Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Define the notion of $\mathcal{C}^{k,\alpha}$ set Ω.

Solution:

See lecture.

[100] 2. Assume that $u \in W^{1,2}(0,\pi)$. Decide whether the following implications are correct. Carefully justify!

70%

$$\int_0^\pi u = 0 \implies \int_0^\pi u^2 \le \int_0^\pi (\partial_x u)^2 \qquad \text{(Case 1)}$$

30%

$$u(0) = 0 \implies \int_0^\pi u^2 \le \frac{7}{2}\int_0^\pi (\partial_x u)^2 \qquad \text{(Case 2)}$$

Solution:

Let us start with observation (the Poincaré inequality) that in both cases there holds

$$\lambda \int_0^\pi u^2 \le \int_0^\pi (\partial_x u)^2,$$

where the constant $\lambda > 0$ is independent of u. (Here λ may be different for each case). Hence, we want to find the optimal (largest value) of λ such that the Poincaré inequality holds true.

Thus, clearly, the optimal value of λ_1 is given by

$$\lambda_1 := \inf_{u\in V} \int_0^\pi (\partial_x u)^2$$

where

$$V := \left\{ u \in W^{1,2}(0,\pi), \int_0^\pi u = 0, \int_0^\pi u^2 = 1 \right\}.$$

Then, it is evident that for all $u \in W^{1,2}(0,\pi)$ fulfilling $\int_0^\pi u = 0$ there holds

$$\lambda_1 \int_0^\pi u^2 \le \int_0^\pi (\partial_x u)^2.$$

Let us find λ_1. Denote $u^n \in V$ the minimizing sequence. Then, one can easily deduce that

$$\|u^n\|_{W^{1,2}(0,\pi)} \le C.$$

Hence, due to the reflexivity we have for a subsequence that

$$u^n \rightharpoonup u \text{ weakly in } W^{1,2}(0,\pi)$$

Due to the compact embedding, we also have

$$u^n \to u \text{ strongly in } L^2(0,\pi)$$

Consequently, we have $u \in V$. In addition, it follows from the weak convergence that

$$\partial_x u^n \rightharpoonup \partial_x u \text{ weakly in } L^2(0,\pi).$$

Thus,

$$\lambda_1 = \lim_{n\to\infty} \int_0^\pi (\partial_x u^n)^2 \ge \lim_{n\to\infty} \int_0^\pi 2\partial_x u^n \partial_x u - (\partial_x u)^2 = \int_0^\pi (\partial_x u)^2$$

where for the last equality, we used the fact that $\partial_x u^n$ converges weakly in L^2. Consequently, it follows from the definition that

$$\lambda_1 = \int_0^\pi (\partial_x u)^2$$

and u is the minimizer. Let us derive the Euler–Lagrange equations. Let $v \in W^{1,2}(0,\pi)$ and fulfills $\int_0^\pi v = 0$. Then

$$\frac{u+\varepsilon v}{\|u+\varepsilon v\|_2} \in V$$

and we have

$$\lambda_1 = \int_0^\pi (\partial_x u)^2 \le \int_0^\pi \left(\frac{\partial_x u + \varepsilon \partial_x v}{\|u+\varepsilon v\|_2}\right)^2$$

and the simple algebraic manipulation gives

$$\lambda_1 \int_0^\pi (u+\varepsilon v)^2 \le \int_0^\pi (\partial_x u + \varepsilon \partial_x v)^2$$

and using the fact that $u \in V$ and that $\lambda_1 = \int_0^\pi (\partial_x u)^2$, we deduce

$$\lambda_1 (1 + \varepsilon \int_0^\pi (2uv + \varepsilon v^2)) \le \lambda_1 + \varepsilon \int_0^\pi (2\partial_x u \partial_x v + \varepsilon (\partial_x v)^2)$$

Dividing by 2ε and letting $\varepsilon \to 0_+$ we deduce (since $-v$ is also a possible choice)

$$\lambda_1 \int_0^\pi uv = \int_0^\pi \partial_x u \partial_x v$$

which is nothing else than the weak formulation of the problem

$$-u'' = \lambda_1 u \text{ in } (0,\pi),\ u'(0) = u'(\pi) = 0$$

and we require $\int_0^\pi u = 0$. We look for possible solutions. A priori we know that $\lambda_1 > 0$. Thus, we can solve the equation and have a solution of the form

$$u(x) = A \sin \sqrt{\lambda_1} x + B \cos \sqrt{\lambda_1} x$$

and using the boundary conditions, we see that the only possible solutions are

$$u(x) = B \cos(kx)\ k \in \mathbb{N}.$$

Hence, the possible values are $\lambda_1 = k^2$ and since we look for the smallest, we know that $\lambda_1 = 1$. Thus (Case 1) holds true.

