Each step must be **carefully** justified. If you use some lemma or theorem do **not** forget to check that all assumptions are satisfied.

Name: ______________________________

Question	1	2	3	**Score**
Maximum points	0	100	100	200
Points				

[0] 1. Define the notion of $\mathcal{C}^{k,\alpha}$ set Ω.

Solution:

See lecture.

[100] 2. Formulate and prove the integration by parts formula in Bochner spaces, i.e., finish the following line

$$\int_0^T \langle \partial_t u, v \rangle \, dt = \dots,$$

specify assumption on V and prove it. Check that everything is well defined!

Solution:

The required theorem is following: Let V, H, V^* be the Gelfand triplet and $p \in (1, \infty)$. Assume that $u, v \in L^p(0, T; V)$ and $\partial_t u, \partial_t v \in L^{p'}(0, T; V^*)$. Then $u, v \in \mathcal{C}([0, T]; H)$ and

$$\int_0^T \langle \partial_t u, v \rangle_V \, dt = -\int_0^T \langle \partial_t v, u \rangle_V \, dt + (u(T), v(T))_H - (u(0), v(0))_H.$$

Proof: see lecture.

[100] 3. Let $\Omega := B_4(1,0,\ldots,0) \subset \mathbb{R}^d$. Assume that $g \in \mathcal{C}_0(B_1(0);\mathbb{R}^d)$, $f \in L^2(\Omega)$ and consider the problem

$$\begin{aligned} -\Delta u(x) + \sum_{i=1}^d g_i(x)\frac{u(x+he_i)-u(x)}{h} &= f(x) \quad \text{in } \Omega, \\ u &= \frac{1}{|x|^d} \quad \text{on } \partial\Omega. \end{aligned} \tag{P$_h$}$$

20% Consider $h \in (0,1)$ and define the notion of weak solution.

40% Show that there exists $\varepsilon > 0$ such that if $\|g\|_\infty \le \varepsilon$ then the above problem has a unique weak solution for every $h \in (0,1)$

40% Consider a sequence of weak solutions u_h to a problem (P$_h$). Find an elliptic problem (P) with unique solution u and show that $u_h \to u$ in a proper topology.

Solution:

Weak solution: First of all we define $V := W_0^{1,2}(\Omega)$. Next, we set $u_0(x) := \frac{\tau(x)}{|x|^d}$, where τ is a smooth function fulfilling $\tau \equiv 0$ in $B_1(0)$, $\tau \equiv 1$ in $\mathbb{R}^d \setminus B_2(0)$. Then evidently u_0 is a smooth function and $u_0(x) = |x|^{-d}$ for all $x \in \partial\Omega$. We say that u is a weak solution to (P$_h$) if $(u - u_0) \in V$ and for all $\varphi \in V$ there holds

$$\int_\Omega \nabla u(x)\cdot\nabla\varphi(x) + \sum_{i=1}^d g_i(x)\varphi(x)\frac{u(x+he_i)-u(x)}{h} = \int_\Omega f(x)\varphi(x). \tag{1}$$

Since g is compactly supported in B_1, all integrals are well defined. Moreover, since $h > 0$ it follows directly from the Hölder inequality that all integrals are finite.

Existence and Uniqueness: We define the bilinear form

$$B(u,\varphi) := \int_\Omega \nabla u(x)\cdot\nabla\varphi(x) + \sum_{i=1}^d g_i(x)\varphi(x)\frac{u(x+he_i)-u(x)}{h}$$

Then to find a weak solution to (P$_h$) is equivalent to finding $v \in V$ such that

$$B(v,\varphi) = \int_\Omega f(x)\varphi(x) - B(u_0,\varphi). \tag{2}$$

Next, we define $F \in V^*$ by formula

$$\langle F,\varphi\rangle := \int_\Omega f(x)\varphi(x) - B(u_0,\varphi)$$

Moreover, we show that the norm of F can be estimated uniformly with respect to h. Indeed,

$$\begin{aligned} \|F\|_{V^*} := \sup_{\|\varphi\|\le 1}\int_\Omega f(x)\varphi(x) - B(u_0,\varphi) &\overset{\text{Hölder}}{\le} \sup_{\|\varphi\|\le 1} (\|f\|_2\|u\|_2 + \|u_0\|_{\mathcal{C}^\infty}\|g\|_2\|\varphi\|_2) \\ &\le \|f\|_2 + \|u_0\|_{\mathcal{C}^\infty}\|g\|_2. \end{aligned} \tag{3}$$

