On the Modeling of Inhomogeneous Incompressible Fluid-like Bodies

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We use a thermodynamic framework that has been recently put into place to describe the dissipative response of materials to develop a model for the response of inhomogeneous incompressible fluid-like bodies whose stored energy depends on gradients of the density. Without appealing to additional balance laws, or concepts such as interstitial working, we are able to obtain models for such bodies that do not need boundary conditions for the density in order to have well posed governing equations. After obtaining a model for the response of inhomogeneous incompressible fluid-like body, we solve a simple boundary value problem and introduce the notion of (weak) solutions to the equations governing the flow of the body that has been developed to illustrate the issues concerning boundary conditions.

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1. Introduction

Bodies such as granular matter respond in a fluid-like manner. They are invariably compressible in virtue of the interstitial space that exists between the grains. However, as the grain size becomes smaller and under certain conditions such as interlocking, they behave as though they are incompressible. Another example of inhomegenous incompressible fluid-like response is that of a complex mixture such as blood which can be modeled as an inhomogeneous incompressible fluid, though in its crudest approximation it is usually modeled as a homogeneous fluid. Of course we recognize that such models are but relatively crude approximations of real bodies, and in this sense the spirit of the approximation is no different than that used to develop models for continua. We tend to think of a material as incompressible when its compressibility is insignificant and more importantly this compressibility has insignificant consequences concerning the response of the body. This paper concerns the response of fluid-like bodies whose stored energy depends on the inhomogeneous nature of the body through the density and the gradient of the density. Such models were introduced first by Korteweg [1901] to describe fluids, but they have been also used to describe granular materials. Since the model such as that being studied have been developed for granular bodies, we motivate the study on the basis of certain issues concerning granular bodies. This paper is not concerned specifically with granular bodies, it is concerned with inhomogeneous incompressible fluid-like bodies. However, in certain limits the behavior of granular bodies can be captured by the models discussed in this study.

Granular Materials, as the nomenclature would suggest, consists of grains†. These grains are discrete particles and when they are very fine, i.e., when their size is very small they are usually referred to as Powders. The interstices between the grains are invariably filled with a fluid, a gas such as air or a liquid such as water. Rarely, if ever, does one find Granular Materials with interstitial vacuum. When the grains themselves and the local volume fraction of the interstices are both sufficiently small, and the interstices are filled with a fluid whose density is small compared to that of grains, we can model the system, on the average, as a single homogenized continuum. The presence of these interstices is the basis for the compressibility exhibited by Granular Materials, even if the grains are incompressible. In virtue of the difference in the shapes and sizes of the grains and granules, Granular Materials exhibit phenomena that have been observed in many non-linear fluids (solids), namely the development of Normal Stress Differences due to simple shear, deformation dependent material properties that lead to phenomena such as Shear-thinning (Shear-softening) and Shear-thickening (Shear-hardening), Non-

† While the primary meaning assigned to the word grain is that of a "seed", the meaning that is pertinent to our topic of discussion is "Any of the irregularly shaped discrete particles or crystals in a rock or a metal, usually but not necessarily small" (see The Oxford English Dictionary [1989]). The earliest use of assigning such a meaning was due to Bakewell: "...composed of grains or crystals closely united without a cement", in 1813 (see The Oxford English Dictionary [1989]).

linear creep and Stress-relaxation. Of course, which, if any, of these features are manifested and to what extent depends on the Granular Material in question.

The grains or granules can be compressible or incompressible, and thus any modeling of Granular Materials has to take cognizance of this issue. Even when dealing with incompressible grains, if the interstitial fluid has inconsequential density compared to that of the grains, we might be justified in developing a model that is compressible as the volume of the interstices can change. However, it is also possible that in certain granular materials the change in the interstitial volume during the motion is small when interlocking takes place, and then the homogenized continuum model can be considered as an incompressible body.

The fact that a fluid-like body exhibits stress relaxation does not imply that the body under consideration is akin to a classical viscoelastic material. Stress relaxation is a rather loose term that implies that the stress that is necessary to engender a certain fixed strain decreases with time. The cause for this drop in stress can have multifarious origins. For instance, this relaxation of the stresses can be due to aging of the material (see Rajagopal and Wineman [2004]), in the absence of the traditional stress relaxation mechanism associated with a viscoelastic material coming into play, and also due to chemical reactions and degradation. That these different types of stress relaxations have completely different origins is made evident by the fact that in the case of aging considered by Rajagopal and Wineman [2004] the relaxation depends on the geometry of the body, while in a classical viscoelastic body it does not. In this paper, we do not consider models for fluid-like bodies that allow for stress relaxation.

Recently, a general thermodynamic framework has been put into place within which a large disparate class of responses of dissipative materials can be studied. The central idea for the framework is the fact that a body can exist in more than one configuration (called natural configuration), modulo rigid body motion, with the material symmetry corresponding to these configurations being different. While the idea of the body having the multiple natural configurations has been well known for a long time and seems to have been first articulated carefully by Eckart [1948], the implications of the natural configurations vis-a-vis the evolution of the material symmetry of the body and its relevance to the development of constitutive theories was not fully understood, nor was Eckart's work given any thermodynamic basis. This aspect of the framework was first discussed in detail by Rajagopal [1995].

Thermodynamical considerations determine the evolution of these natural configurations.

More precisely, the evolution of the natural configuration is determined by the maximization of the rate of entropy production. While the notion of the maximization of the rate of dissipation was advocated first by Ziegler [1983], he did not recognize the role that this plays on the evolution of the material symmetry of the natural configuration, nor did he use it within the context of the thermodynamic framework that we shall employ. However, due credit ought to be given to the seminal contributions of Eckart and Ziegler in the development of our methodology.

