Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Formulate the continuous and compact embedding theorem of the Sobolev space $W^{1,p}(\Omega)$ into the space of Hölder continuous functions.

Solution:

See lecture.

[100] 2. Consider the sets

$$\Omega_+ := (0,1)\times(-1,1), \quad \Omega_- := (-1,0)\times(-1,1), \quad \Omega := (-1,1)\times(-1,1).$$

Assume that $f \in W^{1,1}(\Omega_+)$ is arbitrary and define (here $x = (x_1, x_2)$)

$$f_1(x) := \begin{cases} f(x) & \text{for } x \in \Omega_+, \\ f(-x_1, x_2) & \text{for } x \in \Omega_-, \end{cases}$$

$$f_2(x) := \begin{cases} f(x) & \text{for } x \in \Omega_+, \\ -f(-x_1, x_2) & \text{for } x \in \Omega_-. \end{cases}$$

40% Is it true that $f_1 \in W^{1,1}(\Omega)$? If not, what is the necessary and sufficient additional assumption on f so that $f_1 \in W^{1,1}(\Omega)$?

40% Is it true that $f_2 \in W^{1,1}(\Omega)$? If not, what is the necessary and sufficient additional assumption on f so that $f_2 \in W^{1,1}(\Omega)$?

20% Assume in addition that $f \in W^{2,1}(\Omega_+)$. What is the necessary and sufficient condition for f so that f_1 belongs to $W^{2,1}(\Omega)$? What is the necessary and sufficient condition for f so that f_2 belongs to $W^{2,1}(\Omega)$?

Prove everything in details by using definitions!

Solution:

$W^{1,1}$ regularity:

Let us abbreviate $\partial_1 := \frac{\partial}{\partial x_1}$ and $\partial_2 := \frac{\partial}{\partial x_2}$. First, one should quickly check that $f_i \in W^{1,1}(\Omega_+ \cup \Omega_-)$ and that in $\Omega_- \cup \Omega_+$ there holds

$$\begin{aligned} \partial_i f_j(x_1, x_2) &= \partial_i f(x_1, x_2) & x \in \Omega_+, \\ \partial_i f_j(x_1, x_2) &= (-1)^{i+j+1}\partial_i f(-x_1, x_2) & x \in \Omega_-. \end{aligned}$$

The fist identity is clear and the second relations follow from the following computations - here $\varphi \in \mathcal{C}_0^\infty(\Omega_-)$ and for the second equality we use the definition of weak derivative of f on the set Ω

$$\begin{aligned} \int_{\Omega_-} \partial_i f(-x_1, x_2)\varphi(x_1, x_2) &= \int_{\Omega_+} \partial_i f(x_1, x_2)\varphi(-x_1, x_2) = -\int_{\Omega_+} f(x_1, x_2)\frac{\partial\varphi(-x_1, x_2)}{\partial x_i} \\ &= (-1)^i \int_{\Omega_+} f(x_1, x_2)\partial_i\varphi(-x_1, x_2) = (-1)^{i+j}\int_{\Omega_-} f_j(x_1, x_2)\partial_i\varphi(x_1, x_2) \end{aligned}$$

and thus $\partial_i f_j(x_1, x_2) = (-1)^{i+j+1}\partial_i f(-x_1, x_2)$.

Consequently, we also have that ∇f_i is integrable over Ω_-. All what remains is to check whether for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$ there holds

$$\int_\Omega \partial_i f_j \varphi = -\int_\Omega f_j \partial_i \varphi. \tag{WD}$$

