Transversals in Latin Squares

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A *latin square* of order n is an $n \times n$ matrix in which each of n symbols occurs exactly once in each row and once in each column.

e.g. $\begin{array}{|cccc} \hline 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{array}$ is a latin square of order 4.

Hence a latin square is a 2 dimensional permutation.

Quasigroups are latin squares!

The cancellation laws

$$ax = ay \implies x = y$$

$$xa = ya \implies x = y$$

imply that every quasigroup table is a latin square.

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 1 & 3 \\ 3 & 3 & 1 & 4 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{array} \qquad \text{multiplication in } \mathbb{Z}_5^*.$$

The 16 card trick

Take the picture cards (aces, kings, queens & jacks) from a standard pack and arrange them in a 4×4 array so that each row and column contains one card of each suit and one card of each rank.

There are 6912 ways to do the puzzle, but 20922789881088 ways to fail to do it.

One solution

$$\begin{array}{cccc}
\spadesuit A & \heartsuit K & \diamondsuit J & \clubsuit Q \\
\heartsuit Q & \spadesuit J & \clubsuit K & \diamondsuit A \\
\clubsuit J & \diamondsuit Q & \heartsuit A & \spadesuit K \\
\diamondsuit K & \clubsuit A & \spadesuit Q & \heartsuit J
\end{array}$$

Each solution is the superposition of two latin squares

$$\begin{array}{cccc} \spadesuit & \heartsuit & \diamondsuit & \clubsuit \\ \heartsuit & \spadesuit & \clubsuit & \diamondsuit \\ \clubsuit & \diamondsuit & \heartsuit & \spadesuit \\ \diamondsuit & \clubsuit & \spadesuit & \heartsuit \end{array} \qquad \begin{array}{cccc} A & K & J & Q \\ Q & J & K & A \\ J & Q & A & K \\ K & A & Q & J \end{array}$$

These squares have a special property – they are called *orthogonal mates*.

When we overlay them each ordered pair of symbols occurs once.

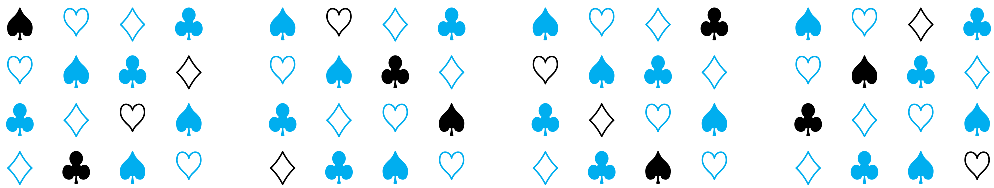
Transversals

A transversal of a latin square is a set of entries which includes exactly one entry from each row and column and one of each symbol.

$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$
$\heartsuit$	$\spadesuit$	$\clubsuit$	$\diamondsuit$
$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$
$\diamondsuit$	$\clubsuit$	$\spadesuit$	$\heartsuit$

A	K	J	Q
Q	J	K	A
J	Q	A	K
K	A	Q	J

Theorem 1 *A latin square has an orthogonal mate iff it can be decomposed into disjoint transversals.*



$$
\begin{array}{cccccccc}
1_a & 2_b & 3_c & 4_d & 5_e & 6_f & 7_g & 8_h \\
7_b & 8_a & 5_d & 6_c & 2_f & 4_e & 1_h & 3_g \\
2_c & 1_d & 6_a & 3_b & 4_g & 5_h & 8_e & 7_f \\
8_d & 7_c & 4_b & 5_a & 6_h & 2_g & 3_f & 1_e \\
4_f & 3_e & 1_g & 2_h & 7_a & 8_b & 5_c & 6_d \\
6_e & 5_f & 7_h & 8_g & 1_b & 3_a & 2_d & 4_c \\
3_h & 6_g & 2_e & 1_f & 8_c & 7_d & 4_a & 5_b \\
5_g & 4_h & 8_f & 7_e & 3_d & 1_c & 6_b & 2_a
\end{array}
\qquad\qquad
\begin{array}{cccccccc}
a & b & c & d & e & f & g & h \\
b & a & d & c & f & e & h & g \\
c & d & a & b & g & h & e & f \\
d & c & b & a & h & g & f & e \\
f & e & g & h & a & b & c & d \\
e & f & h & g & b & a & d & c \\
h & g & e & f & c & d & a & b \\
g & h & f & e & d & c & b & a
\end{array}
$$

More generally, there is interest in sets of *mutually orthogonal latin squares* (MOLS), that is, sets of latin squares in which each pair is orthogonal in the above sense.

