Analytic Filters I

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Plan

Filters

Definition

Let E be a set . A set $\mathscr{F}$ of subsets of E is called a filter if

- $\emptyset \notin \mathscr{F}$
- for $M \in \mathscr{F}$ and $M \subset N \subset E$ we have $N \in \mathscr{F}$
- for M and N in $\mathscr{F}$ we have $M \cap N \in \mathscr{F}$

Definition

A filter $\mathscr{F}$ on E is said to be free if $\bigcap_{M \in \mathscr{F}} M = \emptyset$.

Ideals

Definition

Let E be a set . A set $\mathscr{F}$ of subsets of E is called an ideal si

- $E \notin \mathscr{I}$
- for $N \in \mathscr{I}$ and $M \subset N$ we have $M \in \mathscr{I}$
- for M and N in $\mathscr{I}$ we have $M \cup N \in \mathscr{I}$

Associating an ideal to a filter

Let $\mathscr{F}$ be a filter on the set E. Then the set $\mathscr{F}^*$ of complements in E of members of $\mathscr{F}$ is an ideal called associated to $\mathscr{F}$:

$$\mathscr{F}^* = \{E \setminus M : M \in \mathscr{F}\}$$

Clearly $\mathscr{F}$ and $\mathscr{F}^*$ are disjoint.

The Fréchet filter

In the sequel the domains of the filters will always be countable and most often equal to ω. And all filters will be free.

Example

The set of cofinite subsets of ω is a free filter $\mathscr{N}_1$ called the Fréchet filter.

So, for $M \subset \omega$:

$$M \in \mathscr{N}_1 \iff \exists p \, \forall q \geq p \quad q \in M$$

Katětov's Filters

If $(\mathscr{F}_n)$ is a sequence of filters on a set D, one defines a filter $\mathscr{F}$ on $\omega \times D$ by

$$M \in \mathscr{F} \iff \exists p \, \forall n \geq p \quad M(n) \in \mathscr{F}_n$$

where $M(n) = \{i \in D : (n, i) \in M\}$.
This means that $\{n \in \omega : M(n) \in \mathscr{F}_n\}$ is in $\mathscr{N}_1$.

In particular, if $D = \omega$ and $\mathscr{F}_n = \mathscr{N}_1$ for all n, this construction yields the filter $\mathscr{N}_2$: a subset M of $\omega \times \omega$ belongs to $\mathscr{N}_2$ iff

$$\exists p_0 \, \forall p \geq p_0 \, \exists q_0 \, \forall q \geq q_0 \quad (p, q) \in M$$

And repeating (transfinitely) this operation, one defines

Katětov's filters $\mathscr{N}_\xi$.

Limit along a filter

Let E be a set, $\mathscr{F}$ a filter on E, X a topological space and f a mapping from E to X.

Definition

We say that $\mathscr{F}$ converges to $a \in X$ along $\mathscr{F}$ if for all neighborhood V of a, $f^{-1}(V) \in \mathscr{F}$.

In particular if $\mathscr{F}$ is a filter on ω, and (x_n) a sequence in X, we say that it converges to a along $\mathscr{F}$ if the mapping $n \mapsto x_n$ converges to a along $\mathscr{F}$.

Limit along a filter

A sequence which converges to a along $\mathscr{N}_1$ is a sequence which converges to a in the common sense.

If $(x_{p,q})$ is a (double) sequence of points of the space X such that $x_p = \lim_{q\to\infty} x_{p,q}$ exists for each p and (x_p) converges to a, then $(x_{p,q})$ converges to a along $\mathscr{N}_2$.

One should notice that this condition is not necessary for the convergence of the sequence to a along $\mathscr{N}_2$.

Limit of a sequence of functions

Let $\mathscr{F}$ be a filter on ω and (f_n) a sequence of functions : $X \to Y$.

We say that the sequence (f_n) converges along $\mathscr{F}$ to the function $f : X \to Y$ if for all $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$ along $\mathscr{F}$.

Borel Sets

All spaces under consideration will be metrizable and separable.

Recall that a Polish space is a separable space whose topology can be defined by a complete metric.

If X is a topological space the least σ-algebra of subsets of X containing all open sets is called the Borel σ-algebra of X, and its members are called the Borel subsets of X.

Classification of Borel sets

Let X be a separable metrizable space. We define classes of Borel sets by induction on the countable ordinal ξ :

$\mathbf{\Sigma}^0_1(X)$ is the set of open sets in X, and $\mathbf{\Pi}^0_1(X)$ is the set of closed sets in X

$\mathbf{\Sigma}^0_\xi(X)$ is the set of all countable unions $\bigcup_{n\in\omega} A_n$ where each A_n belongs to $\bigcup_{\eta<\xi} \mathbf{\Pi}^0_\eta(X)$.

$\mathbf{\Pi}^0_\xi(X)$ is the set of all countable intersections $\bigcap_{n\in\omega} A_n$ where each A_n belongs to $\bigcup_{\eta<\xi} \mathbf{\Sigma}^0_\eta(X)$.