In the case of both the compressible and incompressible viscous fluids, the current configuration $\kappa_t(\mathcal{B})$ serves as the natural configuration and there is no necessity to determine how the natural configuration evolves as in the case of viscoelasticity (Rajagopal and Srinivasa [2000, 2001]), classical plasticity (Rajagopal and Srinivasa [1998a,b]), twinning (Rajagopal and Srinivasa [1995, 1997]), solid to solid phase transition Rajagopal and Srinivasa [1999]) or crystallization in polymers (Rajagopal and Rao [2001, 2002]). Thus, maximization of the rate of entropy production, which in the case of a purely mechanical context is due to the rate of dissipation (as this is the only means of entropy production), leads to the constitutive relation for the stress. Our study of granular materials is akin to that of the classical linearly viscous fluid in that the current configuration of the body is assumed to be the natural configuration of the body.

We shall assume that the temperature field is uniform throughout the body. This assumption implies that there is no conduction taking place within the body and thus there is no entropy production such as that associated with conduction. The fact that the body has uniform temperature greatly simplifies our discussion. However, it is not difficult to allow for temperature fields in the body and to consider non-isothermal processes (see Kannan and Rajagopal [2004], Rao and Rajagopal [2002]).

We shall consider a class of constitutive relations for incompressible fluid-like bodies wherein the rate of dissipation depends on the density, the mean normal stress (the pressure) and the symmetric part of the velocity gradient. Such a choice is quite different from the classical incompressible Navier-Stokes fluid wherein the dissipation depends only quadratically on the symmetric part of the velocity gradient. The second law of thermodynamics is automatically met by choosing the rate of dissipation to be non-negative. Our point of departure from standard viscous fluid models is in the assumed form for the specific Helmholtz stored energy, which we suppose depends on the density, the Lagrangian gradient of the density in the reference configuration and the deformation gradient in a special manner. We are thus interested in describing the response of inhomogeneous fluid-like bodies. Furthermore, we shall assume that the body under consideration can only undergo isochoric motions and thus the bodies that are considered are **incompressible inhomogeneous fluid-like bodies**. A few remarks concerning the earlier studies that take into account spatial gradients of the density are in order. The pioneering work of Korteweg [1901] assumed that the stress in the fluid could depend on the density gradients. Korteweg introduced his model without discussing the possibility that it could be used to describe the behavior of inhomogeneous fluids. It was a straightforward extension of the homogeneous compressible classical Euler fluid to one that includes higher Eulerian spatial gradients. The model that we introduce starts with the clear intent of modeling inhomogeneous fluids. It transpires that we can reduce it to a representation in which the model can be expressed in terms of the Eulerian gradient. However, we could choose to use the Lagrangian derivative and study the problem from a Lagrangian point of view. Within such a framework our model will clearly bring out the fact that it is an inhomogeneous body. By Korteweg defining the model in terms of the Eulerian gradient and not explicitly mentioning that he is considering inhomogeneous fluids we can only infer that his intent was not the study of inhomogeneous fluids. The same remarks apply to the models developed by Goodman and Cowin as they do not explicitly develop the model from a Lagrangian perspective, which one would have to do if one is to develop models for inhomogeneous bodies. The difference between the model proposed by Korteweg and that considered in this paper cannot be overemphasized. While the model proposed by Korteweg leads to equations of motion that require boundary conditions for the density, the model developed herein makes no such demands. Dependence of the stresses on the gradients of the density in our model are due to its being inhomogeneous, and the constraint of incompressibility ensures that we do not need additional boundary conditions (how this transpires is discussed in some detail later).

Dunn and Serrin [1985] developed a framework within which they considered bodies wherein the stresses depend on the gradient of the density. Their work however requires notions such as interstitial working etc., that introduces a whole host of ideas that are not appealed to in our work. Moreover, the body that they consider is homogeneous.

The assumption that the body is inhomogeneous and that the specific Helmholtz free energy depends on the gradient of the density leads to models in which the stress depends upon tensor product of the spatial density gradients, among other quantities. This is similar to terms that appear in the model for granular materials due to Goodman and Cowin [1972]. (There are other models for granular materials based on notions in plasticity that do not feature such higher density gradients, e.g., Cowin and Mehrabadi [1978, 1983].) However, a cautionary note is in order. The model developed by Goodman and Cowin [1972] was for compressible granular materials that are homogeneous. Here, we have developed models for incompressible inhomogeneous bodies. Nonetheless, the similarity is quite striking.

Goodman and Cowin [1972] develop their model within a framework that postulates additional balance laws than just those for mass, linear and angular momentum, and energy. They introduce an additional balance law in which primitive notions such as equilibrated forces etc., (see Cowin and Goodman [1972]) are introduced, the physical origins for the same not being transparent. Moreover, they appeal to the Claussius-Duhem inequality to obtain the restrictions on the constitutive relations, a now standard procedure in continuum mechanics. Our development of the model does not appeal to additional balance laws and we do not enforce the second law in the form of the Claussius-Duhem inequality.

The plan of the paper is as follows. First, in Sect. 2, we develop the thermomechanical setting by introducing the elementary kinematical variables, the basic balance laws and the second law of thermodynamics. Then in Sect. 3, we develop the constitutive equations for the Cauchy stress that involve various special cases of incompressible fluid models. In Sect. 4, we discuss mathematical issues connected with the model: we define the notion of solution in the general case and consider steady shear flows of this granular fluid between two parallel plates. In both cases we observe that no boundary conditions for the density are needed to find a solution to the problem.