The literature on MOLS is vast.

A set of MOLS can be thought of as a latin square together with appropriate sets of transversals of that square.

Orthogonal latin squares exist for all orders $n \notin \{2, 6\}$.
For $n = 6$ there is no pair of orthogonal squares, but we get close.

Finney [1945] gives the following example which contains 4 disjoint transversals indicated by the subscripts a, b, c and d.

$$\begin{array}{cccccc}
1_a & 2 & 3_b & 4_c & 5 & 6_d \\
2_c & 1_d & 6 & 5_b & 4_a & 3 \\
3 & 4_b & 1 & 2_d & 6_c & 5_a \\
4 & 6_a & 5_c & 1 & 3_d & 2_b \\
5_d & 3_c & 2_a & 6 & 1_b & 4 \\
6_b & 5 & 4_d & 3_a & 2 & 1_c
\end{array}$$

Some terminology

Each latin square of order n can be thought of as a set of n^2 triples (row,column,symbol).

Let S_n be the symmetric group on n letters.

The natural action of $S_n \times S_n \times S_n$ on latin squares of order n is called *isotopism* (or *isotopy*) and its orbits are called *isotopy classes*.

An important special case of isotopism is the action of the diagonal subgroup of $S_n \times S_n \times S_n$. This action is called *isomorphism* and its orbits are called *isomorphism classes*.

A further group action on latin squares is provided by permutation of the elements of triples. Such images are the *conjugates* (also called *parastrophes*) of the latin square.

An arbitrary combination of a conjugacy and an isotopism is called a *paratopism* (or *paratopy*). The group of all paratopisms is isomorphic to the wreath product $S_n \wr S_3$. The orbits of its action on the set of all latin squares are called *paratopy classes*, *main classes* or *species*.

The number of transversals is a species invariant.

m	$n=4$	5	6	7	8
0	1	0	6	0	33
1	0	1	0	1	0
2	0	0	2	5	7
3	-	0	0	24	46
4	1	-	4	68	712
5	-	1	-	43	71330
6	-	-	0	-	209505
7	-	-	-	6	-
8	-	-	-	-	2024
Total	2	2	12	147	283657

The squares of order n, for $4 \leq n \leq 8$, counted according to their maximum number m of disjoint transversals. The entries in the table are counts of main classes.

Latin squares with no transversals

A latin square of order mq is said to be of *q-step type* if it can be represented by a matrix of $q \times q$ blocks A_{ij} as follows

$$\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{array}$$

where each block A_{ij} is a latin subsquare of order q and two blocks A_{ij} and $A_{i'j'}$ contain the same symbols iff $i + j \equiv i' + j' \mod m$. The following classical theorem is due to Maillet [1894].

Theorem 2 *Suppose that q is odd and m is even. No q-step type latin square of order mq possesses a transversal.*

This rules out many group tables having transversals. In particular, no cyclic group of even order has a transversal.

Odd order squares

By contrast, there is no known example of a latin square of odd order without transversals.

Conjecture 3 *Each latin square of odd order has at least one transversal.*

This conjecture is known to be true for $n \leq 9$. It is attributed to Ryser and has been open for forty years. In fact, Ryser's original conjecture was somewhat stronger: for every latin square of order n, the number of transversals is congruent to n mod 2. In 1990, Balasubramanian proved the even case.

Theorem 4 *In any latin square of even order the number of transversals is even.*

Despite this, there are many counterexamples of odd order to Ryser's original conjecture. Hence the conjecture has now been weakened to Conjecture 3 as stated.

One obstacle to proving Ryser's conjecture was recently revealed:

Theorem 5 *For every $n > 3$ there exists a latin square of order n which contains an entry that is not included in any transversal.*

This latest theorem showed existence for all $n > 3$ of a latin square of order n without an orthogonal mate.