It is clear that for all $\xi < \omega_1$ the $\mathbf{\Pi}^0_\xi$ sets are the complements in X of the $\mathbf{\Sigma}^0_\xi$ sets.

$\mathbf{\Delta}^0_\xi(X)$ is the set of all subsets of X which are both $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$.

Classification of Borel sets

Theorem

If $f : X \to Y$ is continuous and $A \in \mathbf{\Sigma}^0_\xi(Y)$ (resp. $\mathbf{\Pi}^0_\xi(Y)$, $\mathbf{\Delta}^0_\xi(Y)$), then $f^{-1}(A)$ is in $\mathbf{\Sigma}^0_\xi(X)$ (resp. $\mathbf{\Pi}^0_\xi(X)$, $\mathbf{\Delta}^0_\xi(X)$) .

This is true for $\xi = 1$ by definition of the continuity, and clear by induction on ξ.

Analytic sets

Definition

A space X is called analytic if it is the continuous image of some Polish space.

It is clear that the continuous image of an analytic space is analytic too.

Theorem

A Borel subset of an analytic space is itself an analytic space.

Analytic sets

For a Polish space X we denote by $\mathbf{\Sigma}^1_1(X)$ the class of analytic subsets of X.

Theorem

If X is analytic, $f : X \to Y$ is continuous and $A \in \mathbf{\Sigma}^1_1(Y)$ then $f^{-1}(A)$ is in $\mathbf{\Sigma}^1_1(X)$.

The Separation theorem

The most important result (at least here) about analytic sets is the following separation theorem ;

Theorem (Luzin - Suslin)

If A_0 and A_1 are two disjoint analytic subsets of the space X, there exists a Borel subset B of X which separates A_0 from A_1 (it is $A_0 \subset B$ and $A_1 \subset B^c$).

Corollary

If A is an analytic subset of X such that $X \setminus A$ is also analytic, then A is Borel.

Borel functions

A function $f : X \to Y$ will be said to be Borel of class ξ if $f^{-1}(U) \in \mathbf{\Sigma}^0_{1+\xi}(X)$ for every open subset U of Y (equivalently for every open set in a countable basis).

We shall denote by $B_\xi(X, Y)$ the set of Borel functions of class ξ from X to Y. In particular $B_0(X, Y)$ is the set of continuous functions.
Notice that this is not the common definition of the Baire class of a Borel function.

For all Borel function $f : X \to Y$, there is a $\xi \in \omega_1$ such that $f \in B_\xi(X, Y)$.

Borel functions

Theorem

If X and Y are Polish spaces and $f : X \to Y$ is a Borel function, then its graph G is a Borel subset of $X \times Y$.

Let (U_n) be a basis of the topology of Y. Then

$$G = \bigcap_n [(f^{-1}(U_n) \times U_n) \cup (X \times U_n^c)]$$

Theorem

If X and Y are Polish spaces and $f : X \to Y$ is a function, then f is Borel if and only if its graph is an analytic subset of $X \times Y$.

Limits of Borel functions

If the sequence (f_n) of Borel functions of class ξ from X to Y converges pointwise to a function f, then f is of class $\xi + 1$. Moreover if λ is a limit ordinal, if each f_n is of class $< \lambda$ and if the sequence (f_n) converges pointwise to f, then f is of class $\lambda + 1$.

Conversely , if $\xi > 1$, or $Y = \mathbb{R}$, or $dim(X) = 0$, every Borel function : $X \to Y$ of class $\xi + 1$ is the pointwise limit of a sequence of Borel functions of class ξ.
And for limit λ, every f of class $\lambda + 1$ is the pointwise limit of a sequence of Borel functions of classes $< \lambda$.

Limits of Borel functions

The set $C_\xi(X,\mathbb{R})$ of real functions of Baire class ξ is inductively defined as the set of pointwise limits of sequences of functions of Baire classes $< \xi$.
Then, by Lebesgue-Hausdorff theorem we have
$C_n(X,\mathbb{R}) = B_n(X,\mathbb{R})$ for $n \in \omega$, and $C_\xi(X,\mathbb{R}) = B_{\xi+1}(X,\mathbb{R})$ for infinite ξ.

Analytic filters

Identifying the set $\mathscr{P}(E)$ of subsets of the countable set E to the product space $\{0,1\}^E$, we equip $\mathscr{P}(E)$ with a compact metrizable topology. A filter $\mathscr{F}$ is then said to be analytic, Borel, $\mathbf{\Sigma}^0_\xi$, ..., if $\mathscr{F}$ is analytic, Borel, $\mathbf{\Sigma}^0_\xi$, ..., as a subset of the compact space $\mathscr{P}(E)$.

Theorem

If the sequence (f_n) of continuous functions $: X \to Y$ converges to f along an analytic filter $\mathscr{F}$, then the function f is Borel.

