2 Claims Reserving in Non-Life Insurance

2.1 Notation

We separate loss data according to the accident year (year of occurrence) $i$ and the development year $j$. We assume $i \in \{0, \ldots, I\}$, $j \in \{0, \ldots, J\}$, where $I$ denotes the most recent accident year and $J$ denote the last development year. We shall further assume that $I = J$.

We shall denote the incremental data as $X_{ij}$: $X_{ij}$ stands for payments for claims in accidental year $i$ made in year $i + j$ (alternatively, $X_{ij}$ may denote the number of reported claims with delay $j$ or the change of the reported claims amount).

The cumulative amount $C_{ij}$ for accident year $i$ after $j$ development years is then given by

$$C_{ij} = \sum_{k=0}^{j} X_{ik}.$$  

The observations available at time $I$ are represented by the set

$$D_I = \{X_{ij} : i + j \leq I, \ 0 \leq j \leq J\}.$$  

The values for $i + j > I$ need to be estimated or predicted.

The outstanding loss liabilities (claims reserve) for accident year $i$ at time $I$ are given by

$$R_i = \sum_{j=I-i+1}^{J} X_{ij} = C_{iJ} - C_{i,I-i}.$$
In the following sections we shall introduce basic methods used most frequently for estimating claims reserves. These methods consist in simple algorithms applied to claims development triangles. We shall present some stochastic models that can be used to justify the corresponding algorithms and to quantify the uncertainty in the resulting estimates.

2.2 Chain-ladder method

There are several stochastic models that justify the CL method. We start with the distribution-free derivation.

Model assumptions (CL)

- Cumulative claims $C_{ij}$ of different accident years are independent.

- There exist development factors $f_0, \ldots, f_{J-1} > 0$ such that for all $0 \leq i \leq I$ and all $1 \leq j \leq J$ we have

$$E[C_{ij}|C_{i0}, \ldots, C_{i,j-1}] = f_{j-1} C_{i,j-1}.$$ 

(CL) contains only assumptions on the first moments - it is sufficient for estimating (conditionally) expected future claims.

Lemma. Under Model Assumptions (CL) we have

$$E[C_{i,j}|D_I] = E[C_{i,j}|C_{i,I-i}] = C_{i,I-i} f_{I-i} \cdots f_{J-1}$$
for all $1 \leq i \leq I$.

**CL estimator**

The CL estimator for $E[C_{ij}|D_I]$ is given by

$$\hat{C}_{ij}^{CL} = E[C_{ij}|D_I] = C_{i,I-i} \hat{f}_{I-i} \cdots \hat{f}_{j-1}$$

for $i + j > I$, where

$$\hat{f}_j = \frac{\sum_{i=0}^{I-j-1} C_{i,j+1}}{\sum_{i=0}^{I-j-1} C_{i,j}} = \sum_{i=0}^{I-j-1} \frac{C_{i,j}}{\sum_{k=0}^{I-j-1} C_{k,j}} \frac{C_{i,j+1}}{C_{ij}}.$$

*Note.* CL factors $f_j$ are estimated by a weighted average of individual development factors $F_{i, j+1} = \frac{c_{i,j+1}}{c_{ij}}$.

**Properties of CL estimators:**

Denote $B_k = \{C_{ij} : i + j \leq I, \ 0 \leq j \leq k\}$.

We have

(a) given $B_j$, $\hat{f}_j$ is an unbiased estimator for $f_j$, i.e. $E[\hat{f}_j|B_j] = f_j$,

(b) $\hat{f}_j$ is unconditionally unbiased, i.e. $E \hat{f}_j = f_j$,

(c) $E[\hat{f}_0 \cdots \hat{f}_j] = E \hat{f}_0 \cdots E \hat{f}_j$, i.e. $\hat{f}_0, \cdots, \hat{f}_{j-1}$ are uncorrelated,

(d) given $C_{i,I-i}$, $\hat{C}_{i,j}^{CL}$ is an unbiased estimator for $E[C_{i,j}|D_I] = E[C_{i,j}|C_{i,I-i}]$, i.e. $E[\hat{C}_{i,j}^{CL}|C_{i,I-i}] = E[C_{i,j}|D_I]$,

(e) $\hat{C}_{i,j}^{CL}$ is unconditionally unbiased, i.e. $E \hat{C}_{i,j}^{CL} = E C_{i,j}$.  