2. Thermomechanical setting

(a) *Kinematics*

Let $\mathcal{B}$ denote an abstract body and let κ_R and κ_t denote the reference placement and current placement, respectively. Furthermore, let $\kappa_R(\mathcal{B})$ and $\kappa_t(\mathcal{B})$ denote the reference configuration (for the sake of convenience we shall choose the initial configuration as the reference though any configuration even one not taken by the body would suffice) and current configuration (we refer the reader to Truesdell [1991] for a detailed discussion of the notions of body, placement, configuration, etc.). The placements are assumed to be one-to-one and by a motion one means a one parameter family of placements, the parameter being time. One can then associate a mapping $\chi_{\kappa_R} : \kappa_R(\mathcal{B}) \times \mathbb{R} \to \kappa_t(\mathcal{B})$ such that†

$$x = \chi_{\kappa_R}(X,t)\,, \qquad\qquad \text{for } X \in \kappa_R(\mathcal{B})\,, x \in \kappa_t(\mathcal{B})\,. \tag{2.1}$$

We shall suppose that the deformation gradient $\mathbf{F}_{\kappa_R}$ defined through

$$\mathbf{F}_{\kappa_R}(X,t) = \frac{\partial \chi_{\kappa_R}}{\partial X}(X,t) \tag{2.2}$$

is non-singular. In fact, we shall suppose that $\det \mathbf{F}_{\kappa_R} > 0$. Henceforth, for the sake of convenience we shall omit the suffix $_{\kappa_R}$.

We also define the velocity field through

$$\mathbf{v}(X,t) = \frac{\partial \chi}{\partial t}(X,t) \qquad \Longrightarrow \qquad \mathbf{v}(x,t) = \mathbf{v}(\chi^{-1}(x,t),t)\,. \tag{2.3}$$

Any (scalar) property φ associated with an abstract point $P \in \mathcal{B}$ can be expressed as (analogously we proceed for vectors or tensors)

$$\varphi = \varphi(P,t) = \hat{\varphi}(X,t) = \hat{\varphi}(\chi^{-1}(x,t),t) = \tilde{\varphi}(x,t) = \tilde{\varphi}(\chi(X,t),t)\,. \tag{2.4}$$

Thus, $(2.3)_a$ defines $\hat{\mathbf{v}}$ while $(2.3)_b$ defines $\tilde{\mathbf{v}}$. We shall however merely use the notation $\mathbf{v}$ and whether $\hat{\mathbf{v}}$ or $\tilde{\mathbf{v}}$ is intended would be obvious from the context.

We introduce the derivatives

$$\dot{\varphi} := \frac{\partial \hat{\varphi}}{\partial t}\,, \qquad \varphi_{,t} := \frac{\partial \tilde{\varphi}}{\partial t}\,, \qquad \nabla_X \varphi = \frac{\partial \hat{\varphi}}{\partial X}\,, \qquad \nabla_x \varphi := \frac{\partial \tilde{\varphi}}{\partial x}\,. \tag{2.5}$$

It follows from above notations and definitions that

$$\dot{\varphi} = \varphi_{,t} + \nabla_x \varphi \cdot \mathbf{v}\,. \tag{2.6}$$

† It is customary to denote x and X which are points in an Euclidean space as bold face quantities. We however choose not to do so. On the other hand, all vectors, and higher order tensors are indicated by bold face.

(b) Incompressibility

We say that a body is **incompressible** if

$$\int_{\mathcal{P}_R} dX = \int_{\mathcal{P}_t} dx \qquad \text{for all } \mathcal{P}_R \subset \kappa_R(\mathcal{B}) \ \text{ with } \mathcal{P}_t := \chi(\mathcal{P}_R)\,.$$

This immediately implies that

$$\det \mathbf{F}(X,t) = 1 \qquad \text{for all } X \in \kappa_R(\mathcal{B})\,. \tag{2.7}$$

If $\det \mathbf{F}$ is continuously differentiable with respect to time, then since

$$\frac{d}{dt}\det \mathbf{F} = \operatorname{div}\mathbf{v}\det\mathbf{F}\,,$$

it follows from (2.7) that

$$\operatorname{div}\mathbf{v}(x,t) = 0 \quad \text{ for all } t \in \mathbb{R} \text{ and } x \in \kappa_t(\mathcal{B})\,. \tag{2.8}$$

(c) Isothermal Processes

The local forms of the balance laws of continuum physics (for mass, linear momentum, angular momentum, internal energy ϵ) can be written as

$$\dot\varrho + \varrho \operatorname{div}\mathbf{v} = 0\,, \quad \varrho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T} + \varrho\mathbf{b}\,, \quad \mathbf{T} = \mathbf{T}^T\,, \tag{2.9}$$

$$\varrho\dot\epsilon = \mathbf{T}\cdot\nabla\mathbf{v} - \operatorname{div}\mathbf{q} + \varrho r\,, \tag{2.10}$$

and the second law†, involving the specific entropy η, takes the form

$$\varrho\dot\eta + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\varrho r}{\theta} \geq 0\,, \tag{2.11}$$

where θ is the temperature, $\mathbf{q}$ the heat flux, $\mathbf{b}$ and r represent the densities of the the external body forces and the external heat source acting on the body, respectively. The superscript ^{T} denotes the transpose of the linear transformation.

We shall express (2.11) differently in the form

$$\varrho\dot\eta + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) - \frac{\varrho r}{\theta} = \varrho\zeta\,, \tag{2.12}$$

where ζ is the rate of the entropy production per unit mass, and we shall require that

$$\zeta \geq 0\,. \tag{2.13}$$

† Whether the second law can be expressed in a local form or whether it ought to be only enforced for a system, in a global form, is a matter of some debate. Here, we shall express the second law in its local form.

