Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ____________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Define the Sobolev space $W^{1,p}(\Omega)$. For which p's is it separable, reflexive or Hilbert?

Solution:

See lecture.

[100] 2. Formulate and prove the equivalent characterization of Sobolev spaces via difference quotients.

Solution:

This exam was slightly special because there was just one student. Therefore he got some hints for the proof of this theorem during the exam because the proof was not discussed during lectures. But in remaining exams it will not be the case.

[100] 3. Let $\Omega \subset \mathbb{R}^2$ be Lipschitz and $T > 0$. Consider the problem

$$\begin{aligned}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x_1^2} + 10\frac{\partial u}{\partial x_1 \partial x_2} - 26\frac{\partial^2 u}{\partial x_2^2} = \frac{\partial f}{\partial x_1} \quad &\text{in } (0,T)\times\Omega,\\ u = 1 \quad &\text{on } (0,T)\times\partial\Omega,\\ u(0) = u_0 \quad &\text{in } \Omega.\end{aligned}$$

10% Find the optimal (largest) function spaces corresponding to f and u_0 and define a notion of a weak solution. Check that for given data such a definition is meaningful.

30% Prove that for any such f and u there exists unique weak solution. (If you use some theorem, check all assumptions!!!)

25% Show that if $f \in L^2(0,T;W^{1,2}(\Omega))$ and $u_0 \equiv 0$ then $u \in L^\infty_{loc}(0,T;W^{1,2}(\Omega))$. Can be also $u \in L^\infty(0,T;W^{1,2}(\Omega))$.

35% Consider that there exists $\tau > 0$ such that $f(t+\tau,x) = f(t,x)$ for all (t,x). Show that there exists unique u_0 for which the corresponding weak solution is time periodic with period τ. (Hint: Consider the mapping $u_0 \mapsto u(\tau)$ and show that it is a contraction.)

Solution:

First, we deal with the Dirichlet problem, therefore all test functions will have the zero trace. So, we denote $V := W_0^{1,2}(\Omega)$ and the corresponding Gelfand triple is $V \hookrightarrow H(= L^2(\Omega)) = H^* \hookrightarrow V^*$. Since we expect the solution u to be continuous with respect to time with values in $L^2(\Omega)$, we require $u_0 \in L^2(\Omega)$. Finally, to give a good meaning to the right hand side, it is enough to assume that $f \in L^2(\Omega)$. Thus, we say that u is a weak solution iff $(u-1) \in L^2(0,T;V)$ (here we specify the boundary conditions on $(0,T)\times\Omega$), $\partial_t u \in L^2(0,T;V^*)$, $u(0) = u_0$ and for all $\varphi \in V$ and almost all $t \in (0,T)$

$$\langle \partial_t u, \varphi\rangle_V + \int_\Omega \frac{\partial u}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + 26\frac{\partial u}{\partial x_2}\frac{\partial \varphi}{\partial x_2} - A\frac{\partial u}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - (10-A)\frac{\partial u}{\partial x_2}\frac{\partial \varphi}{\partial x_1} = -\int_\Omega f\frac{\partial \varphi}{\partial x_1}. \tag{*}$$

Thanks to the assumptions on f and u all integrals are well defined (Hölder inequality) for almost all $t \in (0,T)$. Note that we "moved"derivative on the right hand side to the test function to keep the minimal assumptions on the regularity of f. Also notice that we have a freedom in the choice of parameter $A \in \mathbb{R}$ (Try to prove that the notion of weak solution is independent of the choice of A!). We just need to specify how u_0 is attained. Since u has not zero trace, we cannot directly use the theory from the lecture. However, we have that $(u-1) \in L^2(0,T;V)$ and also $\partial_t(u-1) = \partial_t u \in L^2(0,T;V^*)$. Therefore we know that $(u-1) \in \mathcal{C}([0,T];L^2(\Omega))$ and consequently also $u \in \mathcal{C}([0,T];L^2(\Omega))$, so it makes sense to talk about the value $u(0)$.