Analytic filters

Theorem

Let $\mathscr{F}$ be an analytic free filter on E. Then $\mathscr{F}$ is meager as a subset of $\mathscr{P}(E)$.

If not, $\mathscr{F}$, which has the Baire property, would be co-meager on a basic open set $U_{I,J} = \{M : I \subset M \text{ and } M \cap J = \emptyset\}$ (where I and J are finite subsets of E). The mapping $\sigma : M \mapsto E \setminus M$ is an auto-homeomorphism of $\mathscr{P}(E)$. Thus $\mathscr{F}^*$ would be co-meager on $\sigma(U_{I,J}) = U_{J,I}$.

The mapping $\tau : M \mapsto M \,\Delta\, (I \cup J)$ is also an auto-homeomorphism of $\mathscr{P}(E)$, and $\tau(\mathscr{F}) = \mathscr{F}$. Hence $\tau(F)$ is co-meager on $\tau(U_{I,J}) = U_{J,I}$. And $\mathscr{F} \cap \mathscr{F}^*$ would be co-meager on $U_{J,I}$, a contradiction. □

An involution on $\mathscr{P}(E)$

The mapping $\sigma : M \mapsto E \setminus M$ is an auto-homeomorphism of $\mathscr{P}(E)$. It follows that $\mathscr{F}^*$ is analytic if and only if so is $\mathscr{F}$, and that if $\mathscr{F}$ is Borel then $\mathscr{F}^*$ is Borel of the same class.

If follows also that if there is a $\mathbf{\Sigma}^0_\xi$ set C which separates $\mathscr{F}$ from $\mathscr{F}^*$, the set $C^* = \sigma(C)$ is a $\mathbf{\Sigma}^0_\xi$ set separating $\mathscr{F}^*$ from $\mathscr{F}$, thus that $\mathscr{P}(E) \setminus C^*$ is a $\mathbf{\Pi}^0_\xi$ set separating $\mathscr{F}$ from $\mathscr{F}^*$.

Separating sets

Let $\mathscr{F}$ be an analytic filter. There exists a Borel set separating $\mathscr{F}$ from $\mathscr{F}^*$, hence a countable ordinal ξ for which exists some $\mathbf{\Sigma}^0_{1+\xi}$ set C separating $\mathscr{F}$ from $\mathscr{F}^*$. There exists then a $\mathbf{\Pi}^0_{1+\xi}$ subset C' of $\mathscr{P}(E)$, which separates $\mathscr{F}$ from $\mathscr{F}^*$.

Theorem

Let $\xi \geq 1$ and $\mathscr{F}$ be a filter on ω. If there is a $\mathbf{\Sigma}^0_{1+\xi}$ set C which separates $\mathscr{F}$ from $\mathscr{F}^*$, every $\mathscr{F}$-limit of a sequence of continuous functions $f_n : X \to Y$ is Borel of class ξ.

Separating sets

Assume first that X is 0-dimensional. We can replace the continuous functions (f_n) by locally constant functions (g_n) such that $d(f_n(x), g_n(x)) \leq 2^{-n}$ for all $x \in X$.
If $U \subset Y$ is an open subset, there are open sets V_p in Y such that $\overline{V_p} \subset U$ and $U = \bigcup_p V_p$.
Then, letting $\Phi_p(x) = \{n : g_n(x) \in V_p\}$, Φ_p is continuous and we get

$$f^{-1}(U) = \bigcup_p \Phi_p^{-1}(S)$$

hence $f^{-1}(U) \in \mathbf{\Sigma}^0_{1+\xi}$, and $f \in B_\xi(X, Y)$.

Separating sets

In the general case we can assume X is Polish. Then there exists a 0-dimensional Polish space X_0 and a continuous open and onto mapping $\pi : x_0 \to X$. We have $f_n \circ \pi \underset{\mathscr{F}}{\to} f \circ \pi$, and $f \circ \pi$ is of class ξ.

Theorem

Let X and X_0 be Polish spaces, $\pi : X_0 \to X$ be continuous, open and onto, and $g : X_0 \to Y$ be Borel of class $\xi \geq 1$. Then there is a Baire-1 section $s : X \to X_0$ of π such that $g \circ s$ be of class ξ.

Take $g = f \circ \pi$. Then $g \circ s = f \circ \pi \circ s = f$ is of class ξ. □

Separating sets

From now on we will denote by $\mathscr{C}_{\mathscr{F}}(X)$ the set of real functions on X that are $\mathscr{F}$-limits of a sequence of continuous functions.

So $\mathscr{C}_{\mathscr{F}}(X) \subset B_{\xi}(X, \mathbb{R})$ whenever $\mathbf{\Sigma}^0_{1+\xi}$ separates $\mathscr{F}$ from $\mathscr{F}^*$.