Multiplying (2.12) by θ and subtracting the product from (2.10) we obtain

$$\varrho(\dot{\epsilon} - \theta\dot{\eta}) = \mathbf{T} \cdot \nabla\mathbf{v} - \frac{1}{\theta}\mathbf{q} \cdot \nabla\theta - \varrho\,\theta\,\zeta\,. \tag{2.14}$$

We shall restrict the body to undergo **isothermal** processes in which the body has uniform temperature. Then, (2.13) and (2.14) lead to

$$\mathbf{T} \cdot \mathbf{D} - \varrho\dot{\psi} = \xi \quad \text{with} \;\; \xi \geq 0\,, \tag{2.15}$$

where we have used the symmetry of $\mathbf{T}$ (see $(2.9)_3$), and introduced the notation

$$\psi := \epsilon - \theta_0\eta\,, \quad \xi := \varrho\,\theta_0\,\zeta \quad \text{and} \;\; 2\mathbf{D} := 2\mathbf{D}(\mathbf{v}) := \nabla_x\mathbf{v} + (\nabla_x\mathbf{v})^T\,,$$

θ_0 being the uniform temperature. We can assume without loss of any generality that the Helmholtz potential ψ is bounded from below, i.e.,

$$\psi \geq 0\,.$$

Taking into the account the constraint of incompressibility (2.8) and that the body is undergoing isothermal processes and that its temperature is uniform, one can show that the system of equations (2.9)-(2.10) governing such flows reduce to

$$\operatorname{tr}\,\mathbf{D} = \operatorname{div}\mathbf{v} = 0\,, \quad \dot{\varrho} = \varrho_{,t} + \nabla_x\varrho \cdot \mathbf{v} = 0\,, \tag{2.16}$$

$$\varrho(\mathbf{v}_{,t} + [\nabla_x\mathbf{v}]\mathbf{v}) = \operatorname{div}\mathbf{T} + \varrho\mathbf{b}\,, \tag{2.17}$$

and the constitutive quantities $\mathbf{T}$, ξ and ψ satisfy (cf. (2.15))

$$\mathbf{T} \cdot \mathbf{D} - \varrho\dot{\psi} = \xi\,, \quad \xi \geq 0\,, \quad \psi \geq 0\,. \tag{2.18}$$

(d) Inhomogeneous incompressible granular materials

The local form for the balance of mass in the form (2.9) implies that

$$\varrho_{,t} + \nabla_x\varrho \cdot \mathbf{v} + \varrho\,\operatorname{div}\mathbf{v} = 0\,,$$

and thus in an incompressible material

$$\dot{\varrho} = \varrho_{,t} + \nabla_x\varrho \cdot \mathbf{v} = 0\,. \tag{2.19}$$

Using notation introduced in (2.4), it follows from (2.19) that

$$\varrho = \tilde{\varrho}(x,t) = \tilde{\varrho}(\chi(X,t),t) = \hat{\varrho}(X,t) = \hat{\varrho}(X,0) =: \varrho_R(X)\,, \tag{2.20}$$

i.e., the density of a fixed particle does not change, though the density can change from point to point in the abstract body. Thus if the body is homogeneous, then the density ϱ is constant for all time at every point in the body. However, it is possible that the body is inhomogeneous, in which case the density can vary in the body in the reference configuration and hence also in the current configuration. This point cannot be overemphasized. Our model is consequence of the inhomogeneity of the body and can be considered purely as a model for an inhomogeneous fluid. However, in view of the resemblance of the model to those developed by Goodman and Cowin [1972] and Mehrabadi and Cowin [1978] we feel that it can also be viewed as a model for granular materials. And given the fact that granular materials are naturally inhomogeneous such an assignment seems apt.

We would like to consider granular materials whose specific Helmholtz potential depends on the density and the gradient of the density in the current configuration, i.e., we are interested in bodies whose Helmholtz potential has the form

$$\psi = \Psi(\tilde{\varrho}(x,t), \nabla_x \tilde{\varrho}(x,t)) = \tilde{\psi}(x,t) \,. \tag{2.21}$$

By virtue of the fact that

$$\nabla_x \tilde{\varrho} = \mathbf{F}^{-T}(X,t) \nabla_X \hat{\varrho} \,,$$

we realize that the assumption (2.21) is tantamount to assuming

$$\psi = \Psi(\varrho_R(X), \mathbf{F}^{-T}(X,t) \nabla_X \varrho_R(X)) \,,$$

and we note that this characterizes a special subclass of bodies whose Helmholtz potential is a function of $\varrho_R(X)$, $\nabla_X \varrho_r(X)$ and $\mathbf{F}(X,t)$, i.e.,

$$\psi = \Psi^*(\varrho_R(X), \nabla_X \varrho_R(X), \mathbf{F}(X,t)) = \hat{\psi}(X,t) \,. \tag{2.22}$$

The form for the stored energy (2.22) is a generalization of inhomogeneous incompressible **elastic** bodies wherein

$$\psi = \Psi^{\#}(\varrho_R(X), \mathbf{F}(X,t)) \,,$$

to an elastic material whose stored energy also depends on $\nabla \varrho_R(X)$. We shall see that this class of constitutive equations includes as a special case models for granular fluid-like materials. Notice that we have chosen a very special subclass of models whose Helmholtz potential depends on $\nabla \varrho_R(X)$ and $\mathbf{F}(X,t)$ in a very special manner which leads to its Eulerian prescription depending only on $\tilde{\varrho}(x,t)$ and $\nabla_x \tilde{\varrho}(x,t)$.