Existence: From now we set $A := 5$ in order to "create"the symmetric operator. First, we show the ellipticity. Consider arbitrary $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$. Then by using the Young inequality, we obtain

$$\begin{aligned}\xi_1^2 + 26\xi_2^2 - 10\xi_1\xi_2 &\geq \xi_1^2 + 26\xi_2^2 - (\sqrt{2}\sqrt{1-\varepsilon}|\xi_1|)\frac{10}{\sqrt{2}\sqrt{1-\varepsilon}}|\xi_2|\\ &\geq \xi_1^2 + 26\xi_2^2 - \frac{2(1-\varepsilon)\xi_1^2}{2} - \frac{100\xi_2^2}{4(1-\varepsilon)}\\ &\geq \varepsilon\xi_1^2 + \frac{1-26\varepsilon}{1-\varepsilon}\xi_2^2 \geq C_1|\xi|^2\end{aligned} \tag{**}$$

with some $C_1 > 0$, which follows e.g. by setting $\varepsilon := 1/50$.

We look for u being of the form $u = 1 + v$. It means we want to find $v \in L^2(0,T;V)$ such that $\partial_t v \in L^2(0,T;V^*)$ fulfilling for all $\varphi \in V$ and almost all $t \in (0,T)$

$$\langle \partial_t v, \varphi \rangle_V + \int_\Omega \frac{\partial v}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + 26\frac{\partial v}{\partial x_2}\frac{\partial \varphi}{\partial x_2} - 5\frac{\partial v}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - 5\frac{\partial v}{\partial x_2}\frac{\partial \varphi}{\partial x_1} = -\int_\Omega f\frac{\partial \varphi}{\partial x_1}, \qquad v(0) = u_0 - 1. \tag{***}$$

The existence of v can follow line by line the proof presented during lectures. So we find a basis $\{w_i\}_{i=1}^\infty$ of V, which is orthonormal in $L^2(\Omega)$ and look for Galerkin approximation $v^n(t,x) = \sum_{i=1}^n a_i^n(t) w_i$ solving for $i = 1, \ldots, n$

$$\int_\Omega \partial_t v^n w_i + \frac{\partial v^n}{\partial x_1}\frac{\partial w_i}{\partial x_1} + 26\frac{\partial v^n}{\partial x_2}\frac{\partial w_i}{\partial x_2} - 5\frac{\partial v^n}{\partial x_1}\frac{\partial w_i}{\partial x_2} - 5\frac{\partial v^n}{\partial x_2}\frac{\partial w_i}{\partial x_1} = -\int_\Omega f\frac{\partial w_i}{\partial x_1}, \qquad v^n(0) = P^n(u_0 - 1) := \sum_{i=1}^n w_i \int_\Omega w_i(u_0 - 1). \tag{****}$$

Next, we can repeat the procedure from the lecture and let $n \to \infty$ and find v solving (***) (see the lecture).

Uniqueness: Assume that u and v are two solutions corresponding to the initial conditions u_0 and v_0. If we denote $w := u - w$ then it follows from (*) that

$$\langle \partial_t w, \varphi \rangle_V + \int_\Omega \frac{\partial w}{\partial x_1}\frac{\partial \varphi}{\partial x_1} + 26\frac{\partial w}{\partial x_2}\frac{\partial \varphi}{\partial x_2} - 5\frac{\partial w}{\partial x_1}\frac{\partial \varphi}{\partial x_2} - 5\frac{\partial w}{\partial x_2}\frac{\partial \varphi}{\partial x_1} = 0$$

for almost all $t \in (0,T)$ and all $\varphi \in V$. In addition (u and v have the same trace) we know $w \in L^2(0,T;V)$, so for almost all time we can set $\varphi := w(t)$ to obtain

$$\langle \partial_t w, w \rangle_V + \int_\Omega \left|\frac{\partial w}{\partial x_1}\right|^2 + 26\left|\frac{\partial w}{\partial x_2}\right|^2 - 10\frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2} = 0$$

Hence, using (**) and the Poincaré inequality, we deduce

$$\langle \partial_t w, w \rangle_V + C_1\|w\|_V^2 \le 0. \tag{(Un)}$$

Hence, integration with respect to t leads directly to

$$\|w(t)\|_2^2 \le \|w_0\|_2^2 = \|u_0 - v_0\|_2^2,$$

so if $u_0 = v_0$ then $w \equiv 0$ and so the solution is unique.