2.3 Bornhuetter-Ferguson method

We present two possibilities of a stochastic model which motivates the BF method.

Model Assumptions (BF1)

- Cumulative claims $C_{ij}$ of different accident years $i$ are independent.

- There exist parameters $\mu_0, \ldots, \mu_I > 0$ and a pattern $\beta_0, \ldots, \beta_J > 0$ with $\beta_J = 1$ such that for all $0 \leq i \leq I$, $0 \leq j \leq J-1$ and $1 \leq k \leq J-j$ we have

$$E C_{i0} = \beta_0 \mu_i$$

$$E [C_{i,j+k} | C_{i0}, \ldots C_{ij}] = C_{ij} + (\beta_{j+k} - \beta_j) \mu_i.$$  

Under (BF1) we have $E C_{ij} = \beta_j \mu_i$ and $E C_{iJ} = \mu_i$. The sequence $(\beta_j)_{j=0,\ldots,J}$ denotes the claims development pattern.

Model Assumptions (BF1) imply the following

Model Assumptions (BF2)

- Cumulative claims $C_{ij}$ of different accident years $i$ are independent.

- There exist parameters $\mu_0, \ldots, \mu_I > 0$ and a pattern $\beta_0, \ldots, \beta_J > 0$ with $\beta_J = 1$ such that for all $0 \leq i \leq I$, $0 \leq j \leq J$ we have

$$E C_{ij} = \beta_j \mu_i.$$
Lemma. Under Model Assumptions (BF1) we have

\[ E[C_{i,j}|D_t] = C_{i,I-i} + (1 - \beta_{I-i}) \mu_i. \]

Proof.

\[
E[C_{i,j}|D_t] = E[C_{i,j}|C_{i0}, \ldots, C_{i,I-i}]
= C_{i,I-i} + E[C_{i,j} - C_{i,I-i}|C_{i0}, \ldots, C_{i,I-i}]
= C_{i,I-i} + (1 - \beta_{I-i}) \mu_i.
\]

Note. The same result we obtain from (BF2) when we assume the independence of \( C_{i,j} - C_{i,I-i} \) of \( C_{i0}, \ldots, C_{i,I-i} \).

BF estimator

The BF estimator for \( E[C_{i,j}|D_t] \) is given by

\[
\hat{C}_{i,j}^{BF} = \hat{E}[C_{i,j}|D_t] = C_{i,I-i} + (1 - \hat{\beta}_{I-i}) \hat{\mu}_i
\]  

(1)

for \( 1 \leq i \leq I \), where \( \hat{\beta}_{I-i} \) is an appropriate estimate for \( \beta_{I-i} \) and \( \hat{\mu}_i \) is a prior estimate for the expected ultimate claim \( E[C_{i,j}] \).

Comparison of BF and CL models

From Model Assumptions (CL) it follows

\[
E[C_{ij}] = E[E[C_{ij}|C_{i,j-1}]] = f_{j-1} E[C_{i,j-1}] = E[C_{i0}] \prod_{k=0}^{j-1} f_k,
\]

\[
E[C_{ij}] = E[C_{i0}] \prod_{k=0}^{J-1} f_k,
\]

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which implies that 

\[ E[C_{ij}] = \prod_{k=1}^{J-1} f_k^{-1} E[C_{i,j}]. \]

It corresponds to Model Assumptions (BF2) with 

\[ \mu_i = E_C i, \beta_j = \prod_{k=j}^{J-1} f_k^{-1}, \ j = 0, \ldots, J - 1, \ \beta_J = 1. \]

Factors \( \beta_j \) in BF method can be estimated by a prior estimates independently on the observations. However, more often we derive the estimates of \( \beta_j \) from data using CL factor estimates \( \hat{f}_j \):

\[ \hat{\beta}^{CL}_j = \prod_{k=1}^{J-1} \frac{1}{\hat{f}_k}. \] (2)