Analytic Filters II

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Plan

Games

We consider games of the following kind : A and B are (countable) sets and two players play alternatively elements a_n of A and b_n of B :

I	II
a_0	b_0
a_1	b_1
a_2	b_2
$\ldots$	$\ldots$
a_n	b_n
$\ldots$	$\ldots$

Games

Player I (resp. Player II) constructs this way $\alpha = (a_n) \in A^\omega$ (resp. $\beta = (b_n) \in B^\omega$). A subset P of $A^\omega \times B^\omega$ is given, and Player I wins the run iff $(\alpha, \beta) \in P$. Else Player II wins the run.

Definition

A strategy for Player I is a mapping $\sigma : B^{<\omega} \to A$. We say that Player I follows the strategy σ in a run $((a_n), (b_n))$ iff for all $n \geq 0$, $a_n = \sigma(\langle b_0, b_1, \ldots, b_{n-1} \rangle)$.

Definition

The strategy σ for Player I is said to be winning if every run following σ is won by Player I.

Games

We define in the same way the notion of (winning) strategy for Player II.

Definition

A strategy for Player II is a mapping $\tau : A^{<\omega} \to B$. We say that Player II follows the strategy τ in a run $((a_n), (b_n))$ iff for all $n \geq 0$, $b_n = \tau(\langle a_0, a_1, \ldots, a_n \rangle)$.

Definition

The strategy τ for Player II is said to be winning if every run following τ is won by Player II.

Player I and Player II cannot have both a winning strategy : if not there would be a run compatible with both strategies, and it would be won simultaneously by both players, a contradiction.

Determinacy

Definition

A game is said to be determined if one of the players has a winning strategy.

The game defined by $P \subset A^\omega \times B^\omega$ is said to be Borel if P is a Borel subset of $A^\omega \times B^\omega$.

Theorem (D.A. Martin)

All Borel games are determined.

Wadge games

Let Z be a Borel subset of A^ω, X and Y be disjoint Borel subsets of B^ω.
In the Wadge game $W(Z; X, Y)$, Player I plays $\alpha \in A^\omega$, Player II plays $\beta \in B^\omega$ and Player II wins the run iff

$$\alpha \in Z \Longrightarrow \beta \in X \quad \wedge \quad \alpha \notin Z \Longrightarrow \beta \in Y$$

it is

$$(\alpha, \beta) \in (Z \times X) \cup (Z^c \times Y)$$

This game is clearly Borel, hence determined.

Wadge games

Theorem

If Player II has a winning strategy in the Wadge game $W(Z; X, Y)$, there exists a continuous function $f : A^\omega \to B^\omega$ such that $f(Z) \subset X$ and $f(Z^c) \subset Y$.

We then say that the function f reduces Z to the pair (X, Y). If $Y = X^c$ we say that f reduces Z to X.

Wadge games

Theorem

If Player I has a winning strategy in the Wadge game $W(Z; X, Y)$ and Z is $\mathbf{\Pi}^0_\xi$ in A^ω, there exists a $\mathbf{\Sigma}^0_\xi$ set S in B^ω which separates X from Y.

If X and Y are analytic disjoint subsets of B^ω, the Wadge game $W(Z; X, Y)$ is no longer Borel. But one can prove that it is still determined and the previous two theorems remain valid.

The class $\mathbf{\Delta}^0_\xi$

Theorem

If $\xi \geq 2$ and $S \in \mathbf{\Delta}^0_\xi(2^\omega)$, there exists $Z \in \mathbf{\Delta}^0_\xi(2^\omega)$ such that no continuous function $f : 2^\omega \to 2^\omega$ reduces Z to S.

Suppose this is false.
Since the space $\mathscr{C}(2^\omega, 2^\omega)$ of continuous functions from 2^ω to itself is Polish (when equipped with the uniform topology), there is a $\mathbf{\Pi}^0_2$ subset G of 2^ω and a continuous onto mapping $\varphi : G \to \mathscr{C}(2^\omega, 2^\omega)$.
Then the sets $X = \{(\alpha, \beta) \in 2^\omega \times 2^\omega : \alpha \in G \text{ and } \varphi(\alpha)(\beta) \in S\}$ and $Y = \{(\alpha, \beta) \in 2^\omega \times 2^\omega : \alpha \in G \text{ and } \varphi(\alpha)(\beta) \in S^c\}$ are both $\mathbf{\Pi}^0_\xi$ and disjoint. Thus they are separated by a $\mathbf{\Delta}^0_\xi$ set $H \subset 2^\omega \times 2^\omega$.

The class $\mathbf{\Delta}^0_\xi$

Claim

The set H is $\mathbf{\Delta}^0_\xi$-universal.

Let Z be a $\mathbf{\Delta}^0_\xi$ subset of 2^ω. There is a continuous function f such that $Z = f^{-1}(S)$, hence an $\alpha \in G$ such that $f = \varphi(\alpha)$. Then

$$H(\alpha) := \{\beta : (\alpha, \beta) \in H\} \supset \{\beta : f(\beta) \in S\} = Z$$

and

$$H(\alpha)^c := \{\beta : (\alpha, \beta) \in H^c\} \supset \{\beta : f(\beta) \in S^c\} = Z^c$$

i.e. $H(\alpha) = Z$.