The Δ function

We define the following function on the elements (x, y, z) of a latin square L of order n.

$$\Delta(x, y, z) = x + y - z \bmod n.$$

Lemma 6 *The sum* (mod n) *of the* Δ *values over the elements of a transversal* T *is* 0 *if* n *is odd, and* $\frac{1}{2}n$ *if* n *is even.*

Proof: By definition, T comprises one element from each row, one element from each column, and one element from each symbol. Hence,

$$\sum_{e \in T} \Delta(e) = \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i - \sum_{i=0}^{n-1} i = \tfrac{1}{2}n(n-1) \qquad \square$$

Corollary 7 *The cyclic square* $\mathbb{Z}_n$ *has no transversal if* n *is even.*

Case: $n \equiv 3 \bmod 4$

We define a latin square L of order $n \geq 7$.

$L[i,j] = i + j \bmod n,$ except for the following entries:

$L[0,0] = 1;$ (-1)

$L[0,1] = 0;$ (1)

for $i = 1, 3, \ldots, \frac{n-5}{2}$

$L[i,0] = i + 2;$ (-2)

$L[i,2] = i;$ (2)

$L[\frac{n-1}{2}, 0] = 0;$ $(\frac{n-1}{2})$

$L[\frac{n-1}{2}, \frac{n+1}{2}] = \frac{n-1}{2};$ $(-\frac{n-1}{2})$

$L[n-1, 1] = 1;$ (-1)

$L[n-1, 2] = \frac{n-1}{2};$ $(-\frac{n-3}{2})$

$L[n-1, \frac{n+1}{2}] = 0.$ $(\frac{n-1}{2})$

Consider a possible transversal which includes $(n-1, \frac{n+1}{2}, 0)$.

Partial transversals

Define a *partial transversal of length k* to be a set of k entries, each selected from different rows and columns of a latin square such that no two entries contain the same symbol.

Since not all squares of order n have a partial transversal of length n, the best we can hope for is to find one of length $n-1$. The following conjecture has been attributed by Brualdi.

Conjecture 8 *Every latin square of order n possesses a partial transversal of length $n-1$.*

There have been several claimed proofs of this conjecture.

The best reliable result to date states that there must be a partial transversal of length at least $n - O(\log^2 n)$. This was shown by Shor [1982].

This improved on a number of earlier bounds including
$\frac{2}{3}n + O(1)$ (Koksma 1969)
$\frac{3}{4}n + O(1)$ (Drake 1977)
$\frac{9}{11}n + O(1)$ (Wang 1978?)
$n - \sqrt{n}$ (Brouwer et al. 1978 and Woolbright 1978)

Erdős and Spencer [1991] showed that any $n \times n$ array in which no entry occurs more than $(n-1)/16$ times has a transversal (in the sense of a diagonal with n different symbols on it).

Cameron and Wanless [2005] showed that every latin square possesses a diagonal in which no symbol appears more than twice.

What is the shortest possible length of a maximal partial transversal?

It is easy to see that no partial transversal of length strictly less than $\frac{1}{2}n$ can be maximal, since there are not enough 'used' symbols to fill the submatrix formed by the 'unused' rows and columns.

However, for all $n > 4$, maximal partial transversals of length $\lceil \frac{1}{2}n \rceil$ can easily be constructed using a square of order n which contains a subsquare S of order $\lfloor \frac{1}{2}n \rfloor$ and a partial transversal containing the symbols of S but not using any of the same rows or columns as S.

Number of transversals

We define $t(n)$ and $T(n)$ to be respectively the minimum and maximum number of transversals among the latin squares of order n.

We have seen that some latin squares have no transversals. Thus for lower bounds on $t(n)$ we cannot do any better than $t(n) \geq 0$, with equality occurring at least when n is even.

Open problem: find an upper bound on $t(n)$ when n is odd.

