1 Multivariate Extreme Value Distribution

Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with joint d.f. $F$ and marginal d.f. $F_1, \ldots, F_d$. $X_i = (X_{i,1}, \ldots, X_{i,d})'$ can be interpreted e.g. as losses of $d$ different types observed in periods $i = 1, \ldots, n$. We are interested in the componentwise block maxima

$$M_{n,j} = \max(X_{1,j}, \ldots, X_{n,j}), \; j = 1, \ldots, d.$$ 

We shall deal with the behavior of the vector

$$M_n = (M_{n,1}, \ldots, M_{n,d})',$$

namely, with the limit behavior of $M_n$ when $n \to \infty$, for an appropriate normalization

$$\frac{M_n - d_n}{c_n} = \left( \frac{M_{n,1} - d_{n,1}}{c_{n,1}}, \ldots, \frac{M_{n,d} - d_{n,d}}{c_{n,d}} \right)'.$$
where $c_n$ and $d_n$ are vectors of constants, satisfying $c_n > 0$ (componentwise).

From the independence of $X_1, \ldots, X_n$ it follows

$$P(M_n \leq x) = P(M_{n,1} \leq x_1, \ldots, M_{n,d} \leq x_d) = F^n(x_1, \ldots, x_d) = F^n(x).$$

If for some choice of vectors $c_n, d_n, \frac{M_n - d_n}{c_n}$ converge in distribution to a random vector with joint d.f. $G$, we have

$$\lim_{n \to \infty} P\left(\frac{M_n - d_n}{c_n} \leq x\right) = \lim_{n \to \infty} F^n(c_n x + d_n) = G(x). \quad (1)$$

We then say that $F$ belongs to the maximum domain of attraction of d.f. $G$ ($F \in \text{MDA}(G)$) and we refer to the distribution with d.f. $G$ as a multivariate extreme value distribution (MEVD).

If the d.f. $G$ has non-degenerate marginals $G_1, \ldots, G_d$, those marginals are, according to the Fisher-Tippet theorem, d.f. of (univariate) extreme value distributions - i.e. continuous distributions of Fréchet, Gumbel or Weibull type.

## 2 Extreme Value Copulas

From the Sklar’s theorem, the limit d.f. $G$ from (1) has a uniquely determined copula $C$. 
**Theorem 1.** If (1) holds for some d.f. $F$ and some d.f. $G$ with EVD marginals, then the unique copula $C$ of $G$ satisfies

$$C(u_1^t, \ldots, u_d^t) = C^t(u_1, \ldots, u_d), \ t > 0. \quad (2)$$

Any copula with the property (2) is called extreme value copula (EV copula).

We provide several examples of copulas satisfying (2).

1) **copula of independence:**

$$C(u_1, \ldots, u_d) = u_1 \cdots u_d.$$

2) **comonotonicity copula** (upper Fréchet bound):

$$C(u_1, \ldots, u_d) = \min(u_1, \ldots, u_d).$$

3) **Gumbel copula**

$$C_{Gu}^\theta(u_1, u_2) = \exp \left\{ - \left( (-\log u_1)^\theta + (-\log u_2)^\theta \right)^{1/\theta} \right\}, \ 1 \leq \theta < \infty.$$

4) **asymmetric Gumbel copula**

$$C_{Gu,\alpha,\beta}^\theta(u_1, u_2) = u_1^{1-\alpha} u_2^{1-\beta} \exp \left\{ - \left( (-\alpha \log u_1)^\theta + (-\beta \log u_2)^\theta \right)^{1/\theta} \right\} \quad (3)$$

$$1 \leq \theta < \infty, \ 0 \leq \alpha, \beta \leq 1.$$

5) **Galambos copula**

$$C_{Gal}^\theta(u_1, u_2) = u_1 u_2 \exp \left\{ (-(\log u_1)^\theta + (\log u_2)^\theta)^{-1/\theta} \right\},$$

$$0 < \theta < \infty.$$
6) asymmetric Galambos copula

\[ C_{\theta,\alpha,\beta}^{Gal}(u_1, u_2) = u_1 u_2 \exp \left\{ \left( -\alpha \log u_1 \right)^{-\theta} + \left( -\beta \log u_2 \right)^{-\theta} \right\}^{1/\theta} \] \quad (4)

\[ 0 < \theta < \infty, \quad 0 \leq \alpha, \beta \leq 1. \]