We will require that Ψ introduced in (2.21) is of such a form that

$$\Psi_{,\mathbf{z}} \otimes \mathbf{z} = \mathbf{z} \otimes \Psi_{,\mathbf{z}} \qquad \text{with } \mathbf{z} = \nabla_x \varrho \,. \tag{2.23}$$

The assumption (2.23) is fulfilled if for example Ψ depends on $\nabla_x \varrho$ through $|\nabla_x \varrho|$ only.

We shall assume that the rate of dissipation is given by the following constitutive relation:

$$\xi = 2\,\nu\,\mathbf{D} \cdot \mathbf{D}\,,$$

where ν can be either a constant, or it can be a function of the density, the symmetric part of the velocity gradient $\mathbf{D}$ specifically through $\mathbf{D} \cdot \mathbf{D}$, or the mean normal stress, i.e. the pressure,

$$p := -\frac{1}{3}\mathrm{tr}\ \mathbf{T}\,,$$

or it can depend on any or all of them. We shall consider the most general case within this setting by assuming that

$$\xi = \Xi(p, \varrho, \mathbf{D}) = 2\nu(p, \varrho, \mathbf{D} \cdot \mathbf{D})\, \mathbf{D} \cdot \mathbf{D}\,. \tag{2.24}$$

Clearly, if $\nu \geq 0$ then automatically $\xi \geq 0$, ensuring that the second law is complied with.

3. Constitutive equation for the Cauchy stress

Since $\mathbf{z} := \nabla_x \varrho$ we first observe that the second equation in (2.16) implies that

$$\dot{\mathbf{z}} = [\nabla_x \varrho]_{,t} + v_k [\nabla_x \varrho]_{,x_k} = -\varrho_{,x_k} \nabla_x v_k = -\nabla_x \mathbf{v}\, \nabla_x \varrho\,. \tag{3.1}$$

This together with (2.21) and the second equation in (2.16) leads to

$$\dot{\psi} = \Psi_{,\varrho}\, \dot{\varrho} + \Psi_{,\mathbf{z}}\, \dot{\mathbf{z}} = -\Psi_{,\mathbf{z}} \otimes \nabla_x \varrho \cdot (\nabla_x \mathbf{v})^T\,. \tag{3.2}$$

Using the symmetry relation (2.23) we then conclude that

$$\mathbf{T} \cdot \mathbf{D} - \varrho\dot{\psi} = \Big(\mathbf{T} + \varrho\, \Psi_{,\mathbf{z}} \otimes \nabla_x \varrho\Big) \cdot \mathbf{D}\,. \tag{3.3}$$

For a fixed $\mathbf{T}$ there are plenty of $\mathbf{D}$'s that satisfy the constitutive equation for the rate of dissipation (2.24) together with the incompressibility constraint

$$\mathrm{tr}\ \mathbf{D} = 0\,, \tag{3.4}$$

and the reduced thermomechanical identity (obtained by inserting (3.3) and (2.24) into (2.18))

$$\Big(\mathbf{T} + \varrho\,\Psi_{,\mathbf{z}} \otimes \nabla_x \varrho\Big) \cdot \mathbf{D} - \Xi = 0\,. \tag{3.5}$$

Following the procedure initiated in Rajagopal [1995], which has led to the development of reasonable constitutive relations in a variety of areas, in addition to yielding most of the classical constitutive models in viscoelasticity, plasticity and other dissipative theories, we will consider such a form for $\mathbf{T}$ that comes out from the maximization of Ξ of the form (2.24) with respect to $\mathbf{D}$ that is subject to the constraints (3.4) and (3.5). This leads to the equation

$$(1+\lambda_1)\Xi_{,\mathbf{D}} - \lambda_1(\mathbf{T} + \varrho\,\Psi_{,\mathbf{z}} \otimes \nabla_x\varrho) - \lambda_2 \mathbf{I} = 0\,, \tag{3.6}$$

where λ_1 and λ_2 are the Lagrange multipliers due to the constraints (3.4) and (3.5). We eliminate them as follows. Taking the scalar product of (3.6) with $\mathbf{D}$, and using (3.4) and (3.5) we obtain

$$\frac{1+\lambda_1}{\lambda_1} = \frac{\Xi}{\Xi_{,\mathbf{D}} \cdot \mathbf{D}}\,. \tag{3.7}$$

Note that

$$\Xi_{,\mathbf{D}} = 4\Big(\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D}) + \nu_{,\mathbf{D}}(p,\varrho,\mathbf{D}\cdot\mathbf{D})\mathbf{D}\cdot\mathbf{D}\Big)\mathbf{D}\,. \tag{3.8}$$

Consequently, $\operatorname{tr}\,\Xi_{,\mathbf{D}} = 0$ by virtue of (3.4). Thus, taking the trace of (3.6) we have

$$-\frac{\lambda_2}{\lambda_1} = -p - \frac{1}{3}\varrho\,\Psi_{,\mathbf{z}} \cdot \nabla_x\varrho \qquad \text{with } p = -\frac{1}{3}\operatorname{tr}\,\mathbf{T}\,. \tag{3.9}$$

Using (3.7)–(3.9), we finally see that (3.6) takes the form

$$\begin{aligned} \mathbf{T} &= -p\mathbf{I} - \varrho\left(\Psi_{,\mathbf{z}} \otimes \nabla_x\varrho - \frac{1}{3}\Psi_{,\mathbf{z}} \cdot \nabla_x\varrho\mathbf{I}\right) + 2\,\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D})\mathbf{D} \\ &=: \tilde{\mathbf{T}}(p,\varrho,\mathbf{D}(\mathbf{v}))\,. \end{aligned} \tag{3.10}$$

Example. Consider the specific Helmholtz potential of the form

$$\Psi(\varrho, \nabla_x\varrho) = \beta_0(\varrho) + \frac{\beta_1}{2\varrho}|\nabla_x\varrho|^2\,, \tag{3.11}$$

where β_0 is a function of ϱ while β_1 is a positive constant. Thus

$$\Psi_{,\mathbf{z}} = \beta_1\mathbf{z} \quad \text{ and } \quad \Psi_{,\mathbf{z}} \otimes \nabla_x\varrho = \frac{\beta_1}{\varrho}\nabla_x\varrho \otimes \nabla_x\varrho\,,$$

clearly satisfying the symmetry condition (2.23).