Regularity

To prove it rigorously, we must start with the approximation (****). We multiply the i-th equation by $\partial_t a_i^n$ and sum the result with respect to $i = 1, \ldots, n$. (It is the same as testing by $\partial_t v^n$)

$$\|\partial_t v^n\|_2^2 + \int_\Omega \frac{\partial v^n}{\partial x_1}\partial_t\frac{\partial v^n}{\partial x_1} + 26\frac{\partial v^n}{\partial x_2}\partial_t\frac{\partial v^n}{\partial x_2} - 5\frac{\partial v^n}{\partial x_1}\partial_t\frac{\partial v^n}{\partial x_2} - 5\frac{\partial v^n}{\partial x_2}\partial_t\frac{\partial v^n}{\partial x_1} = -\int_\Omega f\partial_t\frac{\partial v^n}{\partial x_1}$$

Next, we integrate by parts on the right hand side (we assume that $f \in L^2(0,T;W^{1,2}(\Omega))$) and use the symmetry on the left hand side to obtain

$$\|\partial_t v^n\|_2^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega \left|\frac{\partial v^n}{\partial x_1}\right|^2 + 26\left|\frac{\partial v^n}{\partial x_2}\right|^2 - 10\frac{\partial v^n}{\partial x_1}\frac{\partial v^n}{\partial x_2} = \int_\Omega \frac{\partial f}{\partial x_1}\partial_t v^n \leq \frac{1}{2}\|f\|_{1,2}^2 + \frac{1}{2}\|\partial_t v^n\|_2^2,$$

where we also used the Young inequality on the right hand side. Finally, we move the last term totheleft hand side and integrate the result with respect to time over (τ, t) and use also (**)

$$\begin{aligned}
&\int_\tau^t \|\partial_t v^n\|_2^2 + C_1\|\nabla v^n(t)\|_2^2 \\
&\qquad \overset{(**)}{\leq} \int_\tau^t \|\partial_t v^n\|_2^2 + \int_\Omega \left|\frac{\partial v^n(t)}{\partial x_1}\right|^2 + 26\left|\frac{\partial v^n(t)}{\partial x_2}\right|^2 - 10\frac{\partial v^n(t)}{\partial x_1}\frac{\partial v^n(t)}{\partial x_2} \\
&\qquad \leq \int_\tau^t \|f\|_{1,2}^2 + \int_\Omega \left|\frac{\partial v^n(\tau)}{\partial x_1}\right|^2 + 26\left|\frac{\partial v^n(\tau)}{\partial x_2}\right|^2 - 10\frac{\partial v^n(\tau)}{\partial x_1}\frac{\partial v^n(\tau)}{\partial x_2} \\
&\qquad \leq \int_0^T \|f\|_{1,2}^2 + C\|v^n(\tau)\|_{1,2}^2.
\end{aligned} \tag{A}$$

Next, we set $t := T$ and integrate with respect to $\tau \in (0, \delta)$ to get

$$\int_0^\delta \int_\tau^T \|\partial_t v^n(t)\|_2^2 \, dt \, d\tau \leq \delta \int_0^T \|f\|_{1,2}^2 + \int_0^\delta \|v^n\|_{1,2}^2 \leq C.$$

The last inequality is just the apriori estimate. Since

$$\delta \int_\delta^T \|\partial_t v^n\|_2^2 \leq \int_0^\delta \int_\tau^T \|\partial_t v^n(t)\|_2^2 \, dt \, d\tau$$

it follows

$$\int_\delta^T \|\partial_t v^n\|_2^2 \leq \frac{C}{\delta}.$$

This is the estimate which is independent of n and therefore holds also for the limit v. Since $\partial_t u = \partial_t v$ it gives the desired claim.