Finally, we show that the form B is V-elliptic and V-bounded, which will lead to the existence and uniqueness of a weak solution thanks to the Lax-Milgram theorem. To do so, we first estimate the term with g. We use the Hölder inequality to get (recall that g is supported in B_1)

$$\left|\int_\Omega \sum_{i=1}^d g_i(x)\varphi(x)\frac{u(x+he_i)-u(x)}{h}\right| \leq \|g\|_\infty\|\varphi\|_2\sum_{i=1}^d\left(\int_{B_1(0)}\frac{|u(x+he_i)-u(x)|^2}{h^2}\right)^{\frac{1}{2}}$$

Using the following inequality (stated at the lecture - characterization of Sobolev functions)

$$\int_{B_1(0)}\frac{|u(x+he_i)-u(x)|^2}{h^2} \leq \|\nabla u\|^2_{L^2(B_2(0))},$$

we deduce

$$\left|\int_\Omega \sum_{i=1}^d g_i(x)\varphi(x)\frac{u(x+he_i)-u(x)}{h}\right| \leq d\|g\|_\infty\|\varphi\|_2\|\nabla u\|_2. \tag{4}$$

Thus, using (4) and the Hölder inequality, we have

$$|B(v,\varphi)| \leq \|\nabla v\|_2\|\nabla\varphi\|_2 + d\|g\|_\infty\|\varphi\|_2\|\nabla v\|_2 \leq C\|v\|_V\|\varphi\|_V$$

with C being independent of h. Thus, B is V-bounded. Concerning ellipticity, we first recall the Poincaré inequality valid for all $v \in V$

$$\|\nabla v\|_2^2 \geq c_1\|v\|_V^2 \tag{5}$$

and set $\varepsilon := \frac{c_1}{2d}$. Then assuming that $\|g\|_\infty \leq \varepsilon$, we deduce by using (4) and (5) that

$$B(v,v) \geq \|\nabla v\|_2^2 - d\|g\|_\infty\|v\|_2\|\nabla v\|_2 \geq \|v\|_V^2(c_1 - d\|g\|_\infty) \geq \frac{c_1}{2}\|v\|_V^2$$

and so B is V-elliptic.

Convergence $h \to 0_+$**:** We denote u_h the solution of (P_h), which exists thanks to to the previous step. Moreover, it also follows from the previous step that there exists a constant C independent of h such that

$$\|u_h\|_{1,2} \leq C \qquad \|u_h - u_0\|_V \leq C. \tag{6}$$

Since V is reflexive, it follows from (6) that we can extract a subsequence u_n (corresponding to h_n and $n := h_n^{-1}$) and we can find u such that

$$u_n \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega), \qquad u_n - u_0 \rightharpoonup u - u_0 \text{ weakly in } V$$

In addition, thanks to the compact embedding the above sequence can be chosen such that

$$\|u_n - u\|_2 \to 0 \text{ as } n \to \infty \tag{7}$$

Let us now consider the weak formulation (1) for h_n with $\varphi \in \mathcal{C}_0^\infty(\Omega)$

$$\int_\Omega \nabla u_n(x)\cdot\nabla\varphi(x) + \sum_{i=1}^d g_i(x)\varphi(x)\frac{u_n(x+h_ne_i)-u_n(x)}{h_n} = \int_\Omega f(x)\varphi(x). \tag{8}$$

Next, we pass to the limit in both terms on the left hand side. In the first one, we use the fact that $\varphi \in \mathcal{C}_0^\infty$ and then we can use the definition of weak derivative to get

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \nabla u_n\cdot\nabla\varphi &= -\lim_{n\to\infty}\int_\Omega u_n\Delta\varphi = -\int_\Omega u\Delta\varphi - \lim_{n\to\infty}\int_\Omega (u_n-u)\Delta\varphi\\ &= \int_\Omega \nabla u\cdot\nabla\varphi - \lim_{n\to\infty}\int_\Omega (u_n-u)\Delta\varphi,\end{aligned}$$

where for the last identity we used the definition of weak derivative. Finally, using (7) and the Hölder inequality, we have

$$\lim_{n\to\infty}\left|\int_\Omega (u_n-u)\Delta\varphi\right| \le \lim_{n\to\infty}\|u_n-u\|_2\|\varphi\|_{2,2} = 0$$

and therefore, we can conclude

$$\lim_{n\to\infty}\int_\Omega \nabla u_n\cdot\nabla\varphi = \int_\Omega \nabla u\cdot\nabla\varphi. \tag{9}$$