On choosing Ψ to be that given by (3.11), we find that (3.10) simplifies to

$$\mathbf{T} = -p\mathbf{I} - \beta_1\left(\nabla_x\varrho \otimes \nabla_x\varrho - \frac{1}{3}|\nabla_x\varrho|^2\mathbf{I}\right) + 2\,\nu(p,\varrho,\mathbf{D}\cdot\mathbf{D})\mathbf{D}\,. \tag{3.12}$$

If, in addition, $\nu(p, \varrho, \mathbf{D} \cdot \mathbf{D}) = \nu_0$ we obtain

$$\mathbf{T} = -p\mathbf{I} - \beta_1 \left(\nabla_x \varrho \otimes \nabla_x \varrho - \frac{1}{3} |\nabla_x \varrho|^2 \mathbf{I} \right) + 2\, \nu_0 \mathbf{D} \,. \tag{3.13}$$

It is worth observing that the pressure, unlike that in the case of the compressible Navier-Stokes fluid is not given by an equation of state, nor is it like that in the case of the incompressible Navier-Stokes fluid wherein it is independent of the density.

4. Mathematical issues

We have shown that the class of models that we have considered that have uniform temperature and undergo isothermal processes in Ω_t are governed by the system of equations (2.16) and (2.17) with $\mathbf{T}$ of the form (3.10). To talk about the solution of such a system we need to assign boundary conditions. Here, we assume, for the sake of simplicity, that

$$\mathbf{v} = \mathbf{0} \quad \text{on } [0, T] \times \partial\Omega \,. \tag{4.1}$$

One could also consider other boundary conditions such as Navier's slip, stick-slip, or the material being stress-free.

We also assign the initial conditions

$$\varrho(0, \cdot) = \varrho_0 \quad \text{and} \quad \mathbf{v}(0, \cdot) = \mathbf{v}_0 \quad \text{in } \Omega_0 \,. \tag{4.2}$$

We will introduce a notion of weak solution to such an initial-boundary value problem and find an analytical solution in a special case. In doing so, we shall observe that no boundary conditions on the density are needed even though the equation (2.17) with $\mathbf{T}$ of the form (3.1) represents a second order equation for the density ϱ.

We will also emphasize that the choice of the constitutive equations for ψ and ξ dictates naturally the function spaces where the solution to this boundary value problem should be constructed.

(*a*) *Weak solution*

First, we define the several function spaces needed below.

Let r be such that $1 \le r < \infty$. The Lebesgue spaces $L^r(\Omega_t)$ consists of functions $f : \Omega_t \to \mathbb{R}$ that are Lebesgue-measurable and $\int_{\Omega_t} |f(x)|^r \, dx$ is finite. The Sobolev space $W^{1,r}(\Omega_t)$ is the space of Lebesgue-measurable functions $f : \Omega_t \to \mathbb{R}$ such that $\partial_{x_i} f$ exists in a weak sense, and f and $\partial_{x_i} f$ belong to $L^r(\Omega_t)$. Both $L^r(\Omega_t)$ and $W^{1,r}(\Omega_t)$ are Banach spaces with the norms $\|f\|_r := (\int_{\Omega_t} |f(x)|^r\,dx)^{1/r}$ and $\|f\|_{1,r} := (\|f\|_r + \|\nabla_x f\|_r)^{1/r}$, respectively.

Let $(X(\Omega_t), \|\cdot\|_{X(\Omega)})$ be a Banach space of scalar functions defined in Ω_t. Then $X(\Omega_t)^3$ represents the space of vector-valued functions whose components belong to $X(\Omega_t)$.

We introduce a subspace $W^{1,r}_{0,div}(\Omega_t)$ of $W^{1,r}(\Omega_t)^3$ defined as the closure (w.r.t the norm $\|\mathbf{f}\|_{1,r}$) of all smooth functions $\mathbf{f}$ with compact support in Ω_t such that $\operatorname{div}\mathbf{f} = 0$.

Finally, if $Y(\Omega_t)$ is any Banach space, $T \in (0,\infty)$ and $1 \le q \le \infty$, then $L^q(0,T;Y(\Omega_t))$ denotes the Bochner space formed by functions $g:(0,T)\to Y(\Omega_t)$ such that, for $1 \le q < \infty$, $\|g\|_{L^q(0,T;Y(\Omega_t))} := (\int_0^T \|g(t)\|^q_{Y(\Omega_t)}\,dt)^{1/q}$ is finite. The norm in $L^\infty(0,T;Y(\Omega_t))$ is defined as infimum of $\sup_{t\in[0,T]\setminus E} \|g(t)\|_{Y(\Omega_t)}$, where infimum is taken over all subsets E of $[0,T]$ having zero Lebesgue measure.