However, we cannot obtain it (in principle) up to the boundary. Indeed, if we set $t := T$ and $\tau := 0$ in (A), we have (using that $u_0 = 0$)

$$\begin{aligned}
\int_0^T \|\partial_t v^n\|_2^2 &\leq C(1 + \|v^n(0)\|_{1,2}^2) = C(1 + \|P^n(v_0)\|_{1,2}^2) \\
&= C(1 + \|P^n(u_0 - 1)\|_{1,2}^2) = C(1 + \|P^n(1)\|_{1,2}^2)
\end{aligned}$$

However the right hand side explodes as $n \to \infty$. It is caused by the fact that $1 \notin V$!!! (it does not have the zero trace). It is also visible directly from the beginning, since we prescribe **incompatible** boundary/initial data. We require $u = 1$ on $\partial\Omega$ for all times, but initially (for time $t = 0$), we also require $u \equiv 0$, which is incompatible with desired boundary conditions.

Periodicity: First, we show that $u_0 \mapsto u(\tau)$ is a contraction in $L^2(\Omega)$. Let $u_0, v_0 \in L^2(\Omega)$ be arbitrary. We denote u and v the weak solutions corresponding to u_0, and v_0 respectively and denote $w := u - v$. Then, we can use simple algebraic inequality and ((Un))

$$\langle \partial_t w, w\rangle_V + C_1\|w\|_2^2 \le \langle \partial_t w, w\rangle_V + C_1\|w\|_V^2 \le 0. \tag{(Un2)}$$

Multiplication[1]by $e^{2C_1 t}$ and integration over $(0,\tau)$ we get

$$0 \ge 2\int_0^\tau \langle \partial_t w, we^{2C_1 t}\rangle_V + 2C_1\int_0^\tau e^{2C_1 t}\|w\|_2^2 \tag{1}$$

We evaluate (integrate by parts)

$$\begin{aligned} 2\int_0^\tau \langle \partial_t w, we^{2C_1 t}\rangle_V &= -2\int_0^\tau \langle w, \partial_t(we^{2C_1 t})\rangle_V + 2e^{2C_1\tau}\|w(\tau)\|_2^2 - 2\|w(0)\|_2^2 \\ &= -2\int_0^\tau \langle w, \partial_t we^{2C_1 t}\rangle_V - 4C_1\int_0^\tau \|w\|_2^2 e^{2C_1 t} + 2e^{2C_1\tau}\|w(\tau)\|_2^2 - 2\|w(0)\|_2^2 \end{aligned}$$

which directly implies (comparing both sides)

$$2\int_0^\tau \langle \partial_t w, we^{2C_1 t}\rangle_V = -2C_1\int_0^\tau \|w\|_2^2 e^{2C_1 t} + 2e^{2C_1\tau}\|w(\tau)\|_2^2 - 2\|w(0)\|_2^2$$

which inserted into (1) gives

$$\|w(\tau)\|_2^2 \le e^{-2C_1\tau}\|w(0)\|_2^2. \tag{B}$$

Since $\tau > 0$, then $e^{-2C_1\tau} < 1$ and we see that we obtained a contraction.

Consequently, using the Banach fixed point theorem, there exists unique $u_0 \in L^2(\Omega)$ such that the weak solution satisfies $u(\tau) = u_0$. Finally, since f is τ periodic, we can extend u periodically, i.e., we can set $u(t+\tau) := u(t)$ for $t \in (0,\tau)$, and such extension will solve the original problem on the interval $(0, 2\tau)$. Then we can proceed inductively toextend the solution onto whole interval $(0,\infty)$.

[1]Formally we can rewrite the above inequality as

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + C_1\|w\|_2^2 \le 0$$

which multiplied by $2e^{2C_1 t}$ leads to

$$\frac{d}{dt}\left(e^{2C_1 t}\|w\|_2^2\right) \le 0$$

and integration over $(0,\tau)$ leads to (B).