omega A\nabla u\cdot\nabla\varphi \\
&= \int_0^s \int_\Omega \partial_t u u - \int_0^s \int_\Omega A(\nabla \partial_t \varphi)\cdot \nabla\varphi = \frac{1}{2}\int_0^s \frac{d}{dt}\left(\|u\|_2^2 - \int_\Omega A\nabla\varphi\cdot\nabla\varphi\right) \\
&= \frac{1}{2}\left(\|u(s)\|_2^2 + \int_\Omega A\nabla\varphi(0)\cdot\nabla\varphi(0)\right) \geq \frac{1}{2}\|u(s)\|_2^2.
\end{aligned}$$

Hence, $u(s) = 0$ and since s was arbitrary, we have $u = 0$ almost everywhere in $(0,T)\times\Omega$.

Concerning the regularity statement, we give here just heurestic arguments (which would be however sufficient for the exam). In case that $\partial_{tt} u \in L^\infty(0,T;L^2(\Omega))$ the also $\partial_{tt} u(0) \in L^2(\Omega)$ (this is the heurestic part). Then we can however read from the equation

$$\partial_{tt} u(0) = \operatorname{div} A\nabla u(0) = \operatorname{div} A\nabla u_0 = \sum_{i,j} A_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u_0$$

Since the left hand side is in $L^2(\Omega)$ then also the right hand side must be in $L^2(\Omega)$. However, for this purpose we need that $u_0 \in W^{2,2}(\Omega)$. Since $|\nabla^2 u| \sim |x|^{-1}$ and we are in dimension two, we see that $u_0 \notin W^{2,2}(\Omega)$, which is a contradiction.