We denote the BF estimator of \( E[C_{i,j}|D_I] \) with this special choice of \( \hat{\beta}_j \) as \( \hat{C}^{BF}_{ij} \):

\[ \hat{C}^{BF}_{ij} = C_{i, I-i} + \left( 1 - \hat{\beta}^{CL}_{I-i} \right) \mu_i. \] (3)

\underline{Comparison of CL and BF estimators}

For the CL estimator we have

\[ \hat{C}^{CL}_{ij} = C_{i, I-i} \prod_{j=I-i}^{J-1} \hat{f}_j = C_{i, I-i} + C_{i, I-i} \left( \prod_{j=I-i}^{J-1} \hat{f}_j - 1 \right) \]

\[ = C_{i, I-i} + \frac{\hat{C}^{CL}_{ij}}{\prod_{j=I-i}^{J-1} \hat{f}_j} \left( \prod_{j=I-i}^{J-1} \hat{f}_j - 1 \right) \]

\[ = C_{i, I-i} + \left( 1 - \hat{\beta}^{CL}_{I-i} \right) \hat{C}^{CL}_{ij}. \] (4)

Hence, if we identify the claims development pattern from the CL and BF methods, the difference between the BF and CL estimators is that for the
BF method we use a prior estimate $\hat{\mu}_i$, in the CL method the prior estimate is replaced by the estimate $\hat{C}_{ij}^{CL}$ which is based only on the observations.

2.4 Poisson model for the number of claims

In this chapter we consider the incremental variable $X_{ij}$ as the number of claims in accident year $i$ reported in year $i + j$.

The Poisson model is an example of a claims reserving model that is based on explicit distributional assumptions.

Model Assumptions (Po)

There exist parameters $\mu_0, \ldots, \mu_I > 0$ and $\gamma_0, \ldots, \gamma_J > 0$ such that the incremental claims $X_{ij}$ are independent and Poisson distributed with

$$E[X_{ij}] = \mu_i \gamma_j,$$

for all $0 \leq i \leq I$, $0 \leq j \leq J$, and $\sum_{j=0}^{J} \gamma_j = 1$.

Notes.

1. The Model Assumptions (Po) imply that the increments $X_{ij}$ are non-negative.

2. The cumulative claim in accident year $i$, $C_{ij}$ is Poisson distributed with $E C_{ij} = \mu_i$. (This can be proved by means of the probability generating
function using the independence of the incremental claims:

\[ E e^{r C_{ij}} = \prod_{j=1}^{J} E e^{r X_{ij}} = \prod_{j=1}^{J} e^{\mu_j (r-1)} = e^{\mu_i (r-1)}, \]

which is the p.g.f. of the Poisson distribution with the expectation \( \mu_i \).

3. We have \( \frac{E[X_{ij}]}{E[X_{i0}]} = \frac{\gamma_j}{\gamma_0} \) which is independent of \( i \).

**Lemma.** The Poisson model satisfies Model Assumptions (BF1).

**Proof.** The independence of the cumulative claims of different accident years follows from the independence of \( X_{ij} \).

We have \( E[C_{i0}] = E[X_{i0}] = \mu_i \beta_0 \) with \( \beta_0 = \gamma_0 \), and

\[
E[C_{i,j+k}|C_{i0}, \ldots, C_{ij}] = C_{ij} + \sum_{l=1}^{k} E[X_{i,j+k}|C_{i0}, \ldots, C_{ij}]
\]

\[
= C_{ij} + \mu_i \sum_{l=1}^{k} \gamma_{j+k} = C_{ij} + \mu_i (\beta_{j+k} - \beta_{j})
\]

with \( \beta_{j} = \sum_{l=0}^{j} \gamma_{l} \).

According to the previous lemma we could use BF estimator for the determination of claims reserves in the Poisson model. The distributional assumption allows for estimation of the parameters by means of the maximum likelihood method.