There exists a number of results characterizing MEVD and EV copulas. As an example we present Picands representation:

**Theorem 2.** The copula \( C \) is an EV copula if and only if it has a representation

\[ C(u_1, \ldots, u_d) = \exp \left\{ B \left( \log u_1 \sum_{k=1}^{d} \log u_k, \ldots, \log u_d \sum_{k=1}^{d} \log u_k \right) \sum_{i=1}^{d} \log u_i \right\}, \] \quad (5)

where

\[ B(w_1, \ldots, w_d) = \int_{S_d} \max(x_1 w_1, \ldots, x_d w_d) \, dH(x) \]

and \( H \) is a finite measure on the \( d \)-dimensional simplex

\[ S_d = \left\{ x : x_i \geq 0, i = 1 \ldots, d, \sum_{i=1}^{d} x_i = 1 \right\}. \]

The function \( B \) is sometimes called the dependence function of the extreme value copula.

The representation can be made simpler in the bivariate case. Then we can use instead of \( B \) a function of a scalar argument defined by

\[ A(w) := B(w, 1 - w), \quad w \in [0, 1]. \]
According to the previous theorem, the bivariate EV copula has the representation

\[ C(u_1, u_2) = \exp \left\{ (\log u_1 + \log u_2) A \left( \frac{\log u_1}{\log u_1 + \log u_2} \right) \right\}, \quad (6) \]

where

\[ A(w) = \int_0^1 \max((1 - x)w, x(1 - w)) \, dH(x) \]

for a measure \( H \) on \([0, 1]\).

It can be shown that in this case the dependence function \( A \) must satisfy

\[ \max(w, 1 - w) \leq A(w) \leq 1, \quad 0 \leq w \leq 1, \quad (7) \]

and must be convex.

On the other hand, a convex differentiable function \( A(w) \) satisfying (7) can be used to construct an EV copula by means of (6). It is also seen from (6), how the dependence function can be derived from the EV copula:

\[ A(w) = -\log C \left( e^{-w}, e^{-(1-w)} \right), \quad w \in [0, 1]. \quad (8) \]

We thus obtain for the asymmetric Gumbel copula (3) the dependence function

\[ A(w) = (1 - \alpha)w + (1 - \beta)(1 - w) + \left((\alpha w)^\theta + (\beta (1 - w))^\theta\right)^{1/\theta} \quad (9) \]

and for the asymmetric Galambos copula we have

\[ A(w) = 1 - \left((\alpha w)^{-\theta} + (\beta (1 - w))^{-\theta}\right)^{-1/\theta}. \quad (10) \]
(The symmetric classical Gumbel and Galambos copulas have symmetric
dependence functions obtained by setting $\alpha = \beta = 1$.)

3 Multivariate minima

Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with joint d.f. $F$. We will consider componentwise minima

$$m_{n,j} = \min(X_{1,j}, \ldots, X_{n,j}), \ j = 1, \ldots, d.$$  

The structure of copulas for limiting distributions of multivariate minima can be easily derived from the structure of copulas for limiting distributions of normalized maxima. We shall denote

$$M_{n,j}^* = \max(-X_{1,j}, \ldots, -X_{n,j}), \ j = 1, \ldots, d.$$  

Further, let $\tilde{F}$ be the joint d.f. of random vectors $-X_1, \ldots, -X_n$.