Turning to the maximum number of transversals, it should be clear that $T(n) \leq n!$ since there are only $n!$ different diagonals. An exponential improvement on this trivial bound was obtained by McKay et al. [2006]:

Theorem 9 *For* $n \geq 5$,

$$15^{n/5} \leq T(n) \leq c^n \sqrt{n}\, n!$$

where $c = \sqrt{\frac{3-\sqrt{3}}{6}}\, e^{\sqrt{3}/6} \approx 0.61354.$

The lower bound in Theorem 9 is very simple and would not be too difficult to improve. The upper bound took considerably more work, although it too is probably far from the truth.

In the same paper we reported the results of an exhaustive computation of the transversals in latin squares of orders up to and including 9.

n	$t(n)$	Mean	Std Dev	$T(n)$
2	0	0	0	0
3	3	3	0	3
4	0	2	3.46	8
5	3	4.29	3.71	15
6	0	6.86	5.19	32
7	3	20.41	6.00	133
8	0	61.05	8.66	384
9	68	214.11	15.79	2241

Table 1: Transversals in latin squares of order $n \leq 9$.

This confirms Ryser's conjecture for $n \leq 9$.

A latin square is *semisymmetric* if three of its conjugates are equal. (The corresponding quasigroup satisfies $x(yx) = y$)

The following semisymmetric squares are representatives of the unique main class with $t(n)$ transversals for $n \in \{5, 7, 9\}$. In each case the largest subsquares are shown in **bold**.

1	**2**	**3**	**4**	**5**
2	**1**	4	5	3
3	5	**1**	2	4
4	3	5	**1**	2
5	4	2	3	**1**

3	**2**	**1**	5	4	7	6
2	**1**	**3**	6	7	4	5
1	**3**	**2**	7	6	5	4
5	6	7	4	1	2	3
4	7	6	1	5	3	2
7	4	5	2	3	6	1
6	5	4	3	2	1	7

2	**1**	**3**	6	7	8	9	5	4
1	**3**	**2**	5	4	9	**6**	**7**	**8**
3	**2**	**1**	4	9	5	**7**	**8**	**6**
9	5	4	3	2	1	**8**	**6**	**7**
8	4	6	2	5	7	1	9	3
4	**7**	**9**	**8**	3	6	5	1	2
5	**8**	**7**	**9**	6	**2**	**3**	**4**	1
6	**9**	**8**	**7**	1	**4**	**2**	**3**	5
7	6	5	1	8	**3**	**4**	**2**	9

n	Lower Bound	Upper Bound
10	5504	75000
11	37851	528647
12	198144	3965268
13	1030367	32837805
14	3477504	300019037
15	36362925	2762962210
16	244744192	28218998328
17	1606008513	300502249052
18	6434611200	3410036886841
19	87656896891	41327486367018
20	697292390400	512073756609248
21	5778121715415	6803898881738477

Table 2: Bounds on $T(n)$ for $10 \leq n \leq 21$.

The lower bound in each case is constructive and likely to be very close to the true value.
When $n \not\equiv 2 \mod 4$ the lower bound comes from the group with the highest number of transversals.
When $n \equiv 2 \mod 4$ the lower bound comes from a so-called turn-square.

Turn-squares

A *turn-square* is obtained by starting with the Cayley table of a group (typically a group of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_m$ for some m) and "turning" some of the intercalates (that is, replacing a subsquare of order 2 by the other possible subsquare on the same symbols).

5	**6**	2	3	4	**0**	**1**	7	8	9
6	2	3	4	0	**1**	7	8	9	5
2	3	4	0	1	7	8	9	5	6
3	4	0	1	2	8	9	5	6	7
4	0	1	2	3	9	5	6	7	8
0	**1**	7	8	9	**5**	**6**	2	3	4
1	7	8	9	5	**6**	2	3	4	0
7	8	9	5	6	2	3	4	0	1
8	9	5	6	7	3	4	0	1	2
9	5	6	7	8	4	0	1	2	3

Using the number of transversals as a heuristic in searching for MOLS is not fail-safe. For example, the turn-square of order 14 with the most transversals (namely, 3477504) does not have any orthogonal mates. Meanwhile there are squares of order n with orthogonal mates but which possess only the bare minimum of n transversals.