The likelihood function on the set of observations \( D_I \) is given by

\[
L(\mu_0, \ldots, \mu_I, \gamma_0, \ldots, \gamma_I) = \prod_{i+j \leq I} \left( \exp(-\mu_i \gamma_j) \frac{(\mu_i \gamma_j)^{X_{ij}}}{X_{ij}!} \right).
\]
The log-likelihood function is then
\[ l(\mu_0, \ldots, \mu_I, \gamma_0, \ldots, \gamma_I) = - \sum_{i+j \leq I} \mu_i \gamma_j + \sum_{i+j \leq I} X_{ij} \log(\mu_i \gamma_j) - \sum_{i+j \leq I} \log(X_{ij}!) . \]  \hfill (5)

Setting to zero the partial derivatives of (5) with respect to the unknown parameters \( \mu_i \) and \( \gamma_j \) leads to a system of equations
\[
- \sum_{j=0}^{I-i} \hat{\gamma}_j + \sum_{j=0}^{I-i} \frac{X_{ij}}{\hat{\mu}_i} = 0, \quad i = 0, \ldots, I, \\
- \sum_{i=0}^{I-j} \hat{\mu}_i + \sum_{i=0}^{I-j} \frac{X_{ij}}{\hat{\gamma}_j} = 0, \quad j = 0, \ldots, J,
\]
which can be rewritten as
\[
\sum_{j=0}^{I-i} \hat{\mu}_i \hat{\gamma}_j = \sum_{j=0}^{I-i} X_{ij} = C_{i, I-i}, \quad i = 0, \ldots, I, \tag{6} \\
\sum_{i=0}^{I-j} \hat{\mu}_i \hat{\gamma}_j = \sum_{i=0}^{I-j} X_{ij} = 0, \quad j = 0, \ldots, J. \tag{7}
\]

For the solution of (6) and (7) there is a constraint
\[
\sum_{j=0}^{J} \hat{\gamma}_j = 1. \tag{8}
\]

Poisson ML estimator

The estimators for \( \mathbb{E}[X_{ij}] \) and \( \mathbb{E}[C_{i,j} | D_I] \) for \( i + j > I \) are given by
\[
\hat{X}_{ij}^{Po} = \hat{\mathbb{E}}[X_{ij}] = \hat{\mu}_i \hat{\gamma}_j \\
\hat{C}_{i,j}^{Po} = \hat{\mathbb{E}}[C_{i,j} | D_I] = C_{i, I-i} + \sum_{I-i+1}^{J} \hat{\mu}_i \hat{\gamma}_j.
\]

where $\hat{\mu}_i$, $i = 0, \ldots, I$, and $\hat{\gamma}_j$, $j = 0, \ldots, J$, are ML estimates given by (6) and (7) with the constraint (8).

Note. It holds
\[
\hat{C}_{iJ}^{Po} = C_{i,I-i} + \left(1 - \sum_{j=0}^{I-i} \hat{\gamma}_j\right) \hat{\mu}_i,
\]
hence the Poisson ML estimator has the same form as the BF estimator (1). However, in (9) we use the ML estimates for $\mu_i$ and $\gamma_j$ that depend on the data.

We will show that the CL estimator and the Poisson ML estimator for $E[C_{iJ}|D_I]$ are the same, i.e.
\[
\hat{C}_{iJ}^{CL} = \hat{C}_{iJ}^{Po}.
\]

We assume that there is a positive solution to (6)-(7), $\hat{\mu}_i, \hat{\gamma}_j$, $i = 0, \ldots, I$, $j = 0, \ldots, J$. To prove (10) we will use the following result.

Lemma. Under Model Assumptions (Po) we have
\[
\sum_{i=0}^{I-j} C_{ij} = \sum_{i=0}^{I-j} \hat{\mu}_i \sum_{k=0}^{j} \hat{\gamma}_k.
\]