If $\tilde{F} \in \text{MDA}(G)$ for some non-degenerate d.f. $G$, we have

$$\lim_{n \to \infty} P\left(\frac{M_{n,j}^* - d_n}{c_n} \leq x\right) = \lim_{n \to \infty} \tilde{F}^n(c_n x + d_n) = G(x)$$

for appropriate sequences of normalizing vectors $c_n$ and $d_n$ and a d.f. $G$ of MEVD type,

$$G(x_1, \ldots, x_d) = C\left(G_{\gamma_1}(x_1), \ldots, G_{\gamma_d}(x_d)\right),$$

(12)
where $G_{\gamma_j}$ is a d.f. from EVD family with the shape parameter $\gamma_j$ and $C$ is an EV copula.

Thanks to

$$m_{n,j} = -M_{n,j}^*$$

it follows from (11)

$$\lim_{n \to \infty} P\left( \frac{m_n + d_n}{c_n} \geq x \right) = G(-x),$$

so the normalized minima converge in distribution to a distribution with the survival function

$$G(-x) = C\left( G_{\gamma_1}(-x_1), \ldots, G_{\gamma_d}(-x_d) \right). \quad (13)$$

For the limiting distribution of minima $m_{n,i}$ it holds

$$\lim_{n \to \infty} P \left( M_{n,i}^* \leq d_{n,i} + c_{n,i} x \right) = G_{\gamma_i}(x)$$

$$\iff \lim_{n \to \infty} P \left( m_{n,i} \leq -d_{n,i} + c_{n,i} x \right) = 1 - G_{\gamma_i}(-x).$$

Copulas of limiting distributions for multivariate minima are survival copulas corresponding to MEVD.

In case the distribution of vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is the same as the distribution of $-\mathbf{X}_1, \ldots, -\mathbf{X}_n$ (i.e. radially symmetric about 0), $F = \tilde{F}$ and the copula of the limiting distribution of minima is the same as the survival copula of the limiting distribution for maxima.
4 Copula Domains of Attraction

Theorem 3. Let

\[ F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \]

for continuous marginal d.f. \( F_1, \ldots, F_d \) and some copula \( C \). Let

\[ G(x_1, \ldots, x_d) = C_0(G_1(x_1), \ldots, G_d(x_d)) \]

be a d.f. of MEV distribution with EV copula \( C_0 \). Than \( F \in \text{MDA}(G) \) if and only if \( F_i \in \text{MDA}(G_i) \) for \( i = 1, \ldots, d \) and

\[ \lim_{t \to \infty} C_t \left( u_1^{1/t}, \ldots, u_d^{1/t} \right) = C_0(u_1, \ldots, u_d), \; u \in [0, 1]^d. \] (14)

From the above theorem it is clear, that the copula \( C_0 \) of the limiting MEV distribution is determined solely by the copula \( C \). Marginal d.f. \( F_1, \ldots, F_d \) determine the marginals of limiting MEVD, but they do not influence its dependence structure.

If (14) is satisfied for some copula \( C \) and an EV copula \( C_0 \), we say that \( C \) is in the copula domain of attraction of \( C_0 \), \( C \in \text{MDA}(C_0) \).

There are many equivalent ways for expressing (14).

By taking the logarithm and using the identity \( \log(x) \sim x - 1 \) for \( x \to 1 \),
we obtain for \( u \in [0, 1]^d \)

\[
\lim_{t \to \infty} t \left( 1 - C\left(u_1^{1/t}, \ldots, u_d^{1/t}\right) \right) =
\]

\[
\lim_{s \to 0^+} \frac{1 - C(u_1^s, \ldots, u_d^s)}{s} = -\log C_0(u_1, \ldots, u_d).
\]

(15) (16)

Inserting \( u_i = \exp(-x_i) \) and using \( \exp(-sx) \sim 1 - sx \) for \( s \to 0 \), gives for \( x \in [0, \infty]^d \)

\[
\lim_{s \to 0^+} \frac{1 - C(1 - s x_1, \ldots, 1 - s x_d)}{s} = -\log C_0(e^{-x_1}, \ldots, e^{-x_d}).
\]