Assume that

$$0 \le \sup_{0\le t\le T} \int_{\Omega_t} \Psi(\varrho, \nabla_x \varrho)\,dx < \infty\,. \tag{4.3}$$

Depending on the structure of Ψ, see Example in Sect. 2, one can conclude that

$$\varrho \in L^\infty(0,T;X_\psi)\,. \tag{4.4}$$

Referring again to Example in Sect. 2, $X_\psi = W^{1,2}(\Omega_t)$.

Let us also assume that

$$\int_0^T \int_{\Omega_t} \nu(p,\varrho,\mathbf{D}\cdot\mathbf{D})\,\mathbf{D}\cdot\mathbf{D}\,dx\,dt < \infty\,. \tag{4.5}$$

Again, based on the choice of the apparent viscosity, we pick the space X_{dis} so that

$$\mathbf{v} \in X_{dis}\,. \tag{4.6}$$

Thus, if $\nu \equiv \nu_0$, where ν_0 is a positive constant, then $X_{dis} = L^2(0,T;W^{1,2}_{0,div}(\Omega_t))$; or if $\nu = \nu_0|\mathbf{D}|^{r-2}$ then $X_{dis} = L^r(0,T;W^{1,r}_{0,div}(\Omega_t))$.

Let the functions introduced in the initial conditions (4.2) satisfy (α_1, α_2 being positive constants)

$$\varrho_0 \in X_\psi \quad \text{and } \alpha_1 \le \varrho_0 \le \alpha_2\,, \tag{4.7}$$

$$\mathbf{v}_0 \in L^2(\Omega_0)\,, \quad \operatorname{div}\mathbf{v}_0 = 0 \quad \text{and } \mathbf{v}_0\cdot\mathbf{n} = 0 \ \text{ on } \partial\Omega_0\,. \tag{4.8}$$

We say that the two-tuple of functions $(\varrho, \mathbf{v})$ is a weak solution to (2.16), (2.17), (3.10), (4.1), (4.2) and (4.7)–(4.8) provided that

$$\varrho \in L^\infty(0,T;X_\psi) \quad \text{and } \alpha_1 \le \varrho(x,t) \le \alpha_2 \quad \text{a. a. } (x,t) \in Q_T\,, \tag{4.9}$$

$$\mathbf{v} \in L^\infty(0,T;L^2(\Omega_t)) \cap X_{dis}\,, \tag{4.10}$$

$$\text{Eq. (2.16) is met in the sense of distributions} \tag{4.11}$$

and

$$\int_{\Omega_t} \varrho(\mathbf{v}_{,t} + v_k \mathbf{v}_{,x_k}) \cdot \boldsymbol{\varphi}\, dx + \int_{\Omega_t} \tilde{\mathbf{T}}(p,\varrho,\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx = \int_{\Omega_t} \varrho \mathbf{b} \cdot \boldsymbol{\varphi}\, dx \tag{4.12}$$

is valid for all $\boldsymbol{\varphi}$ that are smooth functions free of divergence, and almost all $t \in [0,T]$.

Note that for such $\boldsymbol{\varphi}$'s we have

$$\begin{aligned} \int_{\Omega_t} \tilde{\mathbf{T}}(p,\varrho,\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx &= \int_{\Omega_t} \nu(p,\varrho,|\mathbf{D}(\mathbf{v})|^2)\mathbf{D}(\mathbf{v}) \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx \\ &\quad - \int_{\Omega_t} \varrho \Psi_{,\mathbf{z}}(\varrho, \nabla_x \varrho) \otimes \nabla_x \varrho \cdot \mathbf{D}(\boldsymbol{\varphi})\, dx \end{aligned} \tag{4.13}$$

The purpose of this note is not to establish the existence of such a solution, rather we just want to point out that the definition of a weak solution does not require any information concerning the density on the boundary.

We also want to emphasize that the choice of the function spaces we have made in the above definition come out naturally from the energy estimates. Formally setting $\boldsymbol{\varphi} = \mathbf{v}$ in (4.12) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} \varrho |\mathbf{v}|^2\, dx + \int_{\Omega} \tilde{\mathbf{T}}(p,\varrho,\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\mathbf{v})\, dx = \int_{\Omega_t} \varrho \mathbf{b} \cdot \mathbf{v}\, dx \tag{4.14}$$

and using Eq. (2.18) instead of Eq. (4.13) we see that the second term in (4.14) can be expressed as

$$\begin{aligned} \int_{\Omega_t} \tilde{\mathbf{T}}(p,\varrho,\mathbf{D}(\mathbf{v})) \cdot \mathbf{D}(\mathbf{v}))\, dx &= \int_{\Omega_t} \Xi(p,\varrho,\mathbf{D}(\mathbf{v}))\, dx + \int_{\Omega_t} \varrho \dot{\Psi}\, dx \\ &= \int_{\Omega_t} \Xi(p,\varrho,\mathbf{D}(\mathbf{v}))\, dx + \int_{\Omega_t} \frac{d}{dt}(\varrho\Psi)\, dx \\ &= \int_{\Omega_t} \nu(p,\varrho,|\mathbf{D}(\mathbf{v})|^2)\, |\mathbf{D}(\mathbf{v})|^2\, dx + \frac{d}{dt}\int_{\Omega_t} \varrho\Psi\, dx\,, \end{aligned}$$

where we used the fact that $\dot{\varrho} = 0$ (see (2.16)). Since ϱ is bounded from below (and above) and the right hand of (4.14) can be easily controlled (if $\mathbf{b}$ is, for example, also bounded), we finally conclude from (4.14) that exactly the information that we used at the beginning of this section (see (4.3) and (4.5)) is that which is obtained.