Proof. (11) is proved by induction. Using (6) for $i = 0$ and $I = J$ gives
\[
C_{0J} = \sum_{j=0}^{J} X_{0j} = \hat{\mu}_0 \sum_{j=0}^{J} \hat{\gamma}_j.
\]
For induction step $j \rightarrow j - 1$ we use the relation between the set of observed incremental claims $\{X_{il}, i = 0, \ldots, I - j, l = 0, \ldots, j\}$ and the set
\{X_{il}, \; i = 0, \ldots, I - j + 1, \; l = 0, \ldots, j - 1\}. The second set is obtained from the first one by adding the accident year \(I - j + 1\) and subtracting the development year \(j:\)

\[
\sum_{i=0}^{I-(j-1)} C_{i,j-1} = \sum_{i=0}^{I-j} C_{ij} - \sum_{i=0}^{I-j} X_{ij} + \sum_{k=0}^{j-1} X_{I-j+1,k}.
\]

Using (6) - (7) and the induction assumption we obtain

\[
\sum_{i=0}^{I-(j-1)} C_{i,j-1} = \sum_{i=0}^{I-j} \hat{\mu}_i \sum_{k=0}^{j} \hat{\gamma}_k - \hat{\gamma}_j \sum_{i=0}^{I-j} \hat{\mu}_i + \hat{\mu}_{I-j+1} \sum_{k=0}^{j-1} \hat{\gamma}_k \\
= \sum_{i=0}^{I-j} \hat{\mu}_i \sum_{k=0}^{j-1} \hat{\gamma}_k + \hat{\mu}_{I-j+1} \sum_{k=0}^{j-1} \hat{\gamma}_k \\
= \sum_{i=0}^{I-(j-1)} \hat{\mu}_i \sum_{k=0}^{j-1} \hat{\gamma}_k.
\]

From the lemma it follows the expression of the CL development factor estimate for \(j \leq J\)

\[
\hat{f}_{j-1} = \frac{\sum_{i=0}^{I-j} C_{ij}}{\sum_{i=0}^{I-j} C_{i,j-1}} = \frac{\sum_{i=0}^{I-j} C_{ij}}{\sum_{i=0}^{I-j} (C_{ij} - X_{ij})} = \frac{\sum_{i=0}^{I-j} \hat{\mu}_i \sum_{k=0}^{j} \hat{\gamma}_k}{\sum_{i=0}^{I-j} \hat{\mu}_i \sum_{k=0}^{j-1} \hat{\gamma}_k} = \frac{\sum_{k=0}^{j} \hat{\gamma}_k}{\sum_{k=0}^{j-1} \hat{\gamma}_k}.
\]

We start the proof of (11) with the expression of the Poisson estimator for \(i > 0\)

\[
\tilde{C}_{i,J}^{Po} = \hat{\mu}_i \sum_{j=I-i+1}^{J} \hat{\gamma}_j + C_{i,I-i} = \hat{\mu}_i \sum_{j=0}^{J} \hat{\gamma}_j,
\]

(12)

where we used (see (6))

\[
C_{i,I-i} = \hat{\mu}_i \sum_{j=0}^{I-i} \hat{\gamma}_j.
\]
From (6) and (7) we have \((I = J)\)

\[
\hat{C}_{iJ}^{\text{Po}} = C_{i,J-i} \left\{ \frac{\sum_{j=0}^{I-1} \hat{\gamma}_j}{\sum_{j=0}^{I-1} \gamma_j} \right\},
\]

which can be rewritten to

\[
\hat{C}_{iJ}^{\text{Po}} = C_{i,J-i} \left\{ \frac{\sum_{j=0}^{I-1} \hat{\gamma}_j}{\sum_{j=0}^{I-1} \gamma_j} \right\} \cdot \hat{f}_{J-i} \cdots \hat{f}_{i-1},
\]

which proves (11).

Comparison of the Poisson ML estimator and the BF estimator

When we use (2) as an estimate of the development pattern for BF estimator, we can show that

\[
\hat{\beta}_{CL}^j = \sum_{k=0}^{j} \hat{\gamma}_k
\]

for all \(0 \leq j \leq J\).

It follows from the equality giving for all \(i \geq 0\)

\[
\frac{\sum_{j=0}^{I-1} \hat{\gamma}_j}{\sum_{j=0}^{I-1} \gamma_j} = \hat{f}_{J-i} \cdots \hat{f}_{i-1} = \left( \hat{\beta}_{CL}^{J-i} \right)^{-1}
\]

and from \(\sum_{j=0}^{I-1} \hat{\gamma}_j = 1\).