Method I: Here, we use the integration by parts formula for Sobolev function. We can do that on both sets $\Omega_\pm$ since $f_i \in W^{1,1}(\Omega_+ \cup \Omega_-)$ to get (note that $|\Omega \setminus (\Omega_+ \cup \Omega_-)| = 0$)

$$\int_\Omega \partial_i f_j \varphi = \int_{\Omega_+} \partial_i f_j \varphi + \int_{\Omega_-} \partial_i f_j \varphi = \int_{\partial\Omega_+} f_j \varphi \nu_i + \int_{\partial\Omega_-} f_j \varphi \nu_i - \int_{\Omega_+} f_j \partial_i \varphi - \int_{\Omega_-} f_j \partial_i \varphi$$
$$= \int_{\partial\Omega_+} f_j \varphi \nu_i + \int_{\partial\Omega_-} f_j \varphi \nu_i - \int_\Omega f_j \partial_i \varphi.$$

Hence, to get (WD), we need to check that

$$\int_{\partial\Omega_+} f_j \varphi \nu_i + \int_{\partial\Omega_-} f_j \varphi \nu_i = 0.$$

Since φ has compact support in Ω we see that the above integrals reduce only on the set where $x_1 = 0$ and therefore it holds trivially true for $i = 2$. For $i = 1$, we denote $(f_j)_+$ the trace of f_j as a function in Ω_+ and similarly also f_-. Then

$$\int_{\partial\Omega_+} f_j \varphi \nu_1 + \int_{\partial\Omega_-} f_j \varphi \nu_1 = -\int_{-1}^1 ((f_j)_+(0,s) - (f_j)_-(0,s))\varphi(0,s)\, ds$$

Since we require that the above integral vanishes for all φ there must hold that $(f_j)_+(0,s) - (f_j)_-(0,s) = 0$ for all $s \in (0,1)$. Due to the definition we however have

$$(f_j)_+(0,s) - (f_j)_-(0,s) = f(0,s) + (-1)^j f(0,s) = f(0,s)(1 + (-1)^j)$$

Note that for $j = 1$ it is always true, while for $j = 2$ we must assume in addition that the trace of f is equal to zero on the set $\{0\} \times (-1,1)$.

Summary: Even extension preserves Sobolev functions. Odd extension preserves Sobolev functions if and only if we extend over the part of the boundary where the function has zero trace.

Method II: We do not use the trace theorem so directly and try to follow the definition. Define η

$$\eta(t) := \begin{cases} 0 & t \in [-\infty, 1] \\ 1 & t \geq 2 \\ t - 1 & t \in (1,2). \end{cases}$$

Then for arbitrary $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have (note that $\varphi(x_1,x_2)\eta(x_1/\varepsilon)$ has compact support in Ω_+, and δ_{ij} denotes the Kronecker delta)

$$\int_{\Omega_+} \partial_i f_j \varphi \, dx + \int_{\Omega_-} \partial_i f_j \varphi \, dx$$
$$= \lim_{\varepsilon \to 0_+} \int_{\Omega_+} \partial_i f(x_1,x_2)\varphi(x_1,x_2)\eta(x_1/\varepsilon)\, dx + (-1)^{i+1+j} \int_{\Omega_-} \partial_i f(-x_1,x_2)\varphi(x_1,x_2)\eta(-x_1/\varepsilon)\, dx$$
$$= \lim_{\varepsilon \to 0_+} \int_{\Omega_+} \partial_i f(x_1,x_2)\left(\varphi(x_1,x_2) + (-1)^{i+1+j}\varphi(-x_1,x_2)\right)\eta(x_1/\varepsilon)\, dx$$
$$= -\lim_{\varepsilon \to 0_+} \int_{\Omega_+} f(x_1,x_2)\partial_i\left(\left(\varphi(x_1,x_2) + (-1)^{i+1+j}\varphi(-x_1,x_2)\right)\eta(x_1/\varepsilon)\right)\, dx$$
$$= -\lim_{\varepsilon \to 0_+} \int_{\Omega_+} f(x_1,x_2)\left(\partial_i\varphi(x_1,x_2) + (-1)^{1+j}\partial_i\varphi(-x_1,x_2)\right)\eta(x_1/\varepsilon)$$
$$+ f(x_1,x_2)\left(\varphi(x_1,x_2) + (-1)^{i+1+j}\varphi(-x_1,x_2)\right)\varepsilon^{-1}\delta_{1i}\eta'(x_1/\varepsilon)\, dx$$
$$= -\int_\Omega f_j \partial_i \varphi - \delta_{i1} \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \int_{-1}^1 \int_\varepsilon^{2\varepsilon} f(x_1,x_2)\left(\varphi(x_1,x_2) + (-1)^j \varphi(-x_1,x_2)\right)\, dx$$

