Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Given name and family name: ______________________________

Question	1	2	3	**Score**
Maximum points	100	100	100	300
Points				

[100] 1. Define the notion of weak derivative and the space $W^{1,p}(\Omega)$.

Solution:

See lecture.

[100] 2. Formulate and prove the Lax-Milgram theorem.

Solution:

See the lecture.

[100] 3. Let $\Omega = B_1(0) \subset \mathbb{R}^2$ be Lipschitz and $T > 0$. Consider the problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x_1^2} + 6\frac{\partial u}{\partial x_1 \partial x_2} - 5\frac{\partial^2 u}{\partial x_2^2} &= f && \text{in } (0,T)\times\Omega,\\ u(t,x) &= 1 && \text{on } (0,T)\times\partial\Omega,\\ u(0,x) &= u_0 && \text{in } \Omega\\ \frac{\partial u(0,x)}{\partial t} &= v_0 && \text{in } \Omega. \end{aligned}$$

10% Find the optimal (minimal) assumptions on f, u_0 and v_0 and define a notion of a weak solution. Check that for given data such a definition is meaningful.

35% Prove that for any such f, u_0 and v_0 there exists unique weak solution. (If you use some theorem, check all assumptions!!!)

20% Under which assumptions on f, u_0 and v_0 you can show that the unique weak solution satisfy $u \in W^{2,\infty}(0,T;L^2(\Omega)) \cap W^{1,\infty}(0,T;W^{1,2}_0(\Omega))$. Prove it.

35% Let $f = 0$. Decide for which sets $\Omega_0 \subset \mathbb{R}^2$ and $Q_0 \subset (0,\infty)\times\mathbb{R}^2$ the following implication holds true:

$$u_0 = v_0 = 0 \text{ in } \Omega_0 \qquad \Longrightarrow \qquad u = 0 \text{ in } Q_0.$$

Prove it[1].

Solution:

Weak solution: First, we deal with the Dirichlet problem (although nonhomogeneous), therefore all test functions will have the zero trace! So, we denote $V := W^{1,2}_0(\Omega)$ and the corresponding Gelfand triple is $V \hookrightarrow H (= L^2(\Omega)) = H^* \hookrightarrow V^*$. Motivated by the lecture and by the prescribed trace on the boundary, we seek for the solution in the following form:
For given $u_0 \in W^{1,2}(\Omega)$ fulfilling $(u_0 - 1) \in V$, given $v_0 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$ to find u satisfying $(u-1) \in L^2(0,T;V) \cap W^{1,\infty}(0,T;H) \cap W^{2,2}(0,T;V^*)$ and fulfilling for all $\varphi \in V$ and almost all $t \in (0,T)$

$$\langle \partial_{tt} u, \varphi\rangle_V + \int_\Omega \mathbb{A}\nabla u \cdot \nabla\varphi = \int_\Omega f\varphi, \tag{*}$$

where the matrix $\mathbb{A}$ is given as

$$\mathbb{A} := \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

Due to the assumptions on u, we also have that $u \in \mathcal{C}([0,T];L^2(\Omega))$ and $\partial_t u \in \mathcal{C}([0,T];V^*)$. Therefore we can talk about values of u and $\partial_t u$ and any time and we require

$$\begin{aligned} \lim_{t\to 0_+} u(t) &= u_0 && \text{in } L^2(\Omega),\\ \lim_{t\to 0_+} \partial_t u(t) &= v_0 && \text{in } V^*. \end{aligned}$$

[1] Hint:

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

Note that in principle we could replace $\mathbb{A}$ by

$$\mathbb{A}_a \begin{pmatrix} 2 & -3+a \\ -3-a & 5 \end{pmatrix}$$

for arbitrary a. But our choice $a = 0$ is motivated by the fact that it is easier to deal with the symmetric matrix. Also notice that $\mathbb{A}$ is elliptic. Indeed for any $\xi \in \mathbb{R}^2$ we have

$$\mathbb{A}\xi \cdot \xi = 2\xi_1^2 + 5\xi_2^2 - 6\xi_1\xi_2 \geq \varepsilon|\xi|^2$$

for some $\varepsilon > 0$.