(17)

**Example.** Let us consider the bivariate Pareto distribution given by the survival function

\[ \bar{F}(x_1, x_2) = \left( \frac{x_1 + \kappa_1}{\kappa_1} + \frac{x_2 + \kappa_2}{\kappa_2} - 1 \right)^{-\alpha}, x_1, x_2 \geq 0, \alpha, \kappa_1, \kappa_2 > 0. \]

Since

\[ \bar{F}(x_1, x_2) = \hat{C} \left( \bar{F}_1(x_1), \bar{F}_2(x_2) \right), \]

we use

\[ \bar{F}_i(x) = \left( \frac{\kappa_i}{\kappa_i + x} \right)^\alpha, i = 1, 2, \]

to derive the survival copula for the bivariate Pareto distribution in the form

\[ \hat{C}(u_1, u_2) = \left( u_1^{-1/\alpha} + u_2^{-1/\alpha} - 1 \right)^{-\alpha}. \]

(Clayton copula)
In the bivariate case the relationship between a copula and its survival copula has the simple form

\[ \hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2). \]

From here we deduce

\[ C(u_1, u_2) = u_1 + u_2 - 1 + \left((1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right)^{-\alpha}. \]

It holds

\[ \lim_{s \to 0^+} \frac{1 - C(1-sx_1, 1-sx_2)}{s} = x_1 + x_2 - \left((x_1)^{-1/\alpha} + (x_2)^{-1/\alpha}\right)^{-\alpha}. \]

From (17) we obtain the expression of the copula of limiting distribution of maxima from the bivariate Pareto distribution in the form

\[ C_0(u_1, u_2) = u_1 u_2 \exp\left((- \log u_1)^{-1/\alpha} + (- \log u_2)^{-1/\alpha}\right)^{-\alpha}. \]

(Galambos copula)

We recall that univariate Pareto distribution is in the maximum domain of attraction of Fréchet distribution with parameter \( \gamma = 1/\alpha \).

To sum up, for the considered bivariate Pareto distribution, the limiting distribution of maxima consists of two univariate Fréchet marginals connected by a Galambos copula.

There is an interesting result concerning the coefficient of upper tail-dependence and copula maximum domain of attraction.
Theorem 4. Let $C$ be a bivariate copula with upper tail-dependence coefficient $\lambda_u$ and assume that $C \in \text{MDA}(C_0)$ for some EV copula $C_0$. Then $\lambda_u$ is also the upper tail-dependence coefficient of $C_0$ and its related to its dependence function by

$$\lambda_u = 2 \left(1 - A(1/2)\right).$$  \hspace{1cm} (18)

Proof. From the connection between a copula $C$ and its survival copula $\hat{C}$ it follows

$$\lambda_u = 2 - \lim_{q \to 1-} \frac{1 - C(q,q)}{1 - q}. \hspace{1cm} (19)$$

From (17) we obtain, using the identity $\log(x) \sim x - 1$ for $x \to 1$

$$\lim_{q \to 1-} \frac{1 - C_0(q,q)}{1 - q} = \lim_{q \to 1-} \lim_{s \to 0+} \frac{1 - C(q^s,q^s)}{-s \log(q^s)}$$

$$= \lim_{v \to 1-} \frac{1 - C(v,v)}{1 - v}.$$  \hspace{1cm} (18) is a consequence of (19) and (6).

In the case $\lambda_u = 0$ we must have $A(1/2) = 1$, which implies due to the convexity of $A$ that in this case $A(w)$ is identically 1, so $C_0$ must be the independence copula. This is the case of the limiting distribution copula for multivariate maxima of multivariate normal random vectors. The limiting d.f. for maxima in this case is a product of univariate Gumbel d. functions.