Hence, for $i = 2$, the second term is zero and there is nothing to solve. For other cases, wee need to show that the second term is equal to zero. For $i = 1$ and $j = 1$ we can use the fact that φ is smooth to estimate

$$\left| \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} f(x_1, x_2) \left(\varphi(x_1, x_2) + (-1)^j \varphi(-x_1, x_2) \right) \, dx \right| \leq 4 \|\nabla \varphi\|_\infty \int_{-1}^{1} \int_{0}^{2\varepsilon} |f(x)| \, dx$$

and the last integral tends to zero as $\varepsilon \to 0_+$. In the second case, i.e., $i = 1$ and $j = 2$, we need to check that

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} f(x_1, x_2) \left(\varphi(x_1, x_2) + \varphi(-x_1, x_2) \right) \, dx = 0$$

By a simple manipulation (and using the computations similar to the one above), we get

$$\begin{aligned}
&\lim_{\varepsilon \to 0_+} \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} f(x_1, x_2) \left(\varphi(x_1, x_2) + \varphi(-x_1, x_2) \right) \, dx \\
&= 2 \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} f(x_1, x_2) \varphi(0, x_2) \, dx \\
&= 2 \lim_{\varepsilon \to 0_+} \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} f(0, x_2) \varphi(0, x_2) \, dx \\
&\qquad + \varepsilon^{-1} \int_{-1}^{1} \int_{\varepsilon}^{2\varepsilon} \int_{0}^{x_1} \partial_1 f(s, x_2) \, ds \varphi(0, x_2) \, dx \\
&= 2 \int_{-1}^{1} f(0, x_2) \varphi(0, x_2) \, dx_2.
\end{aligned}$$

Since the last integral must be equal to zero and φ is arbitrary, we see that in the sense of trace we require $f_2 = 0$ on $\{0\} \times (-1, 1)$.

$W^{2,2}$ **regularity:** In the previous step we identify conditions for $W^{1,1}$ regularity. In order to see that f_j belong also to $W^{2,1}$, it is enough to check whether $\partial_i f_j \in W^{1,1}(\Omega)$. From the previous step, we know that $\partial_i f_j$ is either odd or even extension and therefore we can use the results from previous steps and we see that the additional restriction comes only in the case we consider the odd extension, which is the case of

$$\partial_1 f_1 \quad \text{and} \quad \partial_2 f_2$$

Consequently, for f_1 we need to guarantee that $\partial_1 f = 0$ on $\{0\} \times (-1, 1)$ and for f_2 we need to have $\partial_2 f = 0$ on $\{0\} \times (-1, 1)$. However, the second equality is trivially valid if $f = 0$ on $\{0\} \times (-1, 1)$, which is the assumption for the odd extension.

Summary: The even extension preserves the $W^{2,1}$ regularity provided that the normal derivative is equal to zero on the boundary, the odd extension preserves $W^{2,1}$ regularity provided f is equal to zero on the boundary.

[100] 3. Let $\Omega := B_1(0) \subset \mathbb{R}^2$ and $A \in \mathbb{R}$. Consider the following problem: Find $u : \Omega \to \mathbb{R}$ fulfilling

$$\begin{aligned} -2\Delta u + 2u + A\left(x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2}\right) &= |x|^2 + 1 \qquad \text{in } \Omega, \\ (2x_1 - x_2)\frac{\partial u(x)}{\partial x_1} + (2x_2 + x_1)\frac{\partial u(x)}{\partial x_2} &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{P}$$

20% Define a notion of a weak solution u to (P). Check that it is meaningful.