Theoretical estimates are hard

There are not even good estimates for z_n, the number of transversals of the cyclic group of order n. In 1991, Vardi predicted:

Conjecture 10 *There exist real constants* $0 < c_1 < c_2 < 1$ *such that* $c_1^n n! \leq z_n \leq c_2^n n!$ *for all odd* $n \geq 3$.

Vardi makes this conjecture while considering a variation on the toroidal n-queens problem.

The upper bound is true. We can take

$c_2 = 0.9153$ [Cooper and Kovalenko 1996]

$c_2 = 1/\sqrt{2} \approx 0.7071$ [Kovalenko 1996]

$c_2 = 0.614$ [McKay et al. 2006]

Finding a lower bound of the form given in Conjecture 10 is still an open problem.

Cooper et al. [2000] estimated that perhaps the correct rate of growth for z_n is around $0.39^n\, n!$.

Finite Groups

The study of transversals in groups has been phrased in terms of the equivalent concepts of *complete mapping* and *orthomorphisms*.
A permutation θ of the elements of a quasigroup $(Q, \oplus)$ is a *complete mapping* if $\eta : Q \mapsto Q$ defined by $\eta(x) = x \oplus \theta(x)$ is also a permutation.
The permutation η is known as an *orthomorphism* of $(Q, \oplus)$.
All results on transversals could be rephrased in terms of complete mappings and/or orthomorphisms because:

Theorem 11 *Let $(Q, \oplus)$ be a quasigroup and L_Q its Cayley table. Then $\theta : Q \mapsto Q$ is a complete mapping iff we can locate a transversal of L_Q by selecting, in each row x, the entry in column $\theta(x)$. Similarly, $\eta : Q \mapsto Q$ is an orthomorphism iff we can locate a transversal of L_Q by selecting, in each row x, the entry containing symbol $\eta(x)$.*

The extra structure in groups allows for much stronger results. For example, suppose we know of a transversal of L_G that comprises a choice from each row i of an element g_i. Let g be any fixed element of G. Then if we select from each row i the element $g_i g$ this will give a new transversal and as g ranges over G the transversals so produced will be mutually disjoint. Hence

Theorem 12 *If L_G has a single transversal then it has a decomposition into disjoint transversals.*

Which groups have transversals?

We are very close to answering this question. Consider:

(i) L_G has a transversal.

(ii) L_G can be decomposed into disjoint transversals.

(iii) There exists a latin square orthogonal to L_G.

(iv) There is some ordering of the elements of G, say $a_1, a_2, \ldots, a_n$, such that $a_1 a_2 \cdots a_n = \varepsilon$, where ε denotes the identity element of G.

(v) The Sylow 2-subgroups of G are trivial or non-cyclic.

(i), (ii) and (iii) are equivalent.
Paige [1951] showed that (i) implies (iv). Hall and Paige [1955] then showed that (iv) implies (v). They also showed that (v) implies (i) if G is a soluble, symmetric or alternating group. They conjectured that (v) is equivalent to (i) for all groups.

It was subsequently noted by Dénes and Keedwell [1989] that both (iv) and (v) hold for all non-soluble groups, which proved that (iv) and (v) are equivalent. A more direct and elementary proof of this fact was given by Vaughan-Lee and Wanless [2003].
To summarise:

Theorem 13 *(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv)$\Leftrightarrow$(v)*

Conjecture 14 *(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)*

After decades of incremental progress on Conjecture 14 there has recently been what would appear to be a very significant breakthrough. In a preprint Wilcox has claimed to reduce the problem to showing it for the sporadic simple groups. In another preprint Evans is claiming that there are at most 5 candidates for counterexamples.

The number of transversals through a given entry of L_G is independent of the entry chosen. Hence

Theorem 15 *The number of transversals in L_G is divisible by $|G|$.*

McKay et al. [2006] also showed the following simple results.

Theorem 16 *The number of transversals in any symmetric latin square of order n is congruent to n modulo 2.*

Corollary 17 *If G is abelian or $|G|$ is even then the number of transversals in G is congruent to $|G|$ modulo 2.*

Corollary 17 cannot be generalised to non-abelian groups of odd order, given that the non-abelian group of order 21 has 826814671200 transversals.