To identify also the second limit on the left hand side of (8), we first deduce that for arbitrary $q\in\mathcal{C}_0^1(B_1(0))$

$$\begin{aligned}&\lim_{n\to\infty}\int_\Omega q(x)\frac{u_n(x+h_ne_i)-u_n(x)}{h_n} = \lim_{n\to\infty}\int_\Omega \frac{q(x-h_ne^i)-q(x)}{h_n}u_n(x)\\ &= \lim_{n\to\infty}\int_\Omega \frac{q(x-h_ne^i)-q(x)}{h_n}u(x) + \lim_{n\to\infty}\int_\Omega \frac{q(x-h_ne^i)-q(x)}{h_n}(u_n(x)-u(x)).\end{aligned}$$

Since

$$\frac{q(x-h_ne^i)-q(x)}{h_n}\to -\frac{\partial q(x)}{\partial x_i} \text{ uniformly in } \mathcal{C}(\overline{\Omega})$$

we can use the Lebesgue dominated convergence theorem and (7) to conclude that

$$\lim_{n\to\infty}\int_\Omega q(x)\frac{u_n(x+h_ne_i)-u_n(x)}{h_n} = -\int_\Omega \frac{\partial q}{\partial x_i}u = \int_\Omega q\frac{\partial u}{\partial x_i}, \tag{10}$$

where for thesecond equality we used again the definition of weak derivative.
Finally, since $g_i\in\mathcal{C}_0(B_1(0))$, we can for all $\delta>0$ find $g_i^\delta\in\mathcal{C}_0^1(B_1(0))$ such that

$$\|g_i-g_i^\delta\|\le\delta. \tag{11}$$

Then we also have that $g_i^\delta\varphi\in\mathcal{C}_0^1(B_1(0))$ and we may compute

$$\begin{aligned}&\lim_{n\to\infty}\left|\int_\Omega g_i(x)\varphi(x)\left(\frac{u_n(x+h_ne_i)-u_n(x)}{h_n}-\frac{\partial u(x)}{\partial x_i}\right)\right|\\ &\le \limsup_{n\to\infty}\left|\int_\Omega (g_i(x)-g_i^\delta)\varphi(x)\left(\frac{u_n(x+h_ne_i)-u_n(x)}{h_n}-\frac{\partial u(x)}{\partial x_i}\right)\right|\\ &\quad+\lim_{n\to\infty}\left|\int_\Omega g_i^\delta(x)\varphi(x)\left(\frac{u_n(x+h_ne_i)-u_n(x)}{h_n}-\frac{\partial u(x)}{\partial x_i}\right)\right|\\ &\overset{(10)}{=} \limsup_{n\to\infty}\left|\int_\Omega (g_i(x)-g_i^\delta)\varphi(x)\left(\frac{u_n(x+h_ne_i)-u_n(x)}{h_n}-\frac{\partial u(x)}{\partial x_i}\right)\right|\\ &\le \limsup_{n\to\infty}\|g_i-g_i^\delta\|_\infty\|\varphi\|_2(\|\nabla u_n\|_2+\|\nabla u\|_2) \overset{(11)}{\le} C\delta\end{aligned}$$

Since δ is arbitrary we have

$$\lim_{n\to\infty}\left|\int_\Omega g_i(x)\varphi(x)\left(\frac{u_n(x+h_ne_i)-u_n(x)}{h_n}-\frac{\partial u(x)}{\partial x_i}\right)\right|=0$$

Therefore, we can let $n\to\infty$ in (8) to conclude that for all $\varphi\in\mathcal{C}_0^\infty(\Omega)$ there holds

$$\int_\Omega \nabla u\cdot\nabla\varphi+\sum_{i=1}^d g_i\varphi\frac{\partial u}{\partial x_i}=\int_\Omega f(x)\varphi(x). \tag{12}$$

Since $\mathcal{C}_0^\infty(\Omega)$ is dense in V we see that (12) is valid also for all $\varphi\in V$. Therefore u is a weak solution to

$$\begin{aligned} -\Delta u+g\cdot\nabla u&=f && \text{in } \Omega,\\ u&=\frac{1}{|x|^d} && \text{on } \partial\Omega. \end{aligned} \tag{P}$$

Moreover, using the same procedure as above we see that the solution u to (P) is unique. Therefore even the whole sequence u_h must converge to u.