The class $\mathbf{\Delta}^0_\xi$

Claim

There is no $\mathbf{\Delta}^0_\xi$-universal set.

Suppose $H \subset 2^\omega \times 2^\omega$ is $\mathbf{\Delta}^0_\xi$-universal. Consider $D = \{\alpha : (\alpha, \alpha) \in H^c\}$. Then D is $\mathbf{\Delta}^0_\xi$, and there is an α^* such that $D = H(\alpha^*)$. Then we have

$$\alpha^* \notin D \iff (\alpha^*, \alpha^*) \in H \iff \alpha^* \in H(\alpha^*) \iff \alpha^* \in D$$

a contradiction. □

Rank of an analytic filter

Recall that $\mathscr{C}_{\mathscr{F}}(X)$ denotes the set of real functions on X which are limit along $\mathscr{F}$ of a sequence of continuous functions.

Theorem

Let $\mathscr{F}$ be a filter on ω, X a 0-dimensional space and Z a Borel subset of X. Then $\mathbb{1}_Z \in \mathscr{C}_{\mathscr{F}}(X)$ if and only if Z is reducible to the pair $(\mathscr{F}, \mathscr{F}^*)$ in $\mathscr{P}(\omega) \simeq 2^\omega$.

Corollary

If moreover $\mathbb{1}_Z \in \mathscr{C}_{\mathscr{F}}(X)$ and S separates $\mathscr{F}$ from $\mathscr{F}^*$, then Z is reducible to S.

Rank of an analytic filter

Lemma

Let X be a 0-dimensional space and $\mathscr{F}$ be a $\mathbf{\Sigma}^1_1$ filter on ω. If for some $\eta < \xi$ no $\mathbf{\Sigma}^0_{1+\eta}$ subset of $\mathscr{P}(\omega)$ separates $\mathscr{F}$ from $\mathscr{F}^*$, then for each $Z \in \mathbf{\Pi}^0_{1+\eta}(X)$, $\mathbb{1}_Z$ belongs to $\mathscr{C}_{\mathscr{F}}(X)$.

Lemma

Let $\mathscr{F}$ be a $\mathbf{\Sigma}^1_1$ filter on ω. If $\mathbb{1}_Z \in \mathscr{C}_{\mathscr{F}}(X)$ for all $\eta < \xi$ and all $Z \in \mathbf{\Pi}^0_{1+\eta}(X)$, then for each $Z \in \mathbf{\Delta}^0_{1+\xi}(X)$, $\mathbb{1}_Z$ belongs to $\mathscr{C}_{\mathscr{F}}(X)$.

Rank of an analytic filter

Theorem

Let X be a 0-dimensional space and $\mathscr{F}$ be a $\mathbf{\Sigma}^1_1$ filter on ω. If for all $\eta < \xi$ no $\mathbf{\Sigma}^0_{1+\eta}$ subset of $\mathscr{P}(\omega)$ separates $\mathscr{F}$ from $\mathscr{F}^*$, then every function $f \in B_\xi(X, \mathbb{R})$ is $\mathscr{F}$-limit of a sequence of real continuous functions on X.

We then have

$$B_\xi(X, \mathbb{R}) \subset \mathscr{C}_{\mathscr{F}}(X)$$

Rank of an analytic filter

Definition

We call rank of the analytic filter $\mathscr{F}$ the least ordinal ξ for which $\mathscr{F}$ is separated from $\mathscr{F}^*$ by a $\mathbf{\Sigma}^0_{1+\xi}$ set.

It follows from what precedes that the $\mathscr{F}$-limits of sequences of real continuous functions (on a 0-dimensional space) are exactly the real Borel functions of class $\mathrm{rk}(\mathscr{F})$

Rank of an analytic filter

Filters of rank 0 are the non-free filters.

Theorem

The rank of $\mathscr{N}_1$ is 1.

Every Baire-1 function is limit of continuous functions along $\mathscr{N}_1$, hence $\mathrm{rk}(\mathscr{N}_1) \geq 1$.
$\mathscr{N}_1$ is $\mathbf{\Sigma}^0_2$ and separates $\mathscr{N}_1$ from ${\mathscr{N}_1}^*$, hence $\mathrm{rk}(\mathscr{N}_1) \leq 1$.

Rank of an analytic filter

Theorem

The rank of $\mathscr{N}_2$ is 2.

Since every real Baire-2 function φ can be written as $\lim_{p\to\infty} \varphi_p$ where φ_p is Baire-1, hence $\varphi_p = \lim_{q\to\infty} \varphi_{p,q}$ with $\varphi_{p,q}$ continuous, one has $\varphi_{p,q} \underset{\mathscr{N}_2}{\to} \varphi$, thus $\mathrm{rk}(\mathscr{N}_2) \geq 2$. Moreover, the set

$$S = \{M \subset \omega^2 : \exists n\, \forall p \geq n\, \forall m\, \exists q \geq m \quad (p,q) \in M\}$$

is $\mathbf{\Sigma}^0_3$ and separates $\mathscr{N}_2$ from $\mathscr{N}_2^*$, hence $\mathrm{rk}(\mathscr{N}_2) \leq 2$.