Uniqueness: Assume that u_1 and u_2 are two weak solutions. Denote $w := u_1 - u_2$ and we see that

$$\langle \partial_{tt} w, \varphi \rangle_V + \int_\Omega \mathbb{A}\nabla w \cdot \nabla\varphi = 0,$$

for almost all $t \in (0,T)$ and all $\varphi \in V$. Since $w(0) = \partial_t w(0) = 0$, we can follow the lecture and prove the uniqueness.

Existence: Again, we can denote $U := u - 1$. Then to show the existence of a weak solution, it is equivalent to show the existence of $U \in L^2(0,T;V) \cap W^{1,\infty}(0,T;H) \cap W^{2,2}(0,T;V^*)$ solving

$$\langle \partial_{tt} U, \varphi \rangle_V + \int_\Omega \mathbb{A}\nabla U \cdot \nabla\varphi = \int_\Omega f\varphi,$$

for almost all $t \in (0,T)$ and all $\varphi \in V$. We also require that $U(0) = u_0 - 1 \in V$ and $\partial_t U(0) = v_0$. Now we are exactly at the statement proven at the lecture.

Regularity: We prove the uniform result only for the Galerkin approximation and thanks to the uniqueness it transfers also tothe weak solution. Hence let $\{w_j\}_{j\in\mathbb{N}}$ be a basis of V orthogonal in H and orthonormal in V For which the projection P^n is continuous in H and also in V. We have U^n solving

$$\int_\Omega \partial_{tt} U^n \varphi + \int_\Omega \mathbb{A}\nabla U \cdot \nabla\varphi = \int_\Omega f\varphi, \tag{*n}$$

for all φ in the linear hull of $\{w_j\}_{j=1}^n$ and almost all $t \in (0,T)$. The initial conditions are given as $U^n(0) = P^n(u_0 - 1)$ and $\partial_t U(0) = P^n v_0$ and $U^n(t,x) = \sum_{j=1}^n a_j^n(t) w_j(x)$. Next, we apply the time derivative to (*n) and in the result we set $\varphi := \partial_{tt} U^n$ to get

$$\int_\Omega \partial_{ttt} U^n \partial_{tt} U^n + \int_\Omega \mathbb{A}\nabla\partial_t U \cdot \nabla\partial_{tt} U^n = \int_\Omega \partial_t f \partial_{tt} U^n.$$

Since $\mathbb{A}$ is symmetric and positively definite, we deduce

$$\frac{d}{dt}\underbrace{\left(\int_\Omega |\partial_{tt} U^n|^2 + \mathbb{A}\nabla\partial_t U \cdot \nabla\partial_t U^n\right)}_{=LHS\geq 0} = 2\int_\Omega \partial_t f \partial_{tt} U^n \leq C\|\partial_t f\|_2(1 + LHS).$$

Hence, we may use the Gronwall lemma to get the estimate

$$\begin{aligned} \sup_{t\in(0,T)} (\|\partial_{tt} U^n(t)\|_2^2 + \|\partial_t U^n\|_V^2) &\leq C \sup_{t\in(0,T)} \left(\int_\Omega |\partial_{tt} U^n|^2 + \mathbb{A}\nabla\partial_t U \cdot \nabla\partial_t U^n\right) \\ &\leq Ce^{C\int_0^T \|\partial_t f\|_2} \left(1 + \int_\Omega |\partial_{tt} U^n(0)|^2 + \mathbb{A}\nabla\partial_t U(0) \cdot \nabla\partial_t U^n(0)\right) \end{aligned}$$

So to control the right hand side, we see that we must assume

$$f \in W^{1,1}(0,T;L^2(\Omega)). \tag{1}$$

In addition to control the second term in the integral, we have

$$\int_\Omega \mathbb{A}\nabla\partial_t U(0)\cdot\nabla\partial_t U^n(0) \le C\|\nabla\partial_t U(0)\|_2^2 = C\|\nabla P^n(v_0)\|_2^2 \le C\|v_0\|_V^2.$$