45% For which A can you prove the existence and the uniqueness of the weak solution?

35% For A's from the previous step, find the smallest interval for which you can prove that $u(x) \in [m, M]$ for almost all $x \in \Omega$.

Solution:

Weak solution: Let us define the matrix

$$\mathbb{A} := \begin{pmatrix} 2 & 1 \\ -1 & 2. \end{pmatrix}$$

Then we have that

$$-\text{div } (\mathbb{A}\nabla u) = -2\Delta u.$$

In addition, since the normal vector $\mathbf{n}$ on $\partial\Omega$ is of the form $\mathbf{n} = (x_1, x_2)$, we also get

$$\mathbb{A}\nabla u \cdot \mathbf{n} = (2\partial_{x_1} u + \partial_{x_2} u, -\partial_{x_1} u + 2\partial_{x_2} u) \cdot (x_1, x_2) = \partial_{x_1} u(2x_1 - x_2) + \partial_{x_2} u(2x_2 + x_1).$$

Consequently, (P) can be equivalently rewritten as

$$\begin{aligned} -\text{div } (\mathbb{A}\nabla u) + 2u + A\left(x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2}\right) &= |x|^2 + 1 \qquad \text{in } \Omega, \\ \mathbb{A}\nabla u \cdot \mathbf{n} &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{P2}$$

Hence, we can follow the lecture and define: We say that $u \in W^{1,2}(\Omega)$ is a weak solution to (P) (and also to (P2)) if for all $\varphi \in W^{1,2}(\Omega)$ there holds

$$\int_\Omega \mathbb{A}\nabla u \cdot \nabla \varphi + \left(2u + A\left(x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2}\right)\right)\varphi = \int_\Omega (|x|^2 + 1)\varphi. \tag{W-F}$$

The fact that the definition is meaningful directly follows from the Hölder inequality, the fact that $\mathbb{A}$ is constant and the fact that $|x|^2$ is bounded function.

Existence and uniqueness: We prove the existence and the uniqueness for any $A \in \mathbb{R}$. On the Hilbert space $H := W^{1,2}(\Omega)$, we define the bilinear form

$$\mathcal{B}(u, \varphi) := \int_\Omega \mathbb{A}\nabla u \cdot \nabla \varphi + \left(2u + A\left(x_2 \frac{\partial u}{\partial x_1} - x_1 \frac{\partial u}{\partial x_2}\right)\right)\varphi$$

and $F \in H^*$ as

$$\langle F, \varphi \rangle := \int_\Omega (|x|^2 + 1)\varphi \quad \text{for all } \varphi \in H.$$

Then we want to apply the Lax–Milgram theorem. For that purpose we need to check the assumption on $\mathcal{B}$. First, it directly gollows from the Hölder inequality that

$$|\mathcal{B}(u,\varphi)| \le C\|u\|_H\|\varphi\|_H,$$

so we have the boundedness. To check also the ellipticity/coercivity we shall evaluate it as follows (we use the integration by parts for Sobolev functions)

$$\begin{aligned}\mathcal{B}(u,u) &= 2\|\nabla u\|_2^2 + 2\|u\|_2^2 + A\int_\Omega \left(x_2\frac{\partial u}{\partial x_1} - x_1\frac{\partial u}{\partial x_2}\right)u = 2\|u\|_H^2 + \frac{A}{2}\int_\Omega x_2\frac{\partial u^2}{\partial x_1} - x_1\frac{\partial u^2}{\partial x_2}\\ &= 2\|u\|_H^2 + \frac{A}{2}\int_\Omega \frac{\partial(x_2u^2)}{\partial x_1} - \frac{\partial(x_1u^2)}{\partial x_2}\\ &= 2\|u\|_H^2 + \frac{A}{2}\int_{\partial\Omega} x_2u^2x_1 - x_1u^2x_2 = 2\|u\|_H^2.\end{aligned}$$

Hence, all assumptions of the Lax–Milgram theorem are satisfied and we have the existence and the uniqueness of a weak solution.