More generally, for every countable ordinal ξ there are Borel filters of rank ξ. In particular, $\mathscr{N}_\xi$ has rank ξ for finite ξ, and rank $\xi + 1$ for infinite ξ.

Rank of an analytic filter

Theorem

If $\mathscr{F}$ is an analytic filter of rank $\xi \geq 1$ on ω, no $\mathbf{\Delta}^0_{1+\xi}$ subset of $\mathscr{P}(\omega)$ can separate $\mathscr{F}$ from $\mathscr{F}^*$.

If S were a $\mathbf{\Delta}^0_{1+\xi}$ subset of $\mathscr{P}(\omega)$ separating $\mathscr{F}$ from $\mathscr{F}^*$, every $\mathbf{\Delta}^0_{1+\xi}$ subset Z of 2^ω would satisfy $\mathbb{1}_Z \in \mathscr{C}_{\mathscr{F}}(2^\omega)$, hence would be reducible to S. And this is impossible. □

Refining of analytic filters

Theorem

If $\mathscr{F}$ and $\mathscr{G}$ are analytic filters and if $\mathscr{G}$ is finer than $\mathscr{F}$ (it is $\mathscr{F} \subset \mathscr{G}$), then $\mathrm{rk}(\mathscr{F}) \leq \mathrm{rk}(\mathscr{G})$.

Theorem

If $\mathscr{F}$ is an analytic filter of rank ξ, there exists a Borel filter $\mathscr{G}$ finer than $\mathscr{F}$ which is also of rank ξ.

Analytic Filters III

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Plan

1. Katětov's ordering
   - Definition
   - Lower and Upper Bounds
   - Complexity of Katětov's ordering
2. Embedding of $\mathscr{N}_2$
3. An application

Katětov's ordering

We define on the filters with countable domain an ordering $\leq_\kappa$ (called Katětov's ordering) such that for every set H of continuous functions all $\mathscr{F}$-limit of a sequence of elements of H be $\mathscr{G}$-limit of a sequence of elements of H as soon as $\mathscr{F} \leq_\kappa \mathscr{G}$

Definition

We say that $\mathscr{F} \leq_\kappa \mathscr{G}$ if there is a mapping $g : dom(\mathscr{G}) \to dom(\mathscr{F})$ such that

$$\forall M \in \mathscr{F} \quad g^{-1}(M) \in \mathscr{G}$$

(or equivalently $\forall M \in \mathscr{F}^* \quad g^{-1}(M) \in \mathscr{G}^*$).

$$\varphi n \underset{n,\mathscr{F}}{\rightarrow} \varphi \Longrightarrow \varphi g(m) \underset{m,\mathscr{G}}{\rightarrow} \varphi$$

Katětov's ordering

If $\mathscr{F}$ and $\mathscr{G}$ have the same domain and if $\mathscr{G}$ is finer than $\mathscr{F}$ (i. e. $\mathscr{F} \subset \mathscr{G}$), we have $\mathscr{F} \leq_\kappa \mathscr{G}$ (take $g = Id$).

Theorem

If $\mathscr{F} \leq_\kappa \mathscr{G}$ then $\mathrm{rk}(\mathscr{F}) \leq \mathrm{rk}(\mathscr{G})$.

For $\xi = \mathrm{rk}(\mathscr{F})$ we have

$$B_\xi(2^\omega, \mathbb{R}) = \mathscr{C}_{\mathscr{F}}(2^\omega) \subset \mathscr{C}_{\mathscr{G}}(2^\omega)$$

hence $\mathrm{rk}(\mathscr{G}) \geq \xi$.

Greatest lower bound

Theorem

If $\mathscr{F}$ and $\mathscr{G}$ are two filters with countable domains D and E, there exists a filter $\mathscr{H}$ such that $\mathscr{H} \leq_\kappa \mathscr{F}$ and $\mathscr{H} \leq_\kappa \mathscr{G}$ and that every such $\mathscr{H}'$ satisfies $\mathscr{H}' \leq_\kappa \mathscr{H}$

Take $C = D \dot{\cup} E$ the disjoint sum of D and E, and define

$$\mathscr{H} = \{M \subset C : M \cap D \in \mathscr{F} \text{ and } M \cap E \in \mathscr{G}\}$$

Clearly $\mathscr{F} \wedge \mathscr{G} = \mathscr{H}$ is Borel if so are $\mathscr{F}$ and $\mathscr{G}$. And

$$\mathrm{rk}(\mathscr{H}) = \min(\mathrm{rk}(\mathscr{F}), \mathrm{rk}(\mathscr{G}))$$

Least upper bound

Theorem

If $\mathscr{F}$ and $\mathscr{G}$ are two filters with countable domains D and E, there exists a filter $\mathscr{H}$ such that $\mathscr{F} \leq_\kappa \mathscr{H}$ and $\mathscr{G} \leq_\kappa \mathscr{H}$ and that every such $\mathscr{H}'$ satisfies $\mathscr{H} \leq_\kappa \mathscr{H}'$.