The above result also shows us a way how to obtain the solution of (6) - (8) with the help of the CL factors \(\hat{f}_k\):

\[
\hat{\gamma}_j = \hat{\beta}_{j}^{CL} - \hat{\beta}_{j-1}^{CL} = \prod_{k=j}^{I-1} \frac{1}{\hat{f}_k} \left( 1 - \frac{1}{\hat{f}_{j-1}} \right), \quad j = 1, \ldots, J,
\]

\[
\hat{\mu}_i = \frac{\sum_{j=0}^{I-1} X_{ij}}{\sum_{j=0}^{I-1} \gamma_j}, \quad i = 0, \ldots, I.
\]
2.5 Benktander-Hovinen Method

From the comparison of (3) and (4) we have seen that the BF method uses the prior estimate $\hat{\mu}_i$ for the total claims from year $i$ and completely ignores the observations. On the other hand, the CL estimator ignores the prior estimate and uses only the observations in the development triangle. We could consider a mixture of these two approaches, where the estimate of the total claim amount from the year $i$ has a form

$$u_i(c) = c \hat{\mu}_i^{CL} + (1 - c) \mu_i, \quad 1 \leq i \leq I,$$

where $c \in [0, 1]$ and $\mu_i$ is a known prior estimate for $E[C_{iJ}]$.

The parameter $c$ should increase with the development of $C_{iJ}$ since we obtain better information on $C_{iJ}$ with increasing time $j$. Choosing $c = \beta_{I-i}$ leads to

BH estimator

$$\hat{C}_{iJ}^{BH} = C_{i,I-i} + (1 - \beta_{I-i}) \left( \beta_{I-i} \hat{\mu}_i^{CL} + (1 - \beta_{I-i}) \mu_i \right)$$

for $1 \leq i \leq I$.

In (14) we assume that the claims development pattern $(\beta_j)_{0 \leq j \leq J}$ is known.

If we use Model Assumptions (BF2) we can identify

$$\beta_j = \prod_{k=j}^{J-1} \bar{f}_k^{-1}.$$
Since the development pattern \((\beta_j)_{0 \leq j \leq J}\) is known, the CL factors \((f_j)_{0 \leq j \leq J-1}\) are also known and we can write

\[
\hat{C}_{iJ}^{CL} = \frac{C_{i,I-i}}{\beta_{I-i}}
\]

for \(0 \leq j \leq J - 1\).

Then, the BH estimator (14) can be written in the form

\[
\hat{C}_{iJ}^{BH} = \beta_{I-i} \hat{C}_{iJ}^{CL} + (1 - \beta_{I-i}) \hat{C}_{iJ}^{BF}
\]

\[
= C_{i,I-i} + (1 - \beta_{I-i}) \hat{C}_{iJ}^{BF}.
\]

The BH estimator thus can be seen as an iterated BF estimator using the BF estimate as the new prior estimate.

**Lemma.** Under the assumption that the claims development pattern \((\beta_j)_{0 \leq j \leq J}\) is known and under (15) we have

\[
\hat{C}_{iJ}^{BH} = u_i \left(1 - (1 - \beta_{I-i})^2\right)
\]

for \(1 \leq i \leq I\), where the function \(u_i\) is given by (13).

**Proof.** It holds that

\[
\hat{C}_{iJ}^{BH} = C_{i,I-i} + (1 - \beta_{I-i}) \left(\beta_{I-i} \hat{C}_{iJ}^{CL} + (1 - \beta_{I-i}) \mu_i\right)
\]

\[
= \beta_{I-i} \hat{C}_{iJ}^{CL} + (\beta_{I-i} - \beta_{I-i}^2) \hat{C}_{iJ}^{CL} + (1 - \beta_{I-i})^2 \mu_i
\]

\[
= (1 - (1 - \beta_{I-i})^2) \hat{C}_{iJ}^{CL} + (1 - \beta_{I-i})^2 \mu_i
\]

\[
= u_i \left(1 - (1 - \beta_{I-i})^2\right). \quad \square
\]
We may ask what happens if we further iterate the BF method. Set \( \hat{C}_{ij}^{(0)} = \mu_i \) and

\[
\hat{C}_{ij}^{(m+1)} = C_{i,I-i} + (1 - \beta_{I-i}) \hat{C}_{ij}^{(m)}
\]

for \( m \geq 0 \).