Theorem 18 *If G is a group of order $n \not\equiv 1 \bmod 3$ then the number of transversals in G is divisible by 3.*

Let $z'_n = z_n/n$ denote the number of transversals through any given entry of the cyclic square of order n. Since $z_n = z'_n = 0$ for even n we shall assume for the following discussion that n is odd.
The initial values of z'_n are $z'_1 = z'_3 = 1$, $z'_5 = 3$, $z'_7 = 19$, $z'_9 = 225$, $z'_{11} = 3441$, $z'_{13} = 79259$, $z'_{15} = 2424195$, $z'_{17} = 94471089$, $z'_{19} = 4613520889$, $z'_{21} = 275148653115$, $z'_{23} = 19686730313955$ and $z'_{25} = 1664382756757625$.
If we take these numbers modulo 8 the sequence begins 1,1,3,3,1,1,3,3,1,1,3,3,1. We know from Theorem 16 that z'_n is always odd for odd n, but it is an open question whether there is any deeper pattern. We also know from Theorem 18 that z'_n is divisible by 3 when $n \equiv 2 \bmod 3$. The initial terms of $\{z'_n \bmod 3\}$ are 1,1,0,1,0,0,2,0,0,1,0,0,2.
Interestingly, z_n is the number of *diagonally cyclic latin squares* of order n (ie, the number of quasigroups on the set $\{1, 2, \ldots, n\}$ which have the transitive automorphism $(123 \cdots n)$).

n	Number of transversals in groups of order n
3	3
4	0, 8
5	15
7	133
8	0, 384, 384, 384, 384
9	2025, 2241
11	37851
12	0, 198144, 76032, 46080, 0
13	1030367
15	36362925
16	0, 235765760, 237010944, 238190592, 244744192, 125599744, 121143296, 123371520, 123895808, 122191872, 121733120, 62881792, 62619648, 62357504
17	1606008513
19	87656896891
20	0, 697292390400, 140866560000, 0, 0
21	5778121715415, 826814671200
23	452794797220965

Table 3: Transversals in groups of order $n \leq 23$.

Groups of small order

For groups of order $n \equiv 2 \bmod 4$ there can be no transversals, by Theorem 13.
Bedford and Whitaker [1999] offer an explanation for why all the non-cyclic groups of order 8 have 384 transversals.

The groups of order 4, 9 and 16 with the most transversals are the elementary abelian groups of those orders. Similarly, for orders 12, 20 and 21 the group with the most transversals is the direct sum of cyclic groups of prime order. It is an open question whether such a statement generalises to all n.

Every 2-group of order $n \leq 16$ has a number of transversals which is divisible by 2^{n-1}. It would be interesting to know if this is true for general n.

Generalised transversals

A k-*plex* in a latin square of order n is a set of kn entries which includes k representatives from each row and each column and of each symbol. A transversal is a 1-plex. The marked entries form a 3-plex in the following square:

$$\begin{array}{cccccc}
1^* & 2 & 3 & 4^* & 5 & 6^* \\
2^* & 1 & 4 & 3^* & 6^* & 5 \\
3 & 5^* & 1 & 6 & 2^* & 4^* \\
4 & 6 & 2^* & 5 & 3^* & 1^* \\
5^* & 4^* & 6^* & 2 & 1 & 3 \\
6 & 3^* & 5^* & 1^* & 4 & 2
\end{array}$$

The entries not included in a k-plex of a latin square L of order n form an $(n-k)$-plex of L.

Together the k-plex and its complementary $(n-k)$-plex are an example of what is called an *orthogonal partition* of L.
For our purposes, if L is decomposed into disjoint parts K_1, $K_2, \ldots, K_d$ where K_i is a k_i-plex then we call this a $(k_1, k_2, \ldots, k_d)$-partition of L.
A case of particular interest is when all parts are the same size, k. We call such a partition a *k-partition*. For example, the marked 3-plex and its complement form a 3-partition of the square above.

Finding a 1-partition of a square is equivalent to finding an orthogonal mate.

Some results about transversals generalise directly to other plexes, while others seem to have no analogue. Theorem 4 and Theorem 12 seem to be in the latter class.