Take $C = D \times E$ and define $\mathscr{H}$ as the filter on C generated by the set

$$\{M \times N : M \in \mathscr{F} \text{ and } N \in \mathscr{G}\}$$

Clearly $\mathscr{H} = \mathscr{F} \vee \mathscr{G}$ is $\mathbf{\Sigma}^1_1$ if so are $\mathscr{F}$ and $\mathscr{G}$.

Question

If $\mathscr{F}$ and $\mathscr{G}$ are Borel, is $\mathscr{F} \vee \mathscr{G}$ Borel too ?

Least upper bound

Question

If $\mathscr{F}$ and $\mathscr{G}$ are two filters with countable domains does the equality

$$\mathrm{rk}(\mathscr{F} \vee \mathscr{G}) = \max(\mathrm{rk}(\mathscr{F}), \mathrm{rk}(\mathscr{G}))$$

hold ?

Theorem

If $\mathscr{F}$ and $\mathscr{G}$ are both of rank 1, then $\mathscr{F} \vee \mathscr{G}$ is of rank 1.

Comparison of filters of rank 1

Since $\mathrm{rk}(\mathscr{F}) \leq \mathrm{rk}(\mathscr{G})$ when $\mathscr{F} \leq_\kappa \mathscr{G}$, a (naive) question is :

Question

Does the reverse implication

$$\mathrm{rk}(\mathscr{F}) \leq \mathrm{rk}(\mathscr{G}) \Longrightarrow \mathscr{F} \leq_\kappa \mathscr{G}$$

hold?

In fact the answer is negative. Even on the set of filters of rank 1, Katětov's ordering is very complicated.

Comparison of filters of rank 1

Theorem

There exists a mapping $S \mapsto \mathscr{F}_S$ defined on the set $\mathscr{P}(\omega)$ wich assigns to each $S \subset \omega$ a $\mathbf{\Pi}^0_3$ filter $\mathscr{F}_S$ of rank 1 on ω such that

$$\mathscr{F}_S \leq_\kappa \mathscr{F}_T \iff S \subset^* T$$

(it is $S \setminus T$ is finite).

In particular, $\mathscr{F}_S$ and $\mathscr{F}_T$ are Katětov-equivalent if and only if S and T differ only by a finite set, though all filters $\mathscr{F}_S$ have same rank.

Embedding of $\mathscr{N}_1$

Theorem

If $\mathscr{F}$ is a filter of rank ≥ 1 (a free filter) with countable domain then $\mathscr{N}_1 \leq_\kappa \mathscr{F}$.

In fact, if $\varphi : D \to \omega$ is any enumeration of the domain D of $\mathscr{F}$ and $M \in \mathscr{N}_1^*$ then $\varphi^{-1}(M)$ is finite in D hence belongs to $\mathscr{F}^*$: φ is a bijective Katětov-reduction of $\mathscr{F}$ to $\mathscr{N}_1$.

Diagonalization

Definition

Let D be a set and (M_k) a sequence of subsets of D. We shall say that N diagonalizes the sequence (M_k) if $M_k \setminus N$ is finite for all k (we shall write $M_k \subset^* N$).

Definition

Let $\mathscr{F}$ be a filter on the set D and (M_k) a sequence of elements of the ideal $\mathscr{F}^*$. We shall say that (M_k) is $\mathscr{F}$-diagonalizable if there is some $N \notin \mathscr{F}$ which diagonalizes (M_k).

Diagonalization

Theorem

Let $\mathscr{F}$ be a filter on ω. Then $\mathscr{N}_2 \leq_\kappa \mathscr{F}$ if and only if there exists a sequence (M_k) of elements of $\mathscr{F}^*$ which is not $\mathscr{F}$-diagonalizable.

Filters of rank 2

Theorem

If $\mathcal{F}$ is an analytic filter of rank ≥ 2, we have $\mathcal{N}_2 \leq_\kappa \mathcal{F}$.

If $\mathcal{F}$ has rank ≥ 2, then there exists a continuous mapping $M : 2^\omega \to \mathcal{P}(\omega)$ such that

- if $\alpha \in \mathbb{Q} = \{z : \exists p \, \forall q \geq p \quad z_q = 0\}$, then $M(\alpha) \in \mathcal{F}^*$,
- if $\alpha \in \mathbb{P} = 2^\omega \setminus \mathbb{Q}$, then $M(\alpha) \in \mathcal{F}$,

since $\mathbb{1}_\mathbb{P} \in B_2(2^\omega)$ is $\mathcal{F}$-limit of continuous functions : $2^\omega \to \{0, 1\}$. Then consider the countable family $(M(\alpha))_{\alpha \in \mathbb{Q}}$ of elements of $\mathcal{F}^*$. We claim that this family is not $\mathcal{F}$-diagonalizable.