For \( m = 0 \) we have the BF estimator and for \( m = 1 \) we have the BH estimator. For \( m \geq 1 \) it can be shown by induction that

\[
\hat{C}_{ij}^{(m)} = (1 - (1 - \beta_{I-i})^m) \hat{C}_{ij}^{CL} + (1 - \beta_{I-i})^m \mu_i.
\]

From here it follows under the assumption \( \beta_{I-i} > 0 \)

\[
\lim_{m \to \infty} \hat{C}_{ij}^{(m)} = \hat{C}_{ij}^{CL}.
\]

So, further iteration of the BF method leads to the CL estimator of the ultimate claim.

### 2.6 Cape Cod Model

We shall explain a method that was designed with the aim to eliminate the dependence of the CL estimate on possible outliers on the diagonal of the cumulative development triangle.

**Model Assumptions (CC)**

- Cumulative claims \( C_{ij} \) of different accident years are independent.
• There exist parameters $\Pi_0, \ldots, \Pi_I > 0$, $\kappa > 0$, and a claims development pattern $(\beta_j)_{0 \leq j \leq I}$ with $\beta_I = 1$ such that

$$E[C_{ij}] = \kappa \Pi_i \beta_j$$

for all $i = 0, \ldots, I$.

Observe that Model Assumptions (CC) coincide with (BF2) with $\mu_i = \kappa \Pi_i$.

$\Pi_i$ can be interpreted as the premium received for accident year $i$ and $\kappa$ is the expected loss ratio that is assumed constant for all accident years.

We can estimate for each accident year the loss ratio $\kappa$ using the CL estimate for the ultimate claim:

$$\hat{\kappa}_i = \frac{\hat{C}^{CL}_{i,i}}{\Pi_i} = \frac{C_{i,I-i}}{\beta_{I-i} \Pi_i}.$$  

Obviously, $E[\hat{\kappa}_i] = \kappa$.

Using the data from all the accident years we can estimate the loss ratio $\kappa$ by

$$\hat{\kappa}^{CC} = \sum_{i=0}^{I} \frac{\beta_{I-i} \Pi_i}{\sum_{k=0}^{I} \beta_{I-k} \Pi_k} \hat{\kappa}_i = \frac{\sum_{i=0}^{I} C_{i,I-i}}{\sum_{i=0}^{I} \beta_{I-i} \Pi_i}.$$  

$\hat{\kappa}^{CC}$ is an unbiased estimator for $\kappa$.

We then substitute the diagonal value $C_{i,I-i}$ by a "robusted" value

$$\hat{C}^{CC}_{i,I-i} = \hat{\kappa}^{CC} \Pi_i \beta_{I-i}.$$  

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Cape-Cod estimator

The CC estimator is given by

$$\hat{C}_{iJ}^{CC} = C_{i,I-i} - \hat{C}_{i,I-i}^{CC} + \prod_{j=I-i}^{I-1} f_j \hat{C}_{i,I-i}^{CC}$$

for $1 \leq i \leq I$.

Lemma. Under Model Assumptions (CC) and under (15) it holds

$$\mathbb{E} \left[ \hat{C}_{iJ}^{CC} - C_{i,I-i} \right] = \mathbb{E} \left[ C_{iJ} - C_{i,I-i} \right] = \kappa \Pi_i (1 - \beta_{I-i}).$$

Proof is based on the expression

$$\hat{C}_{iJ}^{CC} - C_{i,I-i} = \hat{C}_{i,I-i}^{CC} \left( \prod_{j=I-i}^{I-1} f_j - 1 \right)$$
$$= \kappa^{CC} \Pi_i \beta_{I-i} \left( \prod_{j=I-i}^{I-1} f_j - 1 \right)$$
$$= \kappa^{CC} \Pi_i (1 - \beta_{I-i}).$$