However, the following are direct analogues of earlier results.

Theorem 19 *Suppose that q and k are odd integers and m is even. No q-step type latin square of order mq possesses a k-plex.*

Theorem 20 *Let G be a group of finite order n with a non-trivial cyclic Sylow 2-subgroup. The Cayley table of G contains no k-plex for any odd k but has a 2-partition and hence contains a k-plex for every even k in the range $0 \leq k \leq n$.*

The situation for even k is quite different to the odd case. Rodney conjectures that every latin square has a duplex. This conjecture was strengthened in [W2002] to the following:

Conjecture 21 *Every latin square has the maximum possible number of disjoint duplexes. In particular, every latin square of even order has a 2-partition and every latin square of odd order has a $(2, 2, 2, \ldots, 2, 1)$-partition.*

Depending on whether a soluble group has a non-trivial cyclic Sylow 2-subgroup, it either has a k-plex for all possible k, or has them for all possible even k but no odd k.
If the Hall-Paige conjecture could be proved it would completely resolve the existence question of plexes in groups, and these would remain the only two possibilities.
Other scenarios occur for latin squares that are not based on groups. For example, the square at the start of this section has no transversal but clearly does have a 3-plex.

Conjecture 22 *For all even $n > 4$ there exists a latin square of order n which has no transversal but does contain a 3-plex.*

Theorem 23 *For all even n there exists a latin square of order n which has k-plexes for every odd value of k between $\lfloor \frac{1}{4}n \rfloor$ and $\lceil \frac{3}{4}n \rceil$, but not for any odd value of k outside this range.*

Interestingly, there is no known example of odd integers $a < b < c$ and a latin square which has an a-plex and a c-plex but no b-plex.

The union of an a-plex and a disjoint b-plex of a latin square L is an $(a+b)$-plex of L. However, it is not always possible to split an $(a+b)$-plex into an a-plex and a disjoint b-plex.

Consider a duplex which consists of $\frac{1}{2}n$ disjoint intercalates (latin subsquares of order 2).

We say that a k-plex is *indivisible* if it contains no c-plex for $0 < c < k$.
For every k there is a indivisible k-plex in some sufficiently large latin square. This was first shown in [W2002], but "sufficiently large" in that case meant quadratic in k. This was recently improved to linear in k as a corollary of the following result.

Theorem 24 *For every $k \geq 2$ there exists a latin square of order $2k$ which contains two disjoint indivisible k-plexes.*

This is slightly surprising.

It is an open question for what values of k and n there is a latin square of order n containing an indivisible k-plex. However, we know the answer when k is small relative to n.

Theorem 25 *Let n and k be positive integers satisfying $5k \leq n$. Then there exists a latin square of order n containing an indivisible k-plex.*

So far we have started with a latin square and asked what sort of plexes it might have. Now we want to start with a plex and ask what latin squares it might be contained in.

A *k-homogeneous partial latin square* of order n is an $n \times n$ array in which each cell is either blank or filled (the latter meaning that it contains one of $\{1, 2, \ldots, n\}$), and which has the properties that (i) no symbol occurs twice within any row or column, (ii) each symbol occurs k times in the array, (iii) each row and column contains exactly k filled cells.

We can then sensibly ask whether this k-homogeneous partial latin square is a k-plex. If it is then we say the partial latin square is *completable* because the blank entries can be filled in to produce a latin square.

Theorem 26 *If $1 < k < n$ and $k > \frac{1}{4}n$ then there exists a k-homogeneous partial latin square of order n which is not completable.*

Burton [1997] and Daykin and Häggkvist [1984] independently conjecture that if $k \leq \frac{1}{4}n$ then every k-homogeneous partial latin square of order n is completable.
It seems certain that for k sufficiently small relative to n, every k-homogeneous partial latin square of order n is completable. This has already been proved when $n \equiv 0 \bmod 16$.

The following partial extension result due to Burton also seems relevant.