Filters of rank 2

If not, let $N \notin \mathscr{F}$ be such that $M(\alpha) \subset^* N$ for all $\alpha \in \mathbb{Q}$: then $F = M(0) \cap N^c$ is finite. And replacing N by $N \cup F \notin \mathscr{F}$, we can assume $F = \emptyset$. Consider the compact set

$$E = \{(\alpha, \beta) \in 2^\omega \times 2^\omega : M(\alpha) \cap M(\beta) \cap N^c = \emptyset\}$$

and the two $\mathbf{\Pi}^0_2$ sets

$$G_0 = \{(\alpha, \beta) \in E : \alpha \in \mathbb{P}\} \quad \text{and} \quad G_1 = \{(\alpha, \beta) \in E : \beta \in \mathbb{P}\}$$

Notice $E \neq \emptyset$ since $(0, 0) \in E$. We show that G_0 is dense in E.

Filters of rank 2

By symmetry, G_1 is dense in E too.
Then $G_0 \cap G_1$ is dense in E, hence non-empty.

We conclude that for $(\alpha, \beta) \in G_0 \cap G_1$ we have $M(\alpha) \in \mathscr{F}$, $M(\beta) \in \mathscr{F}$ and

$$N^c \cap M(\alpha) \cap M(\beta) = \emptyset$$

But then $N \supset M(\alpha) \cap M(\beta) \in \mathscr{F}$, a contradiction. □

Non 0-dimensional spaces

Let $\mathscr{F}$ be an analytic filter of rank ξ on ω. If X is a 0-dimensional Polish space and $f : X \to \mathbb{R}$ a function of Borel class ξ we proved

Theorem

There is a sequence (f_n) of continuous real functions on X which converges to f along $\mathscr{F}$.

But for general Polish spaces, it is an open question whether the above statement is still true. Nevertheless we have

Theorem

Let $\mathscr{F}$ be an analytic filter of rank $\xi \leq 2$ on ω. Then there is a sequence (f_n) of continuous real functions on X which converges to f along $\mathscr{F}$.

And beyond rank 2 ?

A natural (but still open) question:

Question

Let $\mathscr{F}$ be an analytic filter of rank $> \xi$, and $\mathscr{K}_\xi$ be the Katětov's filter of rank $\xi + 1$. Does $\mathscr{K}_\xi$ embed into $\mathscr{F}$?

LUB of filters of rank 1

Theorem

Le $\mathscr{F}$ and $\mathscr{G}$ be filters on ω of rank 1. Then $\mathscr{F} \vee \mathscr{G}$ has rank 1.

Since $\mathrm{rk}(\mathscr{F} \vee \mathscr{G}) \geq \mathrm{rk}(\mathscr{F}) = 1$, it is enough to show $\mathrm{rk}(\mathscr{F} \vee \mathscr{G}) \leq 1$. Assume by contradiction that $\mathrm{rk}(\mathscr{F} \vee \mathscr{G}) \geq 2$. It should exist a non-diagonalizable sequence (C_n) in $(\mathscr{F} \vee \mathscr{G})^*$, hence (A_n) in $\mathscr{F}^*$ and (B_n) in $\mathscr{G}^*$ such that

$$C_n \subset (A_n \times \omega) \cup (\omega \times B_n)$$

Since $\mathrm{rk}(\mathscr{F}) = \mathrm{rk}(\mathscr{G}) < 2$, (A_n) and (B_n) are diagonalizable, and there are $A \notin \mathscr{F}$ and $B \notin \mathscr{G}$ such that $A_n \subset^* A$ and $B_n \subset^* B$ for each n.

LUB of filters of rank 1

Then $\mathscr{F}_1 = \{A^c \cap M : M \in \mathscr{F}\}$ on A^c and $\mathscr{G}_1 = \{B^c \cap N : N \in \mathscr{G}\}$ on B^c are analytic filters.

Recall the following :

Theorem (Godefroy - Talagrand)

Let $\mathscr{H}$ be an meager filter on ω. Then there exists a finite-to-one mapping $h : \omega \to \omega$ such that $h(\mathscr{H}) = \mathscr{N}_1$.

This applies in particular to every analytic free filter $\mathscr{F}$, since every such filter is meager.

LUB of filters of rank 1

Choose $h : B^c \to A^c$ finite-to-one such that

$$\forall N \in \mathscr{G} \quad h(B^c \cap N) \text{ is co-finite in } A^c$$

and define

$$C = (\omega \times \omega) \setminus \{(p, q) \in A^c \times B^c : p = h(q)\}$$

Then

- $\forall n \quad (A_n \times \omega) \setminus C$ is finite.
- $\forall n \quad (\omega \times B_n) \setminus C$ is finite.
- $\forall M \in \mathscr{F} \; \forall N \in \mathscr{G} \quad M \times N \not\subset C$.

So C diagonalizes (C_n) but $C \notin \mathscr{F} \vee \mathscr{G}$, a contradiction. □