(*b*) *Steady simple shear flow*

Consider a **steady** flow between two infinite parallel plates of an incompressible granular material whose stress $\mathbf{T}$ is of the form (3.13). We look for the velocity $\mathbf{v}$ of the form

$$\mathbf{v} = (u(y), 0, 0)\,. \tag{4.15}$$

The mass balance equation then implies

$$u(y)\,\varrho_{,x} = 0\,, \tag{4.16}$$

which implies that ϱ is independent of x. Consequently, the Cauchy stress tensor $\mathbf{T}$ takes the form

$$\begin{bmatrix} -p + \frac{\beta_1}{3}\big([\varrho_{,y}]^2 + [\varrho_{,z}]^2\big) & \nu_0 u' & 0 \\ \nu_0 u' & -p - \frac{2\beta_1}{3}[\varrho_{,y}]^2 + \frac{\beta_1}{3}[\varrho_{,z}]^2 & -\beta_1 \varrho_{,y}\varrho_{,z} \\ 0 & -\beta_1 \varrho_{,y}\varrho_{,z} & -p - \frac{2\beta_1}{3}[\varrho_{,z}]^2 + \frac{\beta_1}{3}[\varrho_{,y}]^2 \end{bmatrix}. \tag{4.17}$$

Note first that

$$T_{11} - T_{22} = \beta_1(\varrho_{,y})^2\,, \quad T_{22} - T_{33} = \beta_1\Big((\varrho_{,z})^2 - (\varrho_{,y})^2\Big)\,, \quad T_{11} - T_{33} = \beta_1(\varrho_{,z})^2\,,$$

which means that the material exhibits normal stress differences in a simple shear flow. It is also obvious that if ν depends on $\mathbf{D}\cdot\mathbf{D}$ the fluid can shear thin (or shear thicken).

Let us for the sake of simplicity assume that ϱ is independent of z, i.e. $\varrho = \varrho(y)$ and $\varrho_{,z} = 0$. Then the governing equations simplify to

$$p_{,x} = \nu_0\, u''\,, \quad p_{,y} + \frac{2}{3}\beta_1\Big((\varrho_{,y})^2\Big)_{,y} = 0\,, \quad p_{,z} = 0\,. \tag{4.18}$$

The second and third equations then imply that

$$p + \frac{2}{3}\beta_1(\varrho_{,y})^2 = h(x)\,. \tag{4.19}$$

Then the first equation in (4.18) leads to

$$h'(x) = \nu_0 u''(y)\,,$$

which implies $h(x) = C_0 x + C_1$ and the velocity subject to the boundary conditions (4.1) takes the familiar classical parabolic profile. Finally, from (4.19) we obtain

$$p(x,y) = C_0 x + C_1 - \frac{2}{3}\beta_1(\varrho_{,y}(y))^2\,. \tag{4.20}$$

We notice that the boundary condition (4.1) completely determines the form of the velocity. On the other hand they are insufficient to determine completely the form of the density and the pressure. Of course, one can augment the boundary conditions with the mass flow rate, or a symmetry conditions, or merely the requirement of boundedness of solutions or an asymptotic structure to the solution can help. However, one cannot expect that these complementary requirements always lead to a determinate system of equations. For instance, one could require that

$$\int_0^h \varrho(y)\, u(y)\, dy = Q\,, \tag{4.21}$$

where Q is the mass flow rate. This however does not pick out a specific solution, and more than one solution is possible that satisfies the integral condition (4.21) and the system of equations we are interested in. In fact, even in the case of compressible granular materials such is indeed the case (see Gudhe *et al.* [1994]).

The inability to fix the density and the pressure stem from our lack of knowledge concerning the precise inhomogeneity of the body in its reference configuration. For this specific problem, if (X, Y, Z) are the coordinates of $\kappa_R(P)$ and (x, y, z) are the coordinates of $\kappa_t(P)$, for some point $P \in \mathcal{B}$, then (4.15) implies $y = Y$ and hence knowing the density in the reference state fixes the density in the current state. We can then proceed to determine the pressure. In fact, if the problem is set within the context of a fully time-dependent structure, information such as the initial density and initial density gradient would be available and would lead to determining the density field. It is also worth observing that the special assumption (4.15) and the fact that we have an incompressible body uncouple the equations so that the velocity can be determined independently thereby rendering a condition such as (4.21) redundant in this case.

It is also worth noting that if we require the Helmholtz potential to be merely a function of the density, then its form will not in any way influence the form of the stress.

5. Concluding remarks

We have developed a model for the fluid-like behavior of materials that are inhomogeneous and incompressible whose stress depends on the density, the Lagrangian gradient of the density in the reference configuration and the deformation gradient in a special manner. The model includes the classical incompressible linearly viscous fluid as a special case. Unlike previous theories for bodies whose stress depends on the spatial gradient of the density where additional primitive notions such as interstitial working (see Dunn and Serrin [1985]) or additional balance laws (see Cowin and Goodman [1972, 1976]) are introduced, the model here has been developed without the necessity for such extraneous notions.

It was assumed that the processes take place so that the rate of entropy production is maximized and this is a much stronger assumption than what is usually required, namely that the rate of production be non-negative. The model that has been developed fits within the framework of bodies that possess multiple natural configurations that has been used to develop models to describe plethora of phenomena: classical plasticity, single crystal plasticity, viscoelasticity, twinning, solid to solid phase transition, crystallization of polymers and others.

It is important to recognize that the model developed here leads to equations of motion that do not require boundary conditions for the density, and this is interestingly a consequence of requiring that the body under consideration be incompressible.

An interesting feature of the work is that the choice of the specific Helmholtz potential ψ and the rate of dissipation ξ determine in a natural manner the function spaces in which the solution lies.

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