Theorem 27 *For $k \leq \frac{1}{4}n$ every k-homogeneous partial latin square of order n is contained in a $(k+1)$-homogeneous partial latin square of order n.*

Covering radii for sets of permutations

A novel approach to Ryser and Brualdi's conjectures has recently been opened up by Andre Kézdy and Hunter Snevily. To explain their approach, we need some terminology.
Consider the *symmetric group* S_n as a metric space equipped with *Hamming distance*. That is, the distance between two permutations $g, h \in S_n$ is the number of points at which they disagree (n minus the number of fixed points of gh^{-1}). Let P be a subset of S_n. The *covering radius* $\mathrm{cr}(P)$ of P is the smallest r such that the balls of radius r with centres at the elements of P cover the whole of S_n. In other words every permutation is within distance r of some member of P, and r is chosen to be minimal with this property.

Theorem 28 *Let $P \subseteq S_n$ be a set of permutations. If $|P| \leq n/2$, then $\mathrm{cr}(P) = n$. However, there exists P with $|P| = \lfloor n/2 \rfloor + 1$ and $\mathrm{cr}(P) < n$.*

Given n and s, what is the smallest m such that there is a set S of permutations with $|S| = m$ and $\mathrm{cr}(S) \leq n - s$? We let $f(n, s)$ denote this minimum value m.
This problem can also be interpreted in graph-theoretic language. Define the graph $G_{n,s}$ on the vertex set S_n, with two permutations being adjacent if they agree in at least s places. Now the size of the smallest dominating set in $G_{n,s}$ is $f(n, s)$.
Theorem 28 shows that $f(n, 1) = \lfloor n/2 \rfloor + 1$. Since any two distinct permutations have distance at least 2, we see that $f(n, n-1) = n!$ for $n \geq 2$. Moreover, $f(n, s)$ is a monotonic increasing function of s.
The next case to consider is $f(n, 2)$. Kézdy and Snevily made the following conjecture in unpublished notes.

Conjecture 29 *If n is even, then $f(n, 2) = n$; if n is odd, then $f(n, 2) > n$.*

This conjecture has several connections with transversals.

The rows of a latin square of order n form a *sharply transitive set* of permutations (that is, for any i and j, exactly one permutation carries i to j); and every sharply transitive set is the set of rows of a latin square.

Theorem 30 *Let S be a sharply transitive subset of S_n. Then S has covering radius at most $n-1$, with equality if and only if the corresponding latin square has a transversal.*

Corollary 31 *If there exists a latin square of order n with no transversal, then $f(n,2) \leq n$. In particular, this holds for n even.*

Hence Conjecture 29 implies Ryser's conjecture, as Kézdy and Snevily observed. In fact a stronger result holds:

Theorem 32 *If S is the set of rows of a latin square L of order n with no transversal, then S has covering radius $n-2$.*

The following result is also due to Kézdy and Snevily.

Theorem 33 *Conjecture* 29 *implies Brualdi's conjecture.*

In other words, to solve the longstanding Ryser and Brualdi conjectures it may suffice to answer this: How small can we make a subset $S \subset S_n$ which has the property that every permutation in S_n agrees with some member of S in at least two places?

We used latin squares to find an upper bound for $f(n, 2)$ when n is even. For odd n we can also find upper bounds based on latin squares. The idea is to choose a latin square with few transversals, or whose transversals have a particular structure, and add a small set of permutations meeting each transversal twice. For $n = 5, 7, 9$, a single extra permutation suffices, showing that $f(n, 2) \leq n + 1$ in these cases.

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1
1	3	4	2	5

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1	2	6	7	4	5
4	5	6	7	1	2	3
5	4	7	1	6	3	2
6	7	4	2	3	5	1
7	6	5	3	2	1	4
3	2	1	7	6	5	4

1	3	2	4	6	5	7	9	8
2	1	3	5	4	6	8	7	9
3	2	1	7	9	8	4	6	5
4	6	5	9	8	7	1	3	2
5	4	6	8	7	9	3	2	1
6	5	4	2	1	3	9	8	7
7	9	8	1	3	2	5	4	6
8	7	9	3	2	1	6	5	4
9	8	7	6	5	4	2	1	3
5	4	6	1	3	2	9	8	7

In general, we have the following:

Theorem 34 *$f(n, 2) \leq \frac{4}{3}n + O(1)$ for all n.*