For (Case 2) we could proceed similarly, but we would obtain a different λ_1 and was not able to show the validity of (Case 2). But we could find that the function $u = \sin \frac{x}{2}$ is the first eigen value that should violate (Case 2). Indeed, u satisfies assumptions and we have

$$\int_0^\pi u^2 = 4 \int_0^\pi (\partial_x u)^2.$$

Hence, we found that (Case 2) is not true.

[100] 3. Let $\Omega := B_1(0) \subset \mathbb{R}^d$ with $d \geq 3$. Denote $V := W_0^{1,2}(\Omega)$ and consider the problem: Fix $\delta \in (0,1)$ and find $u \in W^{1,2}(\Omega)$ such that $u = 1$ on $\partial\Omega$ and such that u fulfills for all $\varphi \in V$

$$\int_\Omega \nabla u \cdot \nabla \varphi + u\varphi - \frac{\varphi x \cdot \nabla u}{\delta + |x|^2} = \int_\Omega \frac{x}{(1+|x|)^d} \cdot \nabla \varphi. \tag{P$_\delta$}$$

Here $x := (x_1, \dots, x_d)$.

10% Write down the classical formulation.

30% For any $\delta \in (0,1)$ show that there exists a unique solution u to (P$_\delta$).

20% Show that $u(x) = \tilde{u}(|x|)$ for some function $\tilde{u}$. (Hint: Show that $v(x) := u(Qx)$ is a solution for arbitrary orthogonal matrix Q.)

20% Prove that $-d \leq u \leq 1$ almost everywhere in Ω.

20% Let $\delta \to 0_+$ and show that there exists unique $u \in W^{1,2}(\Omega)$ fulfilling $u = 1$ on $\partial\Omega$ and satisfying for all $\varphi \in V \cap L^\infty(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla \varphi + u\varphi - \frac{\varphi x \cdot \nabla u}{|x|^2} = \int_\Omega \frac{x}{(1+|x|)^d} \cdot \nabla \varphi. \tag{P}$$

Solution:

Classical formulation: We assume that u is sufficiently smooth and integrate by parts in (P$_\delta$). The boundary terms vanish since φ has zero trace and we get

$$\int_\Omega \left(-\Delta u + u - \frac{x \cdot \nabla u}{\delta + |x|^2} \right) \varphi = \int_\Omega - \operatorname{div} \left(\frac{x}{(1+|x|)^d} \right) \varphi.$$

Since $\varphi \in W_0^{1,2}(\Omega)$ is arbitrary, we see that the classical formulation is the following:

$$\begin{aligned} -\Delta u + u - \frac{x \cdot \nabla u}{\delta + |x|^2} &= - \operatorname{div} \left(\frac{x}{(1+|x|)^d} \right) && \text{in } \Omega, \\ u &= 1 && \text{on } \partial\Omega. \end{aligned}$$

Existence and uniqueness: First, we define the bilinear form on $W^{1,2}(\Omega)$

$$B(u, \varphi) := \int_\Omega \nabla u \cdot \nabla \varphi + u\varphi - \frac{\varphi x \cdot \nabla u}{\delta + |x|^2}$$

and $F \in V^*$ as

$$\langle F, \varphi \rangle := \int_\Omega \frac{x}{(1+|x|)^d} \cdot \nabla \varphi.$$

Since $\delta > 0$ all terms are well defined (use Hölder inequality). Hence, we will look for u of the form $u = 1 + v$, where $v \in V$. Consequently, we v must solve the problem

$$B(v, \varphi) = \langle F, \varphi \rangle - B(1, \varphi) =: \langle \tilde{F}, \varphi \rangle.$$

To solve it, we use the Lax–Milgram theorem. So we need to check that $B(u,v)$ is V-bounded, but it directly follows from the Hölder inequality and that it is V-elliptic, which follows from integration by parts and the following computation

$$\begin{aligned} B(v,v) &= \|\nabla v\|_2^2 + \|v\|_2^2 - \int_\Omega \frac{vx\cdot\nabla v}{\delta+|x|^2} = \|v\|_{1,2}^2 + \frac{1}{2}\int_\Omega v^2 \operatorname{div}\left(\frac{x}{\delta+|x|^2}\right) \\ &= \|v\|_{1,2}^2 + \frac{1}{2}\int_\Omega \frac{(d\delta + (d-2)|x|^2)v^2}{(\delta+|x|^2)^2} \geq \|v\|_{1,2}^2. \end{aligned}$$

Radial symmetry: To prove that u depends only on $|x|$, we proceed as follows. Let $Q\in\mathbb{R}^{d\times d}$ be arbitrary orthogonal matrix, i.e., the matrix representing a rotation in $\mathbb{R}^d$ around an axis through the origin ($QQ^T = I$). If we show that $v(x) := u(Qx)$ is a weak solution, then thanks to uniqueness we have that $u(x) = u(Qx)$ for arbitrary orthogonal matrix and thus u depends just on $|x|$. Hence, let v be given as above and for arbitrary $\varphi\in V$ define $\psi(x) := \varphi(Q^{-1}x) = \psi(Q^Tx)$. Then $\psi\in V$ as well. Note also that $Q(\Omega) = Q^{-1}(\Omega) = \Omega$. Also note that $v=1$ on $\partial\Omega$. Finally, using the chain rule and substitution theorem and the fact that $QQ^T = I$, we have

$$\begin{aligned} &\int_\Omega \nabla v\cdot\nabla\varphi + v\varphi - \frac{\varphi x\cdot\nabla v}{\delta+|x|^2} = \int_\Omega \sum_{i=1}^d \frac{\partial u(Qx)}{\partial x_i}\frac{\partial\psi(Qx)}{\partial x_i} + u(Qx)\psi(Qx) - \frac{\psi(Qx)}{\delta+|x|^2}\sum_{i=1}^d x_i \frac{\partial u(Qx)}{\partial x_i}\,dx \\ &= \int_\Omega \sum_{i,j,k=1}^d \frac{\partial u(Qx)}{\partial y_j} Q_{ji}Q_{ki}\frac{\partial\psi(Qx)}{\partial y_k} + u(Qx)\psi(Qx) - \frac{\psi(Qx)}{\delta+|x|^2}\sum_{i,j=1}^d x_i Q_{ji}\frac{\partial u(Qx)}{\partial y_j}\,dx \\ &= \int_\Omega \nabla u\cdot\nabla\psi + u\psi - \frac{\varphi x\cdot\nabla\psi}{\delta+|x|^2} \end{aligned}$$

Similarly, one can show that

$$\int_\Omega \frac{x}{(1+|x|)^d}\cdot\nabla\varphi = \int_\Omega \frac{x}{(1+|x|)^d}\cdot\nabla\psi$$

Hence, we see that v is a solution and the claim follows.

Minimum/maximum principle: Let us consider $\varphi := (u-1)_+$ in (P_δ). Clearly, $\varphi\in V$ and also $\varphi\geq 0$. So we have (using integration by parts and nonnegativity of φ)

$$\begin{aligned} \|\nabla(u-1)_+\|_2^2 &= \int_\Omega \nabla u\cdot\nabla\varphi = \int_\Omega -u\varphi + \frac{\varphi x\cdot\nabla u}{\delta+|x|^2} + \frac{x}{(1+|x|)^d}\cdot\nabla\varphi \\ &\leq \int_\Omega -\varphi^2 + \frac{x\cdot\nabla\varphi^2}{2(\delta+|x|^2)} + \frac{x}{(1+|x|)^d}\cdot\nabla\varphi \\ &\leq -\int_\Omega \varphi^2\operatorname{div}\left(\frac{x}{2(\delta+|x|^2)}\right) + \varphi\operatorname{div}\left(\frac{x}{(1+|x|)^d}\right) \leq 0 \end{aligned}$$

Where we used the fact that

$$\operatorname{div}\left(\frac{x}{(1+|x|)^d}\right) = \frac{d}{(1+|x|)^{d+1}}. \tag{1}$$

Similarly, we set $\varphi := (u+d)_-$. Again, we have $\varphi \in V$ but now $\varphi \leq 0$. Repeating the procedure, we deduce

$$\begin{aligned}\|\nabla(u+d)_-\|_2^2 = \int_\Omega \nabla u \cdot \nabla \varphi &= \int_\Omega -u\varphi + \frac{\varphi x \cdot \nabla u}{\delta + |x|^2} + \frac{x}{(1+|x|)^d} \cdot \nabla \varphi \\ &\leq \int_\Omega -\varphi^2 + d\varphi + \frac{x \cdot \nabla \varphi^2}{2(\delta + |x|^2)} + \frac{x}{(1+|x|)^d} \cdot \nabla \varphi \\ &\leq -\int_\Omega \varphi^2 \operatorname{div}\left(\frac{x}{2(\delta+|x|^2)}\right) + \varphi\left(\operatorname{div}\left(\frac{x}{(1+|x|)^d}\right) - d\right) \leq 0\end{aligned}$$

where we used (1) and the fact that for all $x \in B_1(0)$ we have

$$\frac{d}{(1+|x|)^{d+1}} - d \leq 0.$$

Existence and uniqueness to (P)**:** Denoting u_δ a solution to (P_δ) and repeating the a priori estimates we get that for all $\delta \in (0,1)$

$$\|u_\delta\|_{1,2} + \|u_\delta\|_\infty \leq C$$

Consequently, there exists a subsequence (not relabeled) such that

$$u_\delta \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega)$$

and in addition we have $u = 1$ on $\partial\Omega$ and $\|u\|_\infty \leq C$. It just remains to pass to the limit $\delta \to 0_+$ in (P_δ). Hence let $\varphi \in V \cap L^\infty(\Omega)$ be arbitrary. Then it just follows from the weak convergence that

$$\lim_{\delta \to 0_+} \int_\Omega \nabla u_\delta \cdot \nabla \varphi + u_\delta \varphi = \int_\Omega \nabla u \cdot \nabla \varphi + u\varphi.$$

Finally, for the last term on the left hand side of (P_δ) we have

$$\lim_{\delta \to 0_+} \int_\Omega \frac{\varphi x \cdot \nabla u_\delta}{\delta + |x|^2} = \lim_{\delta \to 0_+} \int_\Omega \frac{\varphi x \cdot \nabla u_\delta}{|x|^2} + \lim_{\delta \to 0_+} \int_\Omega \varphi \nabla u_\delta \cdot \left(\frac{x}{\delta + |x|^2} - \frac{x}{|x|^2}\right)$$

For the first limit, we can use the weak convergence and for the second limit, we can use the Hölxer inequality and the fact that

$$\int_\Omega \left|\frac{x}{\delta + |x|^2} - \frac{x}{|x|^2}\right|^2 \to 0 \text{ as } \delta \to 0_+,$$

which holds true for $d \geq 3$ and can be justified by the Lebesgue dominated convergence theorem.

Concerning the uniqueness, assume that u_1 and u_2 solve (P) and denote $v := u_1 - u_2$. Then $v \in V$ and for all $\varphi \in V \cap L^\infty(\Omega)$ there holds

$$\int_\Omega \nabla v \cdot \nabla \varphi + v\varphi - \frac{\varphi x \cdot \nabla v}{|x|^2} = 0.$$

We need to test by bounded functions and therefore we set

$$\varphi := T_k(v) = \frac{v}{|v|}\min\{k, |v|\}.$$

Doing so and repeating the procedure from a priori estimates and using integration by parts, we observe that

$$\|T_k(v)\|_{1,2} = 0.$$

Since $k \in \mathbb{R}_+$ can be arbitrary, it follows that $v \equiv 0$ and so we have the uniqueness of a solution.