Minimum/maximum principles: Assume that m^+ and m^- are arbitrary fixed numbers and for arbitrary z let us denote $z_+ := \max\{0,z\}$ and $z_- := \min\{0,z\}$. Then we have $\varphi := (u-m^\pm)_\pm \in W^{1,2}(\Omega)$ (since it is Lipschitz function applied on u) and therefore can be used in (W-F). Doing so, we obtain

$$\begin{aligned}\int_\Omega \mathbb{A}\nabla u\cdot\nabla(u-m^\pm)_\pm + \left(2u + A\left(x_2\frac{\partial u}{\partial x_1} - x_1\frac{\partial u}{\partial x_2}\right)\right)(u-m^\pm)_\pm&\\ = \int_\Omega (|x|^2+1)(u-m^\pm)_\pm.&\end{aligned}\tag{C}$$

We evaluate all terms in the above identity. By using the fact that $\nabla(u-m^\pm)_\pm = \nabla u\chi_{\{\pm(u-m^\pm)\ge 0\}}$ we have

$$\begin{aligned}&\int_\Omega \mathbb{A}\nabla u\cdot\nabla(u-m^\pm)_\pm = 2\|\nabla(u-m^\pm)_\pm\|_2^2,\\ &\int_\Omega\left(x_2\frac{\partial u}{\partial x_1} - x_1\frac{\partial u}{\partial x_2}\right)(u-m^\pm)_\pm = \frac{1}{2}\int_\Omega\left(x_2\frac{\partial(u-m^\pm)_\pm^2}{\partial x_1} - x_1\frac{\partial(u-m^\pm)_\pm^2}{\partial x_2}\right)\\ &\qquad = \frac{1}{2}\int_\Omega \frac{\partial(x_2(u-m^\pm)_\pm^2)}{\partial x_1} - \frac{\partial(x_1(u-m^\pm)_\pm^2)}{\partial x_2} = 0,\\ &\int_\Omega u(u-m^\pm)_\pm = \int_\Omega (u-m^\pm+m^\pm)(u-m^\pm)_\pm = \|(u-m^\pm)_\pm\|_2^2 + \int_\Omega 2m^\pm(u-m^\pm)_\pm.\end{aligned}$$

Hence, using this identities in (C), we have

$$2\|(u-m^\pm)_\pm\|_2^2 = \int_\Omega (|x|^2+1-2m^\pm)(u-m^\pm)_\pm.$$

Consequently, if we are able to set the constants $m^\pm$ such that

$$\pm(|x|^2+1-2m^\pm) \le 0 \quad \text{almost everywhere in } \Omega, \tag{N}$$

then

$$2\|(u-m^{\pm})_{\pm}\|_2^2 = \int_\Omega (|x|^2 + 1 - 2m^{\pm})(u-m^{\pm})_{\pm} \le 0$$

and therefore also $(u-m^{\pm})_{\pm} = 0$ and consequently

$$u \le m^+ \text{ and } u \ge m^- \qquad \text{a.e. in } \Omega.$$

Thus, we need to fix $m^{\pm}$ so that (N) is valid. Since $x \in B_1(0)$, we see that the choice $m^+ = 1$ and $m^- = \frac{1}{2}$ leads to

$$\begin{aligned} (|x|^2 + 1 - 2m^+) &= |x|^2 - 1 \le 0, \\ -(|x|^2 + 1 - 2m^-) &= -|x|^2 \le 0. \end{aligned}$$

Hence, we get that $u(x) \in [\frac{1}{2}, 1]$ for almost all $x \in \Omega$.