Thus, to control it we need,

$$v_0 \in W_0^{1,2}(\Omega). \tag{2}$$

Finally, to control the first term in the above integral, we set $\varphi := \partial_{tt}U(0)$ in (*n) and consider $t=0$ to get

$$\begin{aligned}\int_\Omega |\partial_{tt}U^n(0)|^2 &= \int_\Omega f(0)\partial_{tt}U^n(0) - \mathbb{A}\nabla U^n(0)\cdot\nabla\partial_{tt}U^n(0)\\ &= \int_\Omega f(0)\partial_{tt}U^n(0) + \operatorname{div}(\mathbb{A}\nabla U^n(0))\partial_{tt}U^n(0)\\ &\le C\|\partial_{tt}U^n(0)\|_2(\|f(0)\|_2 + \|\operatorname{div}(\mathbb{A}\nabla U^n(0))\|_2)\\ &\le C\|\partial_{tt}U^n(0)\|_2(\|f\|_{W^{1,1}(0,T;L^2(\Omega))} + \|\operatorname{div}(\mathbb{A}\nabla P^n(u_0-1))\|_2)\end{aligned}$$

Hence to control the right hand side we must assume in addition to (2) also (Note that we are using the continuity of P^n, which is based on the proper choice of the basis.)

$$(u_0 - 1) \in W_0^{1,2}(\Omega)\cap W^{2,2}(\Omega). \tag{3}$$

Propagation: We use the hint and define

$$\mathbb{B} := \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and note that $\mathbb{B}^2 = \mathbb{A}$. We define $\tilde\Omega := \mathbb{B}(x-x_0)$. Then we define $\tilde u(t,x) := u(t,\mathbb{B}(x-x_0))$. Then using the substitution $y := \mathbb{B}(x-x_0)$ and the fact that $\det\mathbb{B} = 1$ and for arbitrary $\tilde\varphi \in W_0^{1,2}(\tilde\Omega)$ we define $\varphi(x) := \tilde\varphi(\mathbb{B}^{-1}x + x_0)$. Then we deduce by using the symmetry of $\mathbb{B}$ and the fact that $\mathbb{B}^2 = \mathbb{A}$ that

$$\begin{aligned}&\langle\partial_{tt}\tilde u, \tilde\varphi\rangle_{W_0^{1,2}(\tilde\Omega)} + \int_{\tilde\Omega}\nabla\tilde u\cdot\nabla\tilde\varphi\\ &= \langle\partial_{tt}u,\varphi\rangle_{W_0^{1,2}(\Omega)} + \int_{\tilde\Omega}\sum_{i=1}^2 \frac{\partial u(t,\mathbb{B}(x-x_0))}{\partial x_i}\frac{\partial\varphi(\mathbb{B}(x-x_0))}{\partial x_i}\\ &= \langle\partial_{tt}u,\varphi\rangle_{W_0^{1,2}(\Omega)} + \int_{\tilde\Omega}\sum_{i,j,k=1}^2 \frac{\partial u(t,\mathbb{B}(x-x_0))}{\partial y_j}\mathbb{B}_{ji}\frac{\partial\varphi(\mathbb{B}(x-x_0))}{\partial y_k}\mathbb{B}_{ki}\\ &= \langle\partial_{tt}u,\varphi\rangle_{W_0^{1,2}(\Omega)} + \int_{\Omega}\sum_{j,k=1}^2 \mathbb{A}_{jk}\frac{\partial u(t,x)}{\partial x_j}\frac{\partial\varphi(x)}{\partial x_k} = 0\end{aligned}$$

Consequently, we can use the lecture and to conclude that if $\tilde u_0 = \tilde v_0 = 0$ on the ball $B_r(0)$ then the solution $\tilde u$ is zero on the set $\tilde Q := \{(t,x); |x| + t \le r\}$. Finally, we transfer this information to the solution u (